Montogue

(AGLA)

Analytic Geometry and Linear Algebra at the AP Level

Syllabus and Sample Problems

Exam Format

→ The AGLA exam consists of two sections. Section I consists of 40 multiple-choice problems; Section II consists of 14 open-ended problems.

→ Section I is divided into two parts: Part A includes problems 1 to 30; Part B includes problems 31 to 40. Use of calculators is prohibited in Part A, but allowed in Part B.

→ Use of calculators is allowed in Section II. The student is asked to solve any 7 of the 14 problems offered in the Section II exam paper.

		Calculators?	**No. of Problems**
Section I	**Part A**	No	30
	Part B	Yes	10
Section II		Yes	14*
**The student shall solve only 7 of the 14 problems.*			

Weight Distribution and Syllabus

Weight Distribution

Topic	***Weight***
1. Lines and Circles on the xy-Plane	12 – 18%
2. Conic Sections	7 – 13%
3. 3D Lines and Planes	9 – 15%
4. Distances	2 – 8%
5. Matrices and Determinants	7 – 13%
6. Systems of Linear Equations	3 – 9%
7. Vector Spaces	12 – 18%
8. Orthogonality	9 – 15%
9. Diagonalization	12 – 18%

Syllabus

1.Lines and circles on the xy-Plane
- **1.1.** Lines on the xy-Plane
- **1.2.** Circles on the xy-Plane

2. Conic sections
- **2.1.** The General Conic Equation
- **2.2.** Ellipses
- **2.3.** Parabolas
- **2.4.** Hyperbolas

Detailed Syllabus + 40 Solved Problems

Topic 1: Lines and Circles on the *xy*-Plane

1.1. Lines on the *xy*-Plane

→ The AGLA student should be able to interpret and use equations of lines in the xy-plane. These equations may occur in four forms:

1. General form

$$ax + by + c = 0 \quad (1)$$

2. Slope-intercept form

$$y = -\frac{a}{b}x - \frac{c}{b} \quad (2)$$

3. Intercept form

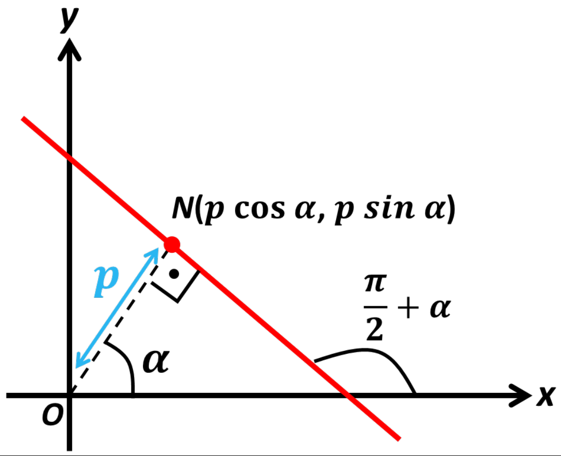
$$\frac{x}{\left(-\frac{c}{a}\right)} + \frac{y}{\left(-\frac{c}{b}\right)} = 1 \quad (3)$$

4. Normal form

$$x\cos\alpha + y\sin\alpha = p \quad (4)$$

Tip

The normal form is usually the one that brings about the most questions from students. In the normal form, p denotes the perpendicular distance from the line to the origin, as illustrated below. Angle α is the angle that this perpendicular makes with the horizontal. The angle that the line itself makes with the horizontal is $\pi/2 + \alpha$, as shown.



→ One formula that may prove useful is the point-slope equation,

$$y - y_0 = m(x - x_0) \quad (5)$$

where (x_0,y_0) is a point on the line and m is the slope.

→ The slope m of a line is given by the ratio of rise over run between any two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$.

→ Bear in mind that, for two parallel lines, the slopes m in equation (5) are equal. For two perpendicular lines, the slope of one is the negative reciprocal of the other:

$$m_2 = -\frac{1}{m_1} \quad (6)$$

Solved Problem 1

The equation of a line that passes through (2,−4) and is perpendicular to the line $3x + 4y = 5$ is:

(A) $2x - y - 8 = 0$

(B) $2x - 2y - 14 = 0$

(C) $3x - 2y - 14 = 0$

(D) $3x - 4y - 22 = 0$

(E) $4x - 3y - 20 = 0$

Solution

For a line to be perpendicular to another, the slope of one line should equal the negative reciprocal of the other. In the present case, the slope of one line is −3/4, so the slope of the other must be $m =$

−1/(−3/4) = 4/3. Further, the line must pass through point (2,−4). The equation of the line is determined with equation (5),

$$y - y_0 = m(x - x_0) \to y - (-4) = \frac{4}{3}(x - 2)$$

$$\therefore y + 4 = \frac{4}{3}(x - 2)$$

$$\therefore 3y + 12 = 4x - 8$$

$$\therefore \boxed{4x - 3y - 20 = 0}$$

▶ **The correct answer is E.**

Solved Problem 2

If the middle points of sides BC, CA, and AB of a triangle ABC are (1,3), (5,7), and (−5,7), respectively, then the equation of side AB is:

(A) $x - y + 8 = 0$

(B) $x - y + 10 = 0$

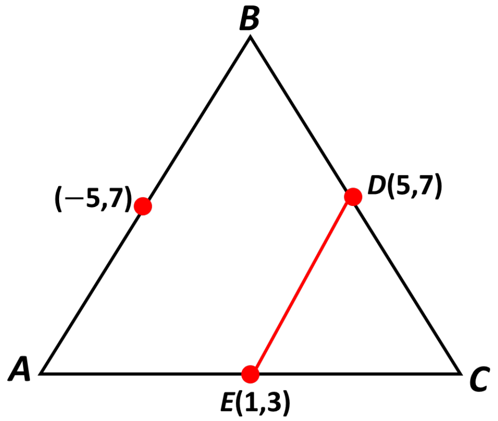
(C) $x - y + 12 = 0$

(D) $2x - y + 10 = 0$

(E) $2x - y + 12 = 0$

Solution

Refer to the following illustration.



As can be seen, the line that includes segment AB passes through point (−5,7) and has the same slope as segment DE, which joins the midpoints of two of the sides of the triangle. This slope is

$$m_{\overline{DE}} = \frac{5-1}{7-3} = 1$$

Thus, the line we aim for is defined by

$$y = m_{\overline{DE}}x + n \to y = x + n$$

Substituting the coordinates of point (−5,7) should yield the value of n,

$$y = x + n \to 7 = -5 + n$$

$$\therefore n = 12$$

Lastly, the line that contains AB is given by

$$y = x + n \to y = x + 12$$

$$\therefore \boxed{x - y + 12 = 0}$$

▶ **The correct answer is C.**

Solved problems 3 and 4 involve the slope-intercept form.

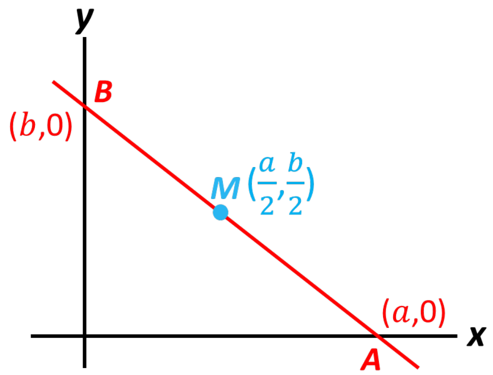
Solved Problem 3

A line is such that the segment that joins the x-intercept to the y-intercept has its middle point located at (3,2). The equation of this line is:

(A) $2x + 3y - 12 = 0$
(B) $3x + 2y - 12 = 0$
(C) $3x + 4y - 17 = 0$
(D) $4x - 3y - 6 = 0$
(E) $5x - 2y - 11 = 0$

Solution

Refer to the following illustration.



For point M to divide the segment that joins the axis intercepts evenly, we must have

$$\left(\frac{a+0}{2}, \frac{0+b}{2}\right) = (3,2) \to \frac{a}{2} = 3, \frac{b}{2} = 2$$

$$, \therefore a = 6,\ b = 4$$

It follows that the line is described by the equation

$$\frac{x}{a} + \frac{y}{b} = 1 \to \frac{x}{6} + \frac{y}{4} = 1$$

$$\therefore 4x + 6y = 24$$

$$\therefore 4x + 6y - 24 = 0$$

$$\therefore \boxed{2x + 3y - 12 = 0}$$

▶ **The correct answer is A.**

Solved Problem 4

For what values of α and β intercepts cut off on the coordinate axes by the line $\alpha x + \beta y + 8 = 0$ are equal in length but opposite in sign to those cut off by the line $2x - 3y + 6 = 0$?

(A) $\alpha = -\frac{8}{3}$; $\beta = -4$

(B) $\alpha = -\frac{8}{3}$; $\beta = 4$

(C) $\alpha = \frac{8}{3}$; $\beta = -4$

(D) $\alpha = \frac{8}{3}$; $\beta = 4$

(E) $\alpha = 8$; $\beta = 4$

Solution

Line $\alpha x + \beta y + 8 = 0$ can be written in the intercept form

$$\alpha x + \beta y + 8 \rightarrow \frac{x}{\left(\frac{-8}{\alpha}\right)} + \frac{y}{\left(\frac{-8}{\beta}\right)} = 1$$

Likewise, line $2x - 3y + 6 = 0$ can be restated as

$$2x - 3y + 6 = 0 \rightarrow 2x - 3y = -6$$

$$\therefore \frac{2x}{-6} - \frac{3y}{-6} = 1$$

$$\therefore \frac{x}{-3} + \frac{y}{2} = 1$$

The intercepts of the first line must be equal in length but opposite in sign to those cut off by the second line; it follows that

$$-\frac{8}{\alpha} = -(-3) \rightarrow \boxed{\alpha = -\frac{8}{3}}$$

and

$$-\frac{8}{\beta} = -2 \rightarrow \boxed{\beta = 4}$$

▶ **The correct answer is B.**

Solved Problem 5 combines use of the line equation with the formula for distance between two points $P_0(x_0,y_0)$ and $P_1(x_1,y_1)$, which should be familiar to the student:

$$\overline{P_0P_1} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \quad (7)$$

Solved Problem 5

Find the point on the line $-4x + 3y + 11 = 0$ that is equidistant from points $A(3,2)$ and $B(-2,3)$.

(A) (−2, −10)
(B) (−1, −5)
(C) (−1/2, −5/2)
(D) (4, 5/3)
(E) (5, 3)

Solution

Let $P(x_1,y_1)$ be the point on line $-4x + 3y + 11 = 0$ that is equidistant to points $A(3,2)$ and $B(-2,3)$. Appealing to the distance formula, we write

$$\overline{AP}^2 = (x_1-3)^2 + (y_1-2)^2 = x_1^2 + y_1^2 - 6x_1 - 4y_1 + 13$$

$$\overline{BP}^2 = (x_1+2)^2 + (y_1-3)^2 = x_1^2 + y_1^2 + 4x_1 - 6y_1 + 13$$

But $\overline{AP} = \overline{BP}$, so

$$\cancel{x_1^2} + \cancel{y_1^2} - 6x_1 - 4y_1 + \cancel{13} = \cancel{x_1^2} + \cancel{y_1^2} + 4x_1 - 6y_1 + \cancel{13}$$

$$\therefore y_1 = 5x_1 \text{ (I)}$$

However, $P(x_1,y_1)$ must also satisfy

$$-4x_1 + 3y_1 + 11 = 0$$

Substituting y_1 from (I) brings to

$$-4x_1 + 3y_1 + 11 = 0 \rightarrow -4x_1 + 3\times(5x_1) + 11 = 0$$

$$\therefore -4x_1 + 15x_1 + 11 = 0$$

$$\therefore 11x_1 = -11$$

$$\therefore \boxed{x_1 = -1}$$

Plugging this result in (I) gives

$$y_1 = 5x_1 = 5\times(-1) = \boxed{-5}$$

The required point is $P(x_1,y_1) = (-1,-5)$.

▶ **The correct answer is B.**

Solved Problem 6 involves use of the point-line formula, which yields the distance from a point $P(x_0,y_0)$ to a line $ax + bx + c = 0$. More on distances in Topic 4.

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \quad (8)$$

Solved Problem 6

If the equation of the base of an equilateral triangle is $2x - y = 1$ and the vertex is $(-1,2)$, then the length of the side of the triangle is:

(A) $4/\sqrt{3}$

(B) $2\sqrt{5}/\sqrt{3}$

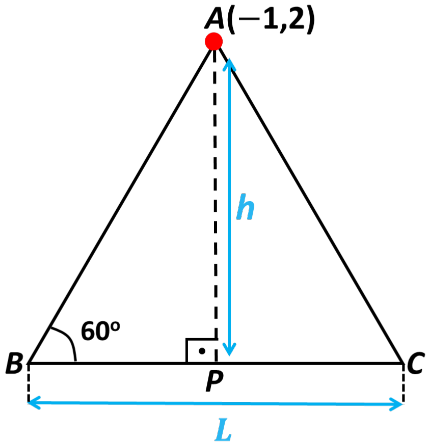
(C) $\sqrt{2}$

(D) $2\sqrt{2}$

(E) $10/\sqrt{3}$

Solution

Refer to the following illustration.



The height AP of the triangle can be determined with the point-line formula,

$$h = \overline{AP} = \frac{|2\times(-1)-(2)-1|}{\sqrt{(-1)^2+2^2}} = \frac{|-5|}{\sqrt{5}} = \sqrt{5}$$

Then, we apply the Pythagorean theorem to triangle ABP, giving

$$\tan 60^\circ = \frac{h}{L/2} \rightarrow L = \frac{2h}{\tan 60^\circ}$$

$$\therefore \boxed{L = \frac{2\sqrt{5}}{\sqrt{3}}}$$

▶ **The correct answer is B.**

→ The angle θ formed by any two lines with slopes m_1 and m_2 can be determined with elementary trigonometry and equals

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \quad (9)$$

Solved Problem 7

If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined relatively to the line $y = mx + 4$, what is the value of m?

(A) −0.749 or 1.36
(B) −0.867 or 1.15
(C) −0.962 or 1.47
(D) −1.11 or 1.96
(E) −1.42 or 2.35

Solution

Let $m_1 = 3$, $m_2 = 1/2$, and $m_3 = m$ denote the slopes of the three lines. Further, let θ_1 be the angle between the first and third lines, and θ_2 be the angle between the second and third lines; mathematically,

$$\tan\theta_1 = \frac{m_1 - m}{1 + m_1 m} \rightarrow \tan\theta_1 = \frac{3-m}{1+3m}\ \text{(I)}$$

and

$$\tan\theta_2 = \frac{m - m_2}{1 + m m_2} \rightarrow \tan\ \theta_2 = \frac{m - 1/2}{1 + m/2}\text{(II)}$$

However, $\theta_1 = \theta_2$, so

$$\theta_1 = \theta_2 \rightarrow \frac{3-m}{1+3m} = \frac{m-1/2}{1+m/2}$$

$$\therefore (3-m)\times\left(1+\frac{m}{2}\right) = (m - 1/2)\times(1+3m)$$

$$\therefore 3 + \frac{3m}{2} - m - \frac{m^2}{2} = m + 3m^2 - \frac{1}{2} - \frac{3m}{2}$$

$$\therefore 7m^2 - 2m - 7 = 0$$

$$\therefore m = \frac{-(-2)\pm\sqrt{4 - 4\times 7\times(-7)}}{14} = \frac{2\pm 2\sqrt{1+49}}{14}$$

$$\therefore m = \frac{1\pm\sqrt{50}}{7} = \boxed{-0.867 \text{ or } 1.15}$$

▶ **The correct answer is B.**

→ The equation of the line that bisects the angle between two given lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, with $c_1 > 0$, $c_2 > 0$, is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad (10)$$

The positive sign is given to the line that bisects the angle containing the origin, while the negative sign applies to the angle not containing the origin. If $a_1a_2 + b_1b_2 > 0$, then the origin is situated in the obtuse angle region and the bisector of this obtuse angle is given a positive sign in equation (10). If $a_1a_2 + b_1b_2 < 0$, then the origin is situated in the acute angle region and the bisector of this angle is given a positive sign in equation (10).

■ 1.2. Circles on the xy-Plane

→ The student should be capable of identifying the center and radius of a circle from its equation. The student should be able to transition from the expanded form of a circle to the reduced form by completing squares, as illustrated in Solved Problem 8.

Solved Problem 8

Find the equation of the circle whose center is (2,−2) and which passes through the center of the circle $x^2 + y^2 - 6x - 8y - 5$.

(A) $x^2 + y^2 - 4x + 4y - 23 = 0$
(B) $x^2 + y^2 - 4x + 4y - 29 = 0$
(C) $x^2 + y^2 - 4x + 4y - 31 = 0$
(D) $x^2 + y^2 + 4x - 4y - 23 = 0$
(E) $x^2 + y^2 + 4x - 4y - 29 = 0$

Solution

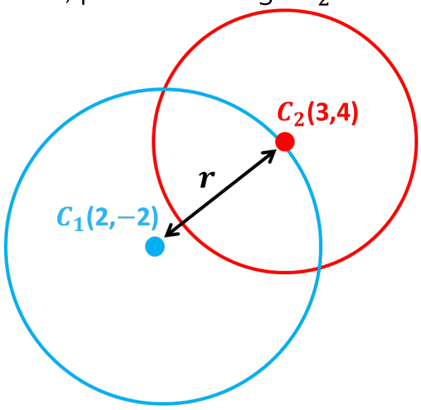
The circle equation we received can be adjusted to yield

$$x^2 + y^2 - 6x - 8y - 5 = 0 \to \left(x^2 - 6x + \_\_\right) + \left(y^2 - 8y + \_\_\right) = 5$$

$$\therefore (x-3)^2 + (y-4)^2 = 5 + 9 + 16$$

$$\therefore (x-3)^2 + (y-4)^2 = 30$$

Clearly, the center of this circle is $C_2(3,4)$. The circle we are looking for, shown in blue below, passes through C_2.



The radius of the blue circle is simply the distance from C_1 to C_2,

$$r = \overline{C_1C_2} = \sqrt{(2-3)^2 + (-2-4)^2} = \sqrt{37}$$

Lastly, the equation of the circle we were asked to describe is

$$(x-2)^2 + (y+2)^2 = \left(\sqrt{37}\right)^2 \to x^2 - 4x + 4 + y^2 + 4y + 4 = 37$$

$$\therefore x^2 + y^2 - 4x + 4y + 8 = 37$$

$$\therefore \boxed{x^2 + y^2 - 4x + 4y - 29 = 0}$$

▶ **The correct answer is B.**

The tangent to a circle is perpendicular to one of its radii. Solved Problem 9 makes use of this property.

Solved Problem 9

Find the equation of the tangent to circle $x^2 + y^2 - 5x + y - 14 = 0$ at point (2,−5).

(A) $2x + 7y + 31 = 0$

(B) $x + 7y + 33 = 0$

(C) $x + 8y + 38 = 0$

(D) $x + 9y + 43 = 0$

(E) $2x + 9y + 41 = 0$

Solution

We first complete squares to write the circle equation in reduced form,

$$\left(x^2 - 5x + \_\_\right) + \left(y^2 + y + \_\_\right) = -14$$

$$\therefore \left(x - 5/2\right)^2 + \left(y + 1/2\right)^2 = -14 + \frac{25}{4} + \frac{1}{4}$$

Thus, the circle is centered at (5/2, −1/2). The slope of the circle radius R extending from (5/2, −1/2) to (2,−5) is

$$m = \frac{-5 + \frac{1}{2}}{2 - \frac{5}{2}} = \frac{-\frac{10}{2} + \frac{1}{2}}{\frac{4}{2} - \frac{5}{2}} = 9$$

The tangent line at (2,−5) must be perpendicular to R; its slope m_T is then

$$m_T = -\frac{1}{m} \to m_T = -\frac{1}{9}$$

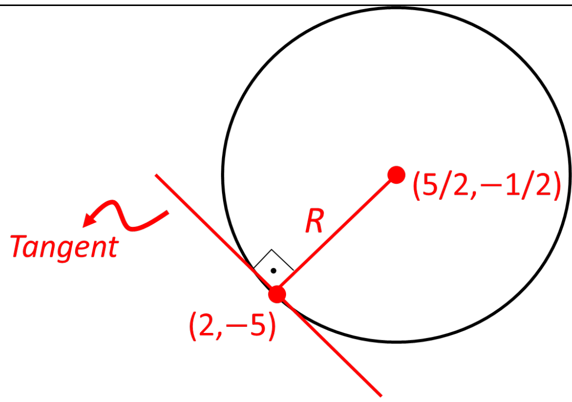
Using the point-slope formula, we can establish the equation of the tangent,

$$y - y_0 = m\left(x - x_0\right) \to y - \left(-5\right) = -\frac{1}{9} \times \left(x - 2\right)$$

$$\therefore 9\left(y + 5\right) = -\left(x - 2\right)$$

$$\therefore 9y + 45 = -x + 2$$

$$\therefore \boxed{x + 9y + 43 = 0}$$



▶ **The correct answer is D.**

Topic 2: Conic Sections

2.1. The General Conic Equation

→ The AGLA student is expected to easily recognize conic section equations and classify the type of section they represent. A conic section equation has the generalized form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (B \neq 0) \quad (11)$$

Such an equation represents:

1. A parabola (or two parallel straight lines, or no locus), if $B^2 - 4AC = 0$;

2. An ellipse (or a point, or no locus) if $B^2 - 4AC < 0$;

3. A hyperbola (or two intersecting straight lines) if $B^2 - 4AC > 0$.

Solved Problem 10

The equation

$$x^2 + 3xy - 8y^2 + 5x + 13 = 0$$

represents:

(A) A parabola, or two parallel straight lines, or no locus at all.

(B) An ellipse.

(C) An ellipse, a point, or no locus at all.

(D) A hyperbola.

(E) A hyperbola or two intersecting straight lines.

Solution

We proceed to determine $B^2 - 4AC$ for the present case,

$$B^2 - 4AC = (-7)^2 - 4 \times 3 \times 4 = 49 - 48 = 1 > 0$$

Since $B^2 - 4AC > 0$, the equation represents a hyperbola or two intersecting straight lines.

▶ **The correct answer is E.**

→ The student may also be asked to perform a simple rotation of axes and obtain the corresponding conic section equation in a rotated system of coordinates. In general, a rotation of θ degrees relatively to the horizontal is such that the (x',y') coordinates in the rotated frame are related to the (x,y) coordinates in the original Cartesian frame by an expression of the form

$$\begin{cases} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{cases} \quad (11)$$

Solved Problem 11

Using equation (11), find an acute angle of rotation θ such that the transformed equation of $2x^2 + \sqrt{3}xy + y^2 = 8$ will have no $x'y'$ term.

Solution

We substitute the coordinate transformations $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ and manipulate, giving

$$2x^2 + \sqrt{3}xy + y^2 = 8$$

$$\therefore 2(x'\cos\theta - y'\sin\theta)^2 + \sqrt{3}(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$$
$$+(x'\sin\theta + y'\cos\theta)^2 = 8$$

We perform the indicated multiplications, collect similar terms, and obtain

$$\left(2\cos^2\theta + \sqrt{3}\sin\theta\cos\theta + \sin^2\theta\right)x'^2 + \left(-2\sin\theta\cos\theta + \sqrt{3}\cos^2\theta - \sqrt{3}\sin^2\theta\right)x'y'$$
$$+\left(2\sin^2\theta - \sqrt{3}\sin\theta\cos\theta + \cos^2\theta\right)y'^2 = 8$$

For the transformed equation to contain no $x'y'$ term, its coefficient must be zero. Accordingly,

$$-\underbrace{2\sin\theta\cos\theta}_{=\sin 2\theta} + \sqrt{3}\cos^2\theta - \sqrt{3}\sin^2\theta \to -\sin 2\theta + \sqrt{3}\underbrace{\left(\cos^2\theta - \sin^2\theta\right)}_{=\cos 2\theta} = 0$$

$$\therefore -\sin 2\theta + \sqrt{3}\cos 2\theta = 0$$

$$\therefore \tan 2\theta = \sqrt{3}$$

so that

$$\tan 2\theta = \sqrt{3} \to 2\theta = 60^\circ$$

$$\therefore \boxed{\theta = 30^\circ}$$

Thus, rotating the coordinate axes by 30 degrees would suffice to make the $x'y'$ term in the equation vanish.

2.2. Ellipses

→ The AGLA student is expected to effortlessly recognize the elements of an ellipse and use the equation for an ellipse with major and minor axes parallel to the coordinate axes. Solved Problem 12 exemplifies the analysis of an ellipse with major axis parallel to the x-axis.

Solved Problem 12

Find the equation of the ellipse whose vertices are (−2,1), (6,1), and one of whose foci is (5,1).

(A) $\frac{(x-1)^2}{12}+\frac{(y-2)^2}{8}=1$ **(B)** $\frac{(x-1)^2}{16}+\frac{(y-2)^2}{7}=1$

(C) $\frac{(x-2)^2}{12}+\frac{(y-1)^2}{8}=1$ **(D)** $\frac{(x-2)^2}{16}+\frac{(y-1)^2}{7}=1$

(E) $\frac{(x-2)^2}{16}+\frac{(y-1)^2}{8}=1$

Solution

Since the vertices are parallel to the x-axis, we know that the major axis is parallel to the x-axis. The distance between the vertices is 8, so a = 4. The center is at the midpoint of the segment that joins the two vertices, so

$$C\left(\frac{-2+6}{2},\frac{1+1}{2}\right)=C(2,1)$$

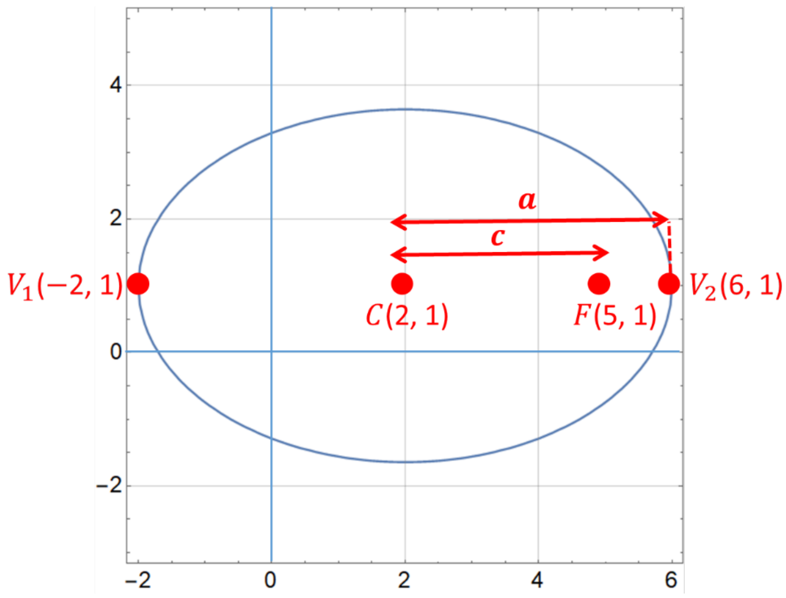
The distance joining the center C to the focus $F(5,1)$ is c = 3. Finally, the length b of the minor axis is determined to be

$$a^2=b^2+c^2\rightarrow b^2=a^2-c^2$$
$$\therefore b^2=4^2-3^2=7$$

The equation of the ellipse is

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1\rightarrow\frac{(x-2)^2}{4^2}+\frac{(y-1)^2}{(\sqrt{7})^2}=1$$

$$\therefore \boxed{\frac{(x-2)^2}{16}+\frac{(y-1)^2}{7}=1}$$



▶ **The correct answer is D.**

■ 2.3. Parabolas

→ The AGLA student is also expected to effortlessly recognize the elements of a parabola and use the equation for a parabola with directrix parallel to one of the coordinate axes. Solved Problem 13 exemplifies the analysis of a parabola with directrix parallel to the x-axis.

Solved Problem 13

Determine the equation of the parabola whose focus is (−2,3) and whose directrix is $y = -3$.

(A) $x^2 - 4x + 8y - 4 = 0$
(B) $x^2 - 4x + 8y - 6 = 0$
(C) $x^2 + 4x - 8y - 4 = 0$
(D) $x^2 + 4x - 8y - 6 = 0$
(E) $x^2 + 4x - 8y - 8 = 0$

Solution

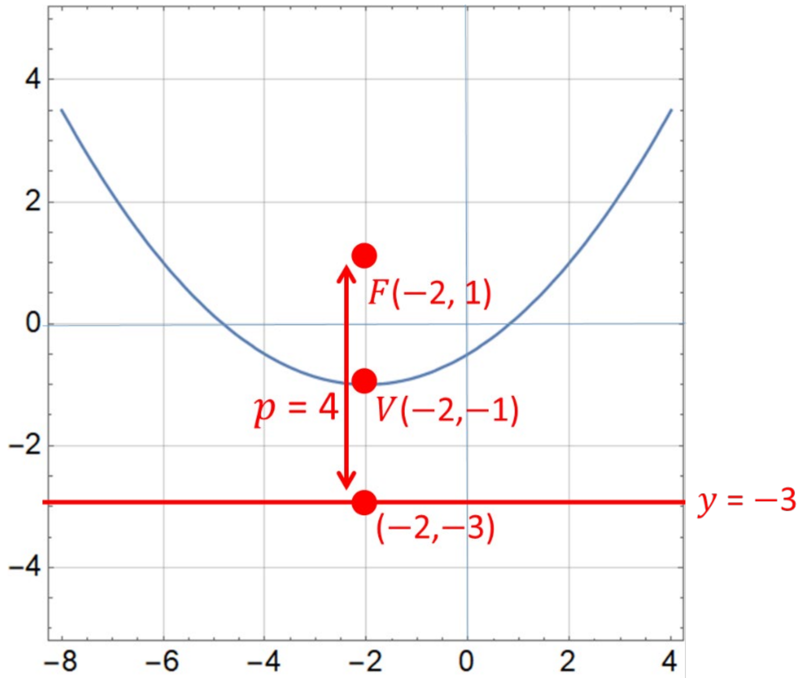
Since the vertex is midway between (−2,3) and $y = -3$, its coordinates are those of the midpoint of the segment whose endpoints are (−2,1) and (−2,−3); hence the vertex is at (−2,−1). The distance from the directrix to the focus is $p = 4$. Thus the required equation is

$$(x-h)^2 = 2p(y-k) \to [x-(-2)]^2 = 2\times 4[y-(-1)]$$

$$\therefore (x+2)^2 = 8(y+1)$$

$$\therefore x^2 + 4x + 4 = 8y + 8$$

$$\therefore \boxed{x^2 + 4x - 8y - 4 = 0}$$



▶ **The correct answer is C.**

2.4. Hyperbolas

→ The AGLA student is also expected to effortlessly recognize the elements of a hyperbola and use the equation for a hyperbola with axes of symmetry parallel to one of the coordinate axes. Solved Problem 14 exemplifies the analysis of a parabola with transverse axis parallel to the y-axis.

Solved Problem 14

Find an equation for the hyperbola with center at the origin, vertices on the y-axis, asymptotes $y = \pm(3/2)x$, and focal width equal to 8.

(A) $\frac{y^2}{64}-\frac{x^2}{25}=1$

(B) $\frac{y^2}{64}-\frac{x^2}{36}=1$

(C) $\frac{y^2}{81}-\frac{x^2}{25}=1$

(D) $\frac{y^2}{81}-\frac{x^2}{36}=1$

(E) $\frac{y^2}{100}-\frac{x^2}{36}=1$

Solution

Since the focus and center are on the y-axis, the hyperbola is described by the relation

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

Further, the asymptotes have the form $y = \pm(a/b)x$, which, comparing with $y = \pm(3/2)x$ obviously yields

$$\frac{a}{b}=\frac{3}{2}\ \text{(I)}$$

Further, from the expression that defines focal width,

$$\frac{2b^2}{a}=8\ \text{(II)}$$

Adjusting this latter expression and substituting from (I), we get

$$\frac{2b^2}{a}=8\rightarrow 2b\times\underbrace{\left(\frac{b}{a}\right)}_{=2/3}=8$$

$$\therefore 2b\times\frac{2}{3}=8$$

$$\therefore b=\frac{24}{4}=6$$

Substituting in (I),

$$\frac{a}{b}=\frac{3}{2}\to a=\frac{3}{2}b$$

$$\therefore a=\frac{3}{2}\times 6=9$$

The equation of the hyperbola is then

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1\to\frac{y^2}{9^2}-\frac{x^2}{6^2}=1$$

$$\therefore\boxed{\frac{y^2}{81}-\frac{x^2}{36}=1}$$

▶ **The correct answer is D.**

→ Solved Problem 15 illustrates the analysis of a hyperbola with center *not* located at the origin.

Solved Problem 15

Determine the equation of the hyperbola with vertices $A_1(1,-2)$ and $A_2(5,-2)$, knowing that one of its foci is $F(6,-2)$.

(A) $\frac{(x-6)^2}{3}-\frac{(y+2)^2}{4}=1$

(B) $\frac{(x-6)^2}{4}-\frac{(y+2)^2}{5}=1$

(C) $\frac{(x-3)^2}{3}-\frac{(y+2)^2}{4}=1$

(D) $\frac{(x-3)^2}{4}-\frac{(y+2)^2}{5}=1$

(E) $\frac{(x-3)^2}{5}+\frac{(y+2)^2}{6}=1$

Solution

The center C of the hyperbola is at the center of segment A_1A_2, so $C(3,-2)$. The distance from the center C to focus F gives c = 3, while the distance from center C to any of the two vertices gives a = 2. It remains to establish the length b of the conjugate axis,

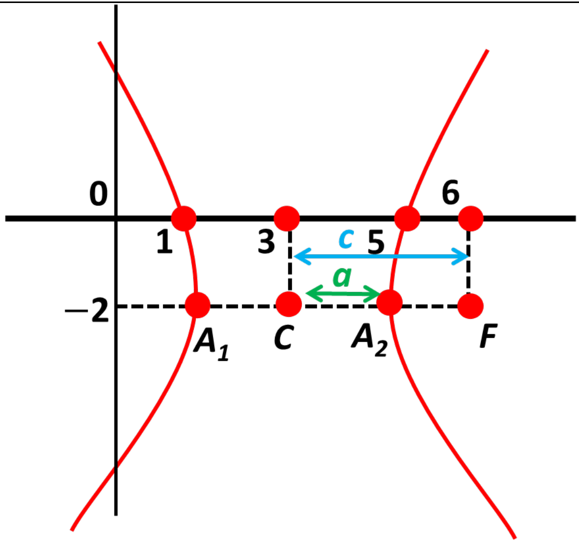
$$c^2=a^2+b^2\to 3^2=2^2+b^2$$

$$\therefore 9=4+b^2$$

$$\therefore b=\sqrt{5}$$

Lastly, the equation of the hyperbola is

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1\to\boxed{\frac{(x-3)^2}{4}-\frac{(y+2)^2}{5}=1}$$

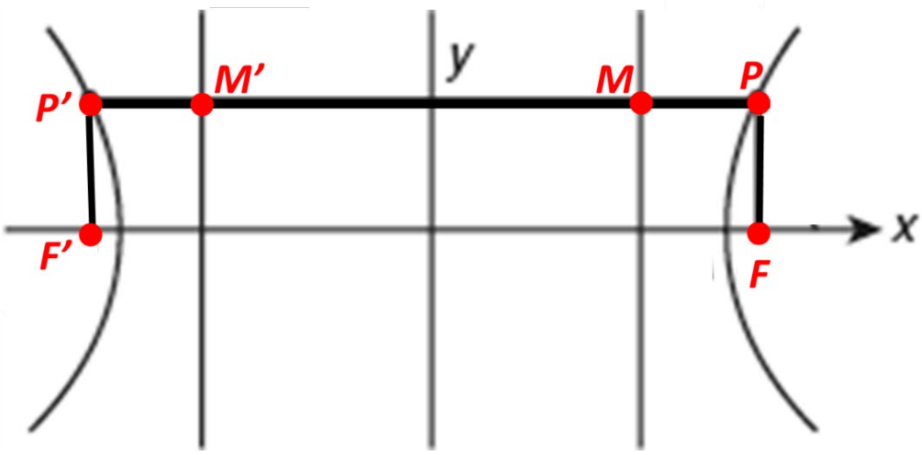


▶ **The correct answer is D.**

→ Calculations involving ellipses and parabolas with axes oblique to the coordinate axes are not directly assessed in the AGLA exam. The course does include, however, parabolas with oblique asymptotes. The equation for hyperbolas of this nature can be determined by noting that the distance from the focus to some point P on the hyperbola equals the distance from point P to the directrix multiplied by the eccentricity e; that is,

$$\overline{FP} = e\overline{PM} \quad (12)$$

Use of equation (12) is illustrated in Solved Problem 16.



Solved Problem 16

Find the equation of the hyperbola with (1,2) as one of its foci, directrix $2x + y = 1$, and eccentricity $\sqrt{3}$.

Solution

Let $P(x,y)$ be a point on the hyperbola, $F(1,2)$ be the focus, and M be the point of intersection of a perpendicular to the directrix and the hyperbola. By definition,

$$\overline{FP}^2 = e^2\overline{PM}^2 \to (x-1)^2 + (y-2)^2 = \left(\sqrt{3}\right)^2 \times \left(\frac{2x+y-1}{\sqrt{1^2+2^2}}\right)^2$$

$$\therefore x^2 - 2x + 1 + y^2 - 4y + 4 = 3 \times \left(\frac{2x+y-1}{\sqrt{5}}\right)^2$$

$$\therefore x^2 - 2x + 1 + y^2 - 4y + 4 = \frac{3}{5} \times \left(4x^2 + y^2 + 4xy - 4x - 2y + 1\right)$$

$$\therefore 5\left(x^2 - 2x + 1 + y^2 - 4y + 4\right) = 3\left(4x^2 + y^2 + 4xy - 4x - 2y + 1\right)$$

$$\therefore 5x^2 - 10x + 5 + 5y^2 - 20y + 20 = 12x^2 + 3y^2 + 12xy - 12x - 6y + 3$$

$$\therefore 12x^2 - 5x^2 + 3y^2 - 5y^2 + 12xy - 12x + 10x - 6y + 20y + 3 - 25$$

$$\therefore \boxed{7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0}$$

Topic 3: 3D Lines and Planes

3.1. Vector Products

→ The AGLA student should master basic vector operations, including use and interpretation of dot, vector, and scalar triple products.

→ Solved Problems 17 and 18 illustrate how the dot product can be used to determine the angle θ between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Problems of this nature may appear in the calculator-free section of the exam.

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \quad (13)$$

Solved Problem 17

Determine the value of α so that vectors $\boldsymbol{u} = 2\boldsymbol{i} + \alpha\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{i} + 3\boldsymbol{j} - 8\boldsymbol{k}$ are orthogonal.

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Solution

For u and v to be orthogonal, their dot product must equal zero:

$$\mathbf{u} \bullet \mathbf{v} = 0 \to 2 \times 1 + \alpha \times 3 + 1 \times (-8) = 0$$

$$\therefore 2 + 3\alpha + -8 = 0$$

$$\therefore \boxed{\alpha = 2}$$

▶ **The correct answer is B.**

Solved Problem 18

Determine the angle between vectors $\boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{v} = 7\boldsymbol{i} + 24\boldsymbol{k}$.

(A) 53.3°
(B) 61.3°
(C) 70.4°
(D) 82.1°
(E) 97.6°

Solution

First, we determine the dot product of vectors u and v,

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2\times 7 + 2\times 0 + (-1)\times 24 = -10$$

Then, we determine the norms of $\boldsymbol{u}$ and $\boldsymbol{v}$,

$$\|\mathbf{u}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

$$\|\mathbf{v}\| = \sqrt{7^2 + 0^2 + 24^2} = \sqrt{625} = 25$$

Finally, we apply equation (13),

$$\cos\theta = \frac{-10}{3\times 25} = -0.133 \rightarrow \theta = \arccos(-0.133)$$

$$\therefore \boxed{\theta = 97.6^{\circ}}$$

▶ **The correct answer is E.**

→ Solved Problem 19 illustrates how the cross product, formula (14), can be used to determine a vector that is simultaneously orthogonal to other two vectors.

$$\mathbf{u}\times\mathbf{v} = (x_1, y_1, z_1)\times(x_2, y_2, z_2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \quad (14)$$

Solved Problem 19

Which of the following vectors is simultaneously orthogonal to vectors $\boldsymbol{u} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{v} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$?

(A) $6\boldsymbol{i} - 10\boldsymbol{j} - 14\boldsymbol{k}$
(B) $-10\boldsymbol{i} + 14\boldsymbol{j} - 6\boldsymbol{k}$
(C) $-14\boldsymbol{i} + 7\boldsymbol{j} + 10\boldsymbol{k}$
(D) $10\boldsymbol{i} + 6\boldsymbol{j} + 7\boldsymbol{k}$
(E) $7\boldsymbol{i} - 14\boldsymbol{j} + 10\boldsymbol{k}$

Solution

The simplest method to find a vector that is simultaneously perpendicular to two other vectors is to use the cross product.

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & -1\\ 3 & -1 & 2\end{vmatrix}=3\mathbf{i}-5\mathbf{j}-7\mathbf{k}$$

The vector in option (A) is parallel to the vector above and simultaneously orthogonal to $\boldsymbol{u}$ and $\boldsymbol{v}$.

▶ **The correct answer is A.**

3.2. Vector Line Equations

→ The student should be familiar with the vector equation for lines. Suppose a line passes through point $A(x_1,y_1,z_1)$ and has the direction of a nonzero vector $\boldsymbol{v}$ = (a,b,c). For a point $P(x,y,z)$ to belong to the line, vectors $\boldsymbol{AP}$ and $\boldsymbol{v}$ must be collinear; that is,

$$\mathbf{AP}=t\mathbf{v}$$

$$\therefore P-A=t\mathbf{v}$$

$$\therefore P=A+t\mathbf{v}$$

$$\therefore (x,y,z)=(x_1,y_1,z_1)+t(a,b,c)$$

$$\therefore \begin{cases} x=x_1+at \\ y=y_1+bt \quad (15) \\ z=z_1+ct \end{cases}$$

Equation (15) is the parametric equation for lines.

→ The student should also be familiar with the symmetric form of the vector line equation. In general, a line that passes through point $A(x_1,y_1,z_1)$ and has the direction of a nonzero vector $\boldsymbol{v}$ = (a,b,c) can be stated as

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \quad (16)$$

Solved Problem 20 illustrates how equations (15) and (16) can be solved simultaneously to yield the point of intersection of two lines in space.

Solved Problem 20

Find the point of intersection of lines r and s defined below.

$$r:\begin{cases} x=5+t \\ y=2-t \\ z=7-2t \end{cases} \; ; \; s:\frac{x-2}{2}=\frac{y}{3}=\frac{z-5}{4}$$

(A) (1,9,3)
(B) (4,7,8)
(C) (4,3,9)
(D) (4,5,8)
(E) (3,4,9)

Solution

To find the desired point of intersection, we first write

$$\frac{x-2}{2}=\frac{y}{3}$$

From the equation that describes line r, however, x = 5 + t and y = 2 − t. Substituting above and solving for t, we obtain

$$\frac{x-2}{2}=\frac{y}{3}\to\frac{(5+t)-2}{2}=\frac{(2-t)}{3}$$

$$\therefore\frac{3+t}{2}=\frac{2-t}{3}$$

$$\therefore 9+3t=4-2t$$

$$\therefore 5t=-5$$

$$\therefore t=-1$$

Substituting t = −1 in the parametric equations of r brings to

$$\begin{cases} x=5+t=5-1=4 \\ y=2-t=2-(-1)=3 \\ z=7-2\times(-1)=9 \end{cases}$$

Thus, the lines intersect at point P(4,3,9). Needless to say, these coordinates must also satisfy the equation of line s; indeed,

$$\frac{x-2}{2}=\frac{y}{3}=\frac{z-5}{4}\to\frac{4-2}{2}=\frac{3}{3}=\frac{9-5}{4}$$

$$\therefore 1=1=1$$

The three sides of the equality check, as expected.

▶ **The correct answer is C.**

3.3. Planes

→ The student should also master the use and interpretation of equations for planes in three-dimensional space. A plane has general equation

$$ax + by + cz + d = 0 \quad (17)$$

Solved Problem 21 is a typical problem combining plane and line equations.

Solved Problem 21

Determine values of α and β for line r to be contained in plane π.

$$r = \begin{cases} x = 2 + t \\ y = 1 + t \\ z = -3 - 2t \end{cases} ; \ \pi : \alpha x + \beta y + 2z - 1 = 0$$

(A) $\alpha = 1, \beta = 3$
(B) $\alpha = 2, \beta = 2$
(C) $\alpha = 2, \beta = 3$
(D) $\alpha = 3, \beta = 1$
(E) $\alpha = 3, \beta = 2$

Solution

Line π crosses point $A(2,1,-3)$ and has the direction of vector $\boldsymbol{v}$ = (1,1,−2). One vector normal to plane π is $\boldsymbol{n} = (\alpha,\beta,2)$. For r to be contained in π, we must have $\boldsymbol{v} \perp \boldsymbol{n}$ and $A \in \pi$. The first condition brings to the dot product relation

$$\mathbf{v} \bullet \mathbf{n} = 0 \to (1,1,-2) \bullet (\alpha, \beta, 2) = 0$$
$$\therefore \alpha + \beta = 4 \text{ (I)}$$

As for the second condition, we substitute A in the equation for π to obtain

$$\alpha x + \beta y + 2 \times (-3) - 1 = 0 \to 2\alpha + \beta = 7 \text{ (II)}$$

Equations (I) and (II) constitute a system of linear equations,

$$\begin{cases} \alpha + \beta = 4 \\ 2\alpha + \beta = 7 \end{cases}$$

Substituting (I) in (II), we get

$$2\alpha + \beta = 7 \to \alpha + \underbrace{(\alpha + \beta)}_{=4} = 7$$
$$\therefore \alpha + 4 = 7$$
$$\therefore \boxed{\alpha = 3}$$

Finally, inserting α in (I) yields $\beta = 1$.

▶ **The correct answer is D.**

Topic 4: Distances

→ The student should be able to determine the following distances effortlessly:

1. Distance from a line to a point.
2. Distance between two parallel lines.
3. Distance from a plane to a point.
4. Distance from a point to a plane.
5. Distance between two parallel planes.

Calculation of distance between two points is considered prior knowledge and will not be assessed directly. Solved Problem 22 illustrates a calculation of type 1.

Solved Problem 22

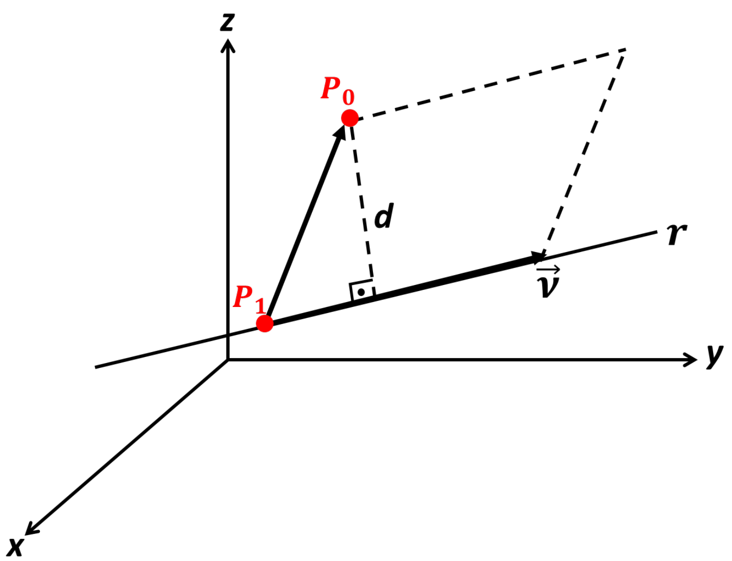
Determine the distance from point $P_0(2,0,7)$ to line r defined below.

$$r: \frac{x}{2} = \frac{y-2}{2} = \frac{z+3}{1}$$

(A) 1.8
(B) 3.6
(C) 5.4
(D) 7.1
(E) 9.8

Solution

To determine the distance from a point P_0 to a line r, consider the following illustration. We have a line r defined by a point $P_1(x_1,y_1,z_1)$ and by a direction vector $\boldsymbol{v} = (a,b,c)$. Consider also a point $P_0(x_0,y_0,z_0)$ anywhere in space. Vectors $\boldsymbol{v}$ and $\boldsymbol{P_1P_0}$ define a parallelogram whose height d corresponds to the distance between P_0 and r, which is the quantity we aim for.



The area A of a parallelogram is given by the product of base and height; in the case at hand,

$$A = |\mathbf{v}|d$$

However, we also know that, from the definition of cross-product,

$$A = |\mathbf{v} \times \mathbf{P_1 P_0}|$$

Combining the two results, we get,

$$|\mathbf{v}|d = |\mathbf{v} \times \mathbf{P_1 P_0}| \rightarrow \boxed{d = \frac{|\mathbf{v} \times \mathbf{P_1 P_0}|}{|\mathbf{v}|}}$$

The equation above gives the distance from line r to point P_0.

In the present case, line r crosses point $P_1(0,2,-3)$ and has the direction of vector $\boldsymbol{v}$ = (2,2,1). We define vector $\boldsymbol{P_0 P_1}$ = (0−2, 2−0,−3−7) = (−2,2,−10) and apply the formula for point-line distance, giving

$$d(P_0, r) = \frac{|\mathbf{v} \times \mathbf{P_1 P_0}|}{|\mathbf{v}|} = \frac{|(2,2,1) \times (-2,2,-10)|}{|(2,2,1)|}$$

$$\therefore d(P_0, r) = \frac{|(-22,18,8)|}{|(2,2,1)|}$$

$$\therefore d(P_0, r) = \frac{29.5}{3} = \boxed{9.83}$$

▶ **The correct answer is E.**

Topic 5: Matrices and Determinants

5.1. Matrix Algebra and Determinants

→ The student should master basic operations involving matrices, especially matrix multiplication, without use of calculators.

→ The student should master calculations of determinants via the Laplace expansion theorem.

5.2. Matrix Inversion

→ The student is asked to know by heart the inversion formula for 2×2 matrices:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (18)$$

→ The student should be able to find the inverse of a matrix using the inversion algorithm. Solved Problem 23 illustrates this process.

Solved Problem 23

Find the inverse of matrix A.

$$A=\begin{pmatrix}1&2&3\\2&5&3\\1&0&8\end{pmatrix}$$

(A) $\begin{pmatrix}-38&15&9\\8&-4&-3\\7&2&-1\end{pmatrix}$ **(B)** $\begin{pmatrix}-39&17&8\\11&3&-2\\6&-1&-1\end{pmatrix}$

(C) $\begin{pmatrix}-40&16&9\\13&-5&-3\\5&-2&-1\end{pmatrix}$ **(D)** $\begin{pmatrix}-41&14&8\\9&-3&-4\\8&2&-2\end{pmatrix}$

(E) Matrix A is singular.

Solution

Let us apply the inversion algorithm.

$$A=\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&5&3&0&1&0\\1&0&8&0&0&1\end{array}\right)$$

Add −2 times the first row to the second row and −1 times the first row to the third row.

$$A=\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\2-2\times1&5-2\times2&3-2\times3&0-2\times1&1&0\\1-1&0-2&8-3&0-1&0&1\end{array}\right)$$

$$\therefore A=\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&-3&-2&1&0\\0&-2&5&-1&0&1\end{array}\right)$$

Next, add 2 times the second row to the third.

$$A=\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&-3&-2&1&0\\0&-2+2\times1&5+2\times(-3)&-1+2\times(-2)&0+2\times1&1\end{array}\right)$$

$$\therefore A=\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&-3&-2&1&0\\0&0&-1&-5&2&1\end{array}\right)$$

Multiply the third row by −1.

$$A=\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&-3&-2&1&0\\0&0&-1&-5&2&1\end{array}\right)\to A=\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&-3&-2&1&0\\0&0&1&5&-2&-1\end{array}\right)$$

Next, add 3 times the third row to the second and −3 times the third row to the first.

$$A=\left(\begin{array}{ccc|ccc}1 & 2 & 3-3\times 1 & 1-3\times 5 & 0-3\times(-2) & 0-3\times(-1)\\ 0 & 1 & -3+3\times 1 & -2+3\times 5 & 1+3\times(-2) & 0+3\times(-1)\\ 0 & 0 & 1 & 5 & -2 & -1\end{array}\right)$$

$$\therefore A=\left(\begin{array}{ccc|ccc}1 & 2 & 0 & -14 & 6 & 3\\ 0 & 1 & 0 & 13 & -5 & -3\\ 0 & 0 & 1 & 5 & -2 & -1\end{array}\right)$$

Lastly, add −2 times the second row to the first.

$$A=\left(\begin{array}{ccc|ccc}1 & 2-2\times 1 & 0 & -14-2\times 13 & 6-2\times(-5) & 3-2\times(-3)\\ 0 & 1 & 0 & 13 & -5 & -3\\ 0 & 0 & 1 & 5 & -2 & -1\end{array}\right)$$

$$\therefore A=\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -40 & 16 & 9\\ 0 & 1 & 0 & 13 & -5 & -3\\ 0 & 0 & 1 & 5 & -2 & -1\end{array}\right)$$

We conclude that

$$\boxed{A^{-1}=\begin{pmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{pmatrix}}$$

▶ **The correct answer is C.**

→ In addition to the inversion algorithm, the student must be able to determine inverse matrices through the classical adjoint matrix. Recall that the inverse of a matrix A can be established as

$$A=\frac{1}{|A|}\text{adj}\,A \quad (19)$$

Solved Problem 24

Find the inverse of matrix B.

$$B=\begin{pmatrix}0 & 2 & 0\\ 1 & 0 & 1\\ 2 & 3 & 1\end{pmatrix}$$

(A) $\begin{pmatrix}-3/2 & -1 & 1\\ 1/2 & 0 & 0\\ 3/2 & 2 & -1\end{pmatrix}$ **(B)** $\begin{pmatrix}-3/2 & -1/2 & 1\\ 1/2 & 3 & -2\\ 0 & 1 & -1\end{pmatrix}$

(C) $\begin{pmatrix} 0 & 16 & 0 \\ 1/2 & -5 & -3/2 \\ 1/2 & -3/2 & 0 \end{pmatrix}$ **(D)** $\begin{pmatrix} -3/2 & 2 & 1 \\ -1/2 & 0 & 0 \\ 3/2 & -1 & -1 \end{pmatrix}$

(E) Matrix A is singular.

Solution

The adjoint matrix is the transpose of the cofactor matrix; that is,

$$\text{adj } B = \begin{pmatrix} (-1)^{1+1}\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \end{pmatrix}^T$$

$$\therefore \text{adj } B = \begin{pmatrix} -3 & 1 & 3 \\ -2 & 0 & 4 \\ 2 & 0 & -2 \end{pmatrix}^T = \begin{pmatrix} -3 & -2 & 2 \\ 1 & 0 & 0 \\ 3 & 4 & -2 \end{pmatrix}$$

The determinant of B is 2; try calculating it yourself. It remains to apply equation (19):

$$B^{-1} = \frac{1}{|B|}\text{adj}\, B = \frac{1}{2}\begin{pmatrix} -3 & -2 & 2 \\ 1 & 0 & 0 \\ 3 & 4 & -2 \end{pmatrix} = \boxed{\begin{pmatrix} -\frac{3}{2} & -1 & 1 \\ \frac{1}{2} & 0 & 0 \\ \frac{3}{2} & 2 & -1 \end{pmatrix}}$$

▶ **The correct answer is A.**

5.3. Hermitian Matrices

→ The student should be able to effortlessly identify a Hermitian matrix, a skew-Hermitian matrix, a unitary matrix, and their respective properties.

Solved Problem 25

A. Show that matrix A is Hermitian.

$$A = \begin{pmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{pmatrix}$$

B. If $\boldsymbol{u}^H\boldsymbol{u} = 1$, where $\boldsymbol{u}$ is a complex vector and $\boldsymbol{u}^H$ is its Hermitian, and show that $I - 2\boldsymbol{u}\boldsymbol{u}^H$ is Hermitian and also unitary. The rank-one matrix $\boldsymbol{u}\boldsymbol{u}^H$ is the projection onto what line in $\mathbb{C}^n$?

Solution

Part A: The conjugate A' of the matrix at hand is

$$A' = \begin{pmatrix} 1 & 1+i & 2 \\ 1-i & 3 & -i \\ 2 & i & 0 \end{pmatrix}$$

The transpose $(A')^{\mathrm{T}}$ is, in turn,

$$A' = \begin{pmatrix} 1 & 1+i & 2 \\ 1-i & 3 & -i \\ 2 & i & 0 \end{pmatrix} \rightarrow \left(A'\right)^T = \begin{pmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{pmatrix} = A$$

Since the transpose conjugate of A equals A itself, the matrix is Hermitian.

Part B: Simply verify that

$$\left(I - 2\mathbf{u}\mathbf{u}^H\right)^H = I - 2\mathbf{u}\mathbf{u}^H$$

and

$$\left(I - 2\mathbf{u}\mathbf{u}_H\right)^2 = I - 4\mathbf{u}\mathbf{u}^H + 4\mathbf{u}\left(\mathbf{u}^H\mathbf{u}\right)\mathbf{u}^H = I$$

The rank-1 matrix $\boldsymbol{uu^H}$ projects onto the line through $\boldsymbol{u}$.

Topic 6: Systems of Linear Equations

→ The student should master solving systems of linear equations through row reduction and Cramer's rule.

Solved Problem 26

Consider the following system of linear equations,

$$\begin{cases} x + y + 7z = -7 \\ 2x + 3y + 17z = -16 \\ x + 2y + \left(a^2 + 1\right)z = 3a \end{cases}$$

Which of the following is true?

(A) The system is inconsistent for $a = -3$.
(B) The system has infinitely many solutions for $a = 3$.
(C) The system has exactly one solution for $a = 0$.
(D) The system is inconsistent for any $a \neq -1$.
(E) The system has infinitely many solutions for all $a \in \mathbb{R}$.

Solution

The augmented matrix of the system is

$$\begin{pmatrix} 1 & 1 & 7 & -7 \\ 2 & 3 & 17 & -16 \\ 1 & 2 & a^2+1 & 3a \end{pmatrix}$$

This reduces to

$$\begin{pmatrix} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & a^2-9 & 3a+9 \end{pmatrix}$$

The last row corresponds to $(a^2 - 9)z = 3a + 9$. If $a = -3$ this becomes $0 = 0$ and the system will have infinitely many solutions. If $a = 3$, then the last row corresponds to $0 = 18$ and the system will be inconsistent. If $a \neq \pm 3$, then

$$\left(a^2-9\right)z = 3a+9 \rightarrow z = \frac{3\left(a+3\right)}{a^2-9}$$

$$\therefore z = \frac{3\cancel{\left(a+3\right)}}{\cancel{\left(a+3\right)}\left(a-3\right)} = \frac{3}{a-3}$$

and, from back substitution, y and z will be uniquely determined as well; the system has exactly one solution.

▶ **The correct answer is C.**

Topic 7: Vector Spaces

7.1. Subspaces

→ The student should be able to identify whether given sets form subspaces of $\mathbb{R}^n$. Recall that if S is a nonempty subset of a vector space V, and S satisfies the conditions

1. $\alpha\boldsymbol{x} \in S$ whenever $\boldsymbol{x} \in S$ for any scalar α;

2. $\boldsymbol{x} + \boldsymbol{y} \in S$ whenever $\boldsymbol{x} \in S$ and $\boldsymbol{y} \in S$;

then S is said to be a subspace of V.

Solved Problem 27

Consider the following sets:

I. $\{(x_1,x_2)^{\mathrm{T}} \mid x_1 + x_2 = 0\}$

II. $\{(x_1,x_2)^{\mathrm{T}} \mid x_1x_2 = 0\}$

III. $\{(x_1,x_2)^{\mathrm{T}} \mid x_1x_2 = 0\}$

Which of these sets form subspaces of $\mathbb{R}^2$?

(A) I and II only.

(B) I and III only.

(C) II and III only.

(D) I, II and III.

(E) None of the sets form subspaces of $\mathbb{R}^2$.

Solution

Consider first set I. Given a vector $\boldsymbol{x} = (a,b)^{\mathrm{T}}$, we have, from the definition of this set, $a + b = 0$ and $b = -a$. Accordingly, every vector in S has the form $(a,-a)^{\mathrm{T}}$. Given some scalar α, we have $\alpha\boldsymbol{x} = (\alpha a,-\alpha a)$. Here,

$$\alpha a+\alpha(-a)=\alpha[a+(-a)]=\alpha(0)=0$$

Thus, $\alpha\boldsymbol{x} \in S$ for any scalar α. Next, let $\boldsymbol{x} = (a,-a)^{\mathrm{T}}$ and $\boldsymbol{y} = (b,-b)^{\mathrm{T}}$ be arbitrary elements of S. It follows that

$$\mathbf{x}+\mathbf{y}=(a,-a)^T+(b,-b)^T=(a+b,-(a+b))^T$$

Here,

$$(a+b)+[-(a+b)]=a+b-a-b$$
$$\therefore a-a+b-b=0$$

Thus, $\boldsymbol{x}+\boldsymbol{y} \in S$ whenever $\boldsymbol{x} \in S$ and $\boldsymbol{y} \in S$. In summary, set I satisfies both conditions to form a subspace of $\mathbb{R}^2$. We leave it to the reader to show that set II does *not* constitute a subspace of $\mathbb{R}^2$ because it does not satisfy the second closure condition, while set III does constitute a subspace of $\mathbb{R}^2$ because it satisfies both closure conditions.

▶ **The correct answer is B.**

■ 7.2. Linear Dependence, Span, Basis and Dimension

→ The student should be able to test vectors for linear dependence. Further, the student should be acquainted with the notions of span, basis, and dimension. The student should bear in mind that sometimes spanning implies linear independence and vice versa; indeed:

1. A set of k linearly independent vectors in a nonzero k-dimensional subspace of $\mathbb{R}^n$ is a basis for that subspace.

2. A set of k vectors that span a nonzero k-dimensional subspace of $\mathbb{R}^n$ is a basis for that subspace.

3. A set of fewer than k vectors in a nonzero k-dimensional subspace of $\mathbb{R}^n$cannot span that subspace.

4. A set with more than k vectors in a nonzero k-dimensional subspace of $\mathbb{R}^n$ is linearly dependent.

Solved Problem 28

Consider the vectors

$$\mathbf{x}_1=\begin{pmatrix}2\\1\end{pmatrix};\ \mathbf{x}_2=\begin{pmatrix}4\\3\end{pmatrix};\ \mathbf{x}_3=\begin{pmatrix}7\\-3\end{pmatrix}$$

A. Show that $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ form a basis for $\mathbb{R}^2$.
B. Why must $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, and $\boldsymbol{x_3}$ be linearly dependent?
C. What is the dimension of Span($\boldsymbol{x_1}$,$\boldsymbol{x_2}$,$\boldsymbol{x_3}$)?

Solution

Part A: The first step is to show that the two vectors are linearly independent. A given linear combination of $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ must have the form

$$c_1\mathbf{x}_1+c_2\mathbf{x}_2=0\rightarrow c_1\begin{pmatrix}2\\1\end{pmatrix}+c_2\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\therefore \begin{cases} 2c_1 + 4c_2 = 0 \ \ \text{(I)} \\ c_1 + 3c_2 = 0 \ \ \text{(II)} \end{cases}$$

Subtracting 2×(II) from (I) brings to

$$2c_1 + 4c_2 - 2(c_1 + 3c_2) = 0 \rightarrow \cancel{2c_1} + 4c_2 - \cancel{2c_1} - 6c_2 = 0$$

$$\therefore -2c_2 = 0$$

$$\therefore c_2 = 0$$

Substituting c_2 = 0 into either (I) or (II) gives c_1 = 0. Since $c_1 = c_2$ = 0, vectors $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are linearly independent. Further, it is easy to see that these two linearly independent vectors span $\mathbb{R}^2$. Thus, both conditions for $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ to form a basis in $\mathbb{R}^2$ are satisfied.

The first step is to show that the two vectors are linearly independent. A given linear combination of $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ must have the form

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = 0 \rightarrow c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{cases} 2c_1 + 4c_2 = 0 \ \ \text{(I)} \\ c_1 + 3c_2 = 0 \ \ \text{(II)} \end{cases}$$

Subtracting 2×(II) from (I) brings to

$$2c_1 + 4c_2 - 2(c_1 + 3c_2) = 0 \rightarrow \cancel{2c_1} + 4c_2 - \cancel{2c_1} - 6c_2 = 0$$

$$\therefore -2c_2 = 0$$

$$\therefore c_2 = 0$$

Substituting c_2 = 0 into either (I) or (II) gives c_1 = 0. Since $c_1 = c_2$ = 0, vectors $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are linearly independent. Further, it is easy to see that these two linearly independent vectors span $\mathbb{R}^2$. Thus, both conditions for $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ to form a basis in $\mathbb{R}^2$ are satisfied.

Part B: To show that $\boldsymbol{x_1}$, $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$ are linearly dependent, simply recall that, if {$\boldsymbol{v_1}$, $\boldsymbol{v_2}$, ..., $\boldsymbol{v_n}$}, then any collection of m vectors in V such that $m > n$ is linearly dependent. In the present case, m = 3, n = 2 and, since $m > n$, the result specified above holds.

Part C: Since $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ form a basis for $\mathbb{R}^2$, x_3 can be expressed as a linear combination of $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$. Indeed,

$$\begin{pmatrix} 7 \\ -3 \end{pmatrix} = a\begin{pmatrix} 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} 4 \\ 3 \end{pmatrix} \rightarrow \begin{cases} 2a + 4b = 7 \ \ \text{(I)} \\ a + 3b = -3 \ \ \text{(II)} \end{cases}$$

Subtracting 2×(II) from (I) gives

$$2a + 4b - 2(a + 3b) = 7 - 2\times(-3) \rightarrow \cancel{2a} + 4b - \cancel{2a} - 6b = 13$$

$$\therefore -2b = 13$$

$$\therefore b = -\frac{13}{2}$$

Substituting b into (I) brings to

$$2a + 4b = 7 \rightarrow 2a + 4\times\left(-\frac{13}{2}\right) = 7$$

$$\therefore 2a - 26 = 7$$

$$\therefore a = \frac{33}{2}$$

Thus, we may write

$$\mathbf{x}_3 = \frac{33}{2}\mathbf{x}_1 - \frac{13}{2}\mathbf{x}_2$$

and it can be concluded that $\dim(\text{Span}(\mathbf{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3})) = 2$.

7.3. Change of Basis

→ The student should be able to assemble the transition matrix associated with a change of basis.

Solved Problem 29

Given

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}\ ;\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}\ ;\ S = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}$$

find two vectors $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ so that S will be the transition matrix from ($\boldsymbol{v_1}$,$\boldsymbol{v_2}$) to ($\boldsymbol{u_1}$,$\boldsymbol{u_2}$). Which of the following is one such vector?

(A) (−1,0)

(B) (−1, 1)

(C) (0, 5)

(D) (1, −1)

(E) (1, 5)

Solution

Suppose U denotes the matrix that has ($\boldsymbol{u_1}$,$\boldsymbol{u_2}$) as columns and V denotes the matrix that has ($\boldsymbol{v_1}$,$\boldsymbol{v_2}$) as columns. The transition matrix S from ($\boldsymbol{u_1}$,$\boldsymbol{u_2}$) to ($\boldsymbol{v_1}$,$\boldsymbol{v_2}$) is such that

$$S = U^{-1}V$$

This expression can be restated as

$$S = U^{-1}V \rightarrow V = US$$

Let $\boldsymbol{u_1}$ = (a,b) and $\boldsymbol{u_2}$ = (c,d). Substituting in the relation above brings to

$$V = US \rightarrow \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 4a + 2c & a + c \\ 4b + 2d & b + d \end{pmatrix}$$

Equating entry by entry gives the systems of equations

$$\begin{cases} 4a + 2c = 2\ \text{(I)} \\ a + c = 1\ \ \text{(II)} \end{cases} ;\ \begin{cases} 4b + 2d = 6\ \ \text{(III)} \\ b + d = 4\ \text{(IV)} \end{cases}$$

Manipulating (I),

$$4a + 2c = 2 \rightarrow 2a + 2\underbrace{(a + c)}_{=1} = 2$$

$$\therefore 2a + 2\times 1 = 2$$
$$\therefore a = 0$$

Substituting a in (II),

$$a + c = 1 \rightarrow 0 + c = 1$$
$$\therefore c = 1$$

Next, manipulating (III),

$$4b + 2d = 6 \rightarrow 2b + 2\underbrace{(b+d)}_{=4} = 6$$
$$\therefore 2b + 2\times 4 = 6$$
$$\therefore b = -1$$

Substituting b in (IV),

$$b + d = 4 \rightarrow -1 + d = 4$$
$$\therefore d = 5$$

Lastly, we assemble the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \; ; \; \mathbf{u}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

▶ **The correct answer is E.**

7.4. The Dimension Theorem for Matrices

→ The student should be well aware that the four fundamental subspaces of a matrix A are:

1. The row space $\text{row}(A)$.

2. The column space $\text{col}(\text{A})$.

3. The null space $N(A)$.

4. The null space $N(A^T)$.

The student should be familiar with the definitions of rank and nullity. A crucial result is the *dimension theorem for matrices*, according to which, if A is an $m \times n$ matrix,

$$\text{rank}(A) + \text{nullity}(A) = n \quad (20)$$

Indeed, if an $m \times n$ matrix has rank k, then:

1. A has nullity $n - k$.

2. Every row echelon form of A has k nonzero rows.

3. Every row echelon form of A has $m - k$ zero rows.

4. The homogeneous system $Ax = 0$ has k pivot variables (leading variables) and $n - k$ free variables.

Solved Problem 30

A 5×6 matrix has null space with dimension equal to 2. The rank of this matrix is:

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Solution

According to the dimension theorem for matrices, if A is a $m \times n$ matrix, then $rank(A) + nullity(A) = n$. In the present case, $nullity(A) = 2$, $n = 6$, and

$$\text{rank}(A) + \text{nullity}(A) = n \to \text{rank}(A) = n - \text{nullity}(A)$$

$$\therefore \text{rank}(A) = 6 - 2 = \boxed{4}$$

▶ **The correct answer is D.**

Topic 8: Orthogonality

8.1. Inner Products

→ The student should know how to apply a given inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Further, the student should be able to find the norm $\|\boldsymbol{u}\|$,

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} \quad (21)$$

Solved Problems 31 and 32

Given two vectors $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ in $V \in \mathbb{R}^2$, suppose we define an inner product with the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

Determine the inner product of vectors $\boldsymbol{u} = (2,1)$ and $\boldsymbol{v} = (-1,3)$.

(A) 3
(B) 6
(C) 9
(D) 12
(E) 15

Compute the angle formed by vectors u and v using the inner product defined in Problem 31.

(A) 41°
(B) 58°
(C) 68°
(D) 75°
(E) 84°

Solution

Problem 31: All we have to do is substitute $\boldsymbol{u}$ and $\boldsymbol{v}$ into the expression that defines the IP,

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2 \times 2 \times 3 + 3 \times 1 \times (-1) = 12 - 3 = \boxed{9}$$

▶ **The correct answer is C.**

Problem 32: First, note that the norm induced by the inner product in question is given by, for $\boldsymbol{u}$,

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{3u_1u_1 + 2u_2u_2} = \sqrt{3u_1^2 + 2u_2^2}$$

$$\therefore \|\mathbf{u}\| = \sqrt{3 \times 2^2 + 2 \times 1^2} = \sqrt{14}$$

while, for $\boldsymbol{v}$,

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{3v_1v_1 + 2v_2v_2} = \sqrt{3v_1^2 + 2v_2^2}$$

$$\therefore \|\mathbf{v}\| = \sqrt{3\times(-1)^2 + 2\times 3^2} = \sqrt{21}$$

Further, we also have $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 9$, which was determined in the previous problem. The final step is to apply the angle formula,

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{9}{\sqrt{14}\times\sqrt{21}} = 0.525$$

$$\therefore \theta = \cos^{-1} 0.525 = \boxed{58.3^{\circ}}$$

▶ **The correct answer is B.**

8.2. Least Squares

→ The student should be able to fit a set of (x,y) data points to a linear equation $y = ax + b$ using the least-squares technique. Problems involving this technique are likely to appear in the calculator-free section of the exam.

Solved Problem 33

Find the least-squares straight-line fit $y = a + bx$ to the given points.

x	0	2	3	3
y	1	0	1	2

(A) $y = \frac{2}{3} + \frac{x}{6}$ **(B)** $y = \frac{3}{4} + \frac{x}{6}$ **(C)** $y = \frac{2}{3} + \frac{x}{3}$

(D) $y = \frac{3}{4} + \frac{x}{3}$ **(E)** $y = \frac{4}{5} + \frac{x}{3}$

Solution

The linear model for the given data is $M\boldsymbol{v} = \boldsymbol{y}$, where

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} \; ; \; y = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

The least squares solution is obtained by solving the normal system $M^T M\boldsymbol{v} = M^T\boldsymbol{y}$, which is

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 4 & 8 \\ 8 & 22 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

Since the matrix on the left-hand side is nonsingular, the system has a unique solution given by

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 8 & 22 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{11}{12} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 4 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{6} \end{pmatrix}$$

Thus the least-squares line fit to the given data is $y = 2/3 + x/6$.

▶ **The correct answer is A.**

8.3. Gram-Schmidt Orthonormalization

→ The student must be able to convert a given basis to an orthonormal basis via the Gram-Schmidt technique.

Solved Problem 34

Use the Gram-Schmidt process to transform the given basis into an orthonormal basis.

$$\mathbf{w}_1 = (1,0,0) \; ; \; \mathbf{w}_2 = (3,7,-2) \; ; \; \mathbf{w}_3 = (0,4,1)$$

Solution

Let $\boldsymbol{v_1} = \boldsymbol{w_1}$. To determine $\boldsymbol{v_2}$, we calculate

$$\therefore \mathbf{v}_2 = \mathbf{w}_2 - \frac{\mathbf{w}_2 \bullet \mathbf{v}_1}{\|\mathbf{v}_1\|^2} = (3,7,-2) - \frac{[3\times1 + 7\times0 - 2\times0]}{1^2 + 0^2 + 0^2}(1,0,0)$$

$$\therefore \mathbf{v}_2 = (3,7,-2) - \frac{3}{1}(1,0,0) = (0,7,-2)$$

Then, we determine $\boldsymbol{v_3}$,

$$\mathbf{v}_3 = \mathbf{w}_3 - \frac{\mathbf{w}_3 \bullet \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{w}_3 \bullet \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

$$\therefore \mathbf{v}_3 = (0,4,1) - \frac{0\times1 + 4\times0 + 1\times0}{1^2 + 0^2 + 0^2}(1,0,0) - \frac{0\times0 + 4\times7 + 1\times(-2)}{0^2 + 7^2 + (-2)^2}(0,7,-2)$$

$$\therefore \mathbf{v}_3 = (0,4,1) - 0\times(1,0,0) - \frac{26}{53}(0,7,-2)$$

$$\therefore \mathbf{v}_3 = \left(0 - 0, 4 - \frac{26\times7}{53}, 1 - \frac{26\times(-2)}{53}\right) = \left(0, \frac{30}{53}, \frac{105}{53}\right)$$

The penultimate step is to determine the norms of vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$,

$$\|\mathbf{v}_1\| = \sqrt{1^2+0^2+0^2} = 1$$

$$\|\mathbf{v}_2\| = \sqrt{0^2+7^2+(-2)^2} = \sqrt{53}$$

$$\|\mathbf{v}_3\| = \sqrt{\left(\frac{30}{53}\right)^2+\left(\frac{105}{53}\right)^2} = \frac{15}{\sqrt{53}}$$

Lastly, we conclude that $\{\boldsymbol{v_1},\boldsymbol{v_2},\boldsymbol{v_3}\}$ is an orthogonal basis in $\mathbb{R}^3$, and the vectors $\{\boldsymbol{q_1},\boldsymbol{q_2},\boldsymbol{q_3}\}$, such that

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1,0,0)}{1} = (1,0,0)$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(0,7,-2)}{\sqrt{53}} = \left(0,\frac{7}{\sqrt{53}},-\frac{2}{\sqrt{53}}\right)$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(0,\frac{30}{53},\frac{105}{53}\right)}{\frac{15}{\sqrt{53}}} = \left(0,\frac{2}{\sqrt{53}},\frac{7}{\sqrt{53}}\right)$$

constitute an orthonormal basis for $\mathbb{R}^3$.

Topic 9: Diagonalization

9.1. Eigenvalues and Eigenvectors

→ The student should be able to write the characteristic polynomial of a matrix A, solve it for the eigenvalues, determine the corresponding eigenvectors, and find an invertible matrix P such that $P^{-1}AP$ is diagonal. In short, the student should have mastered the entire diagonalization process. The student must know by heart the general equations for characteristic polynomials of order-2 matrices,

$$\Delta(t) = t^2 - \operatorname{tr}(A)t + \det A \quad (22)$$

and the corresponding expression for order-3 matrices,

$$\Delta(t) = t^3 - (a_{11}+a_{22}+a_{33})t^2 + \left(\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right)t - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad (23)$$

Solved Problem 35 illustrates use of equation (23).

Solved Problem 35

What is the coefficient of power t^2 in the characteristic polynomial of matrix A?

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 7 & 1 \\ 1 & 6 & 2 \end{pmatrix}$$

(A) 2
(B) 4
(C) 6
(D) 8
(E) 10

Solution

The term of power t^2 equals the trace of the matrix,

$$a_{11} + a_{22} + a_{33} = 1 + 7 + 2 = \boxed{10}$$

▶ **The correct answer is E.**

Solved Problem 36 illustrates the full diagonalization process.

Solved Problem 36

Consider matrix A.

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$$

A. Determine the characteristic polynomial of A.
B. Determine the eigenvalues of A.
C. Find two eigenvectors for A, one belonging to each of the eigenvalues you determined in Part (B).
D. Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

Solution

Part A: To establish the characteristic polynomial, we compute $tI - A$, where I is the identity matrix, giving

$$tI - A = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$$

$$\therefore tI - A = \begin{pmatrix} t-1 & -3 \\ -2 & t+4 \end{pmatrix}$$

The characteristic polynomial is given by the determinant of the matrix above,

$$|tI - A| = \begin{vmatrix} t-1 & -4 \\ -2 & t-3 \end{vmatrix} = (t-1)\times(t+4) - (-2)\times(-3)$$

$$\therefore |tI - A| = t^2 + 3t - 4 - 6$$

$$\therefore |tI - A| = \boxed{t^2 + 3t - 10}$$

Another way to arrive at the same result is to apply equation (21),

$$\Delta(t) = t^2 - \text{tr}(A)t + \det A = t^2 - (1-4)t + (-4 - 2\times 3)$$

$$\therefore \Delta(t) = t^2 + 3t - 10$$

Part B: The eigenvectors are the solutions of the equation $\Delta(t) = 0$, namely

$$\Delta(t) = 0 \to t^2 + 3t - 10 = 0$$

$$\therefore t = \frac{-3 \pm \sqrt{9 - 4\times 1\times(-10)}}{2} = \frac{-3 \pm 7}{2} = \boxed{-5}, \boxed{2}$$

Part C: Substitute $t = -5$ in the matrix $tI - A$ to obtain the matrix

$$\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} -5-1 & 0-3 \\ 0-2 & -5-(-4) \end{pmatrix} = \begin{pmatrix} -6 & -3 \\ -2 & -1 \end{pmatrix}$$

The eigenvectors belonging to $\lambda_1 = -5$ form the solution of the homogeneous system $MX = 0$, that is,

$$MX = 0 \to \begin{pmatrix} -6 & -3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{cases} -6x - 3y = 0 \\ -2x - y = 0 \end{cases} \to x = -\frac{y}{2}$$

One solution to the homogeneous system above is $x = 1$, $y = -2$. Thus, $\boldsymbol{u} = (1,-2)$ is an eigenvector that spans the eigenspace of $\lambda_1 = -5$.

Moving on to eigenvalue $\lambda_2 = 2$, we proceed as we did above,

$$tI - A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$$

The eigenvectors belonging to $\lambda_2 = 2$ form the solution of the homogeneous system $MX = 0$, that is,

$$MX = 0 \to \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{cases} x - 3y = 0 \\ -2x + 6y = 0 \end{cases} \to x = 3y$$

One solution to this homogeneous system is $x = 3$, $y = 1$. Thus, $\boldsymbol{v}$ = (3,1) is an eigenvector that spans the eigenspace of $\lambda_2 = 2$.

Part D: Let P be the matrix whose columns are the above eigenvectors,

$$P = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$$

Then, $B = P^{-1}AP$ is the diagonal matrix whose diagonal entries are the respective eigenvalues,

$$B = P^{-1}AP = \begin{pmatrix} \frac{1}{7} & -\frac{3}{7} \\ \frac{2}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix}$$

Solved Problem 37 illustrates a simple problem that may appear in the objective section of the exam: finding the eigenvalue that corresponds to a given eigenvector.

Solved Problem 37

Vector $\boldsymbol{v}$ = (1,1,1) is an eigenvector of A and corresponds to an eigenvalue equal to:

$$A = \begin{pmatrix} -1 & 1 & 2 \\ -6 & 2 & 6 \\ 0 & 1 & 1 \end{pmatrix}$$

(A) −3
(B) −2
(C) −1
(D) 1
(E) 2

Solution

For $\boldsymbol{v}$ to be an eigenvector of A, we must have $A\boldsymbol{v} = \lambda\boldsymbol{v}$, where λ is an eigenvalue of A; that is,

$$A\mathbf{v} = \begin{pmatrix} -1 & 1 & 2 \\ -6 & 2 & 6 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1\times1+1\times1+2\times1 \\ -6\times1+2\times1+6\times1 \\ 0\times1+1\times1+1\times1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Thus, λ = 2 is an eigenvalue of A.

▶ **The correct answer is E.**

■ 9.2. Quadratic Forms

→ The student should be acquainted with the basic matrix theory of quadratic forms. In particular, a quadratic form in two variables x and y is an expression of the form

$$q(x,y) = ax^2 + 2cxy + by^2 \quad (24)$$

and has matrix representation

$$q = \mathbf{x}^T A\mathbf{x} = (x \;\; y)\begin{pmatrix} a & c \\ c & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad (25)$$

The classification of the quadratic form hinges on the nature of matrix A. Indeed:

1. q is positive definite if and only if all eigenvalues of A are positive.
2. q is positive semi-definite if and only if all eigenvalues of A are non-negative.
3. q is negative definite if and only if all eigenvalues of A are negative.
4. q is negative semi-definite if and only if all eigenvalues of A are non-positive.
5. q is indefinite if and only if some eigenvalues of A are negative and some are positive.

For the two-variable quadratic form $q(x,y)$ given in (24), whose matrix A has eigenvalues λ_1 and λ_2, we can conclude:
1. If $|A| > 0$ and $a > 0$, then $\lambda_1 > 0$, $\lambda_2 > 0$ and A is positive definite.
2. If $|A| > 0$ and $a < 0$, then $\lambda_1 < 0$, $\lambda_2 < 0$ and A is negative definite.
3. If $|A| < 0$, then λ_1 and λ_2 have opposite signs and A is indefinite.

In general, a quadratic form will be positive definite if the diagonal entries of A are all positive, and negative definite if the principal minors of A alternate in sign, the first one being negative. Solved Problem 38 is an illustrative quadratic form classification problem.

Solved Problem 38

The quadratic form g described below is:

$$g(x,y,z) = 2xy - 4yz + 6xz - 4x^2 - 2y^2 - 4z^2$$

(A) Positive definite.
(B) Negative definite.
(C) Indefinite.
(D) Nothing can be concluded.

Solution

The equation for the quadratic form at hand is repeated below.

$$g(x,y,z) = 2xy - 4yz + 6xz - 4x^2 - 2y^2 - 4z^2$$

The matrix G that represents this quadratic form is given in continuation. The entries are shown with colors corresponding to the equation above.

$$G = \begin{pmatrix} -4 & 1 & 3 \\ 1 & -2 & -2 \\ 3 & -2 & -4 \end{pmatrix}$$

The three principal minors are

$$a_{11} = -4 \;;\; \begin{vmatrix} -4 & 1 \\ 1 & -2 \end{vmatrix} = -4\times(-2) - 1\times 1 = 7 \;;\; \det A = -6$$

The principal minors alternate signs, with the even ones (i.e., the second) being positive and the odd ones (i.e., the first and third) negative; it follows that the quadratic form is negative definite.

▶ **The correct answer is B.**

Solved Problem 39 is an illustrative open-ended quadratic form classification problem.

Solved Problem 39

Let A be a $n \times n$ symmetric negative definite matrix.

A. What will the sign of $\det A$ be if n is even? If n is odd?

B. Show that the leading principal submatrices of A are negative definite.

C. Show that the determinants of the leading principal minors of A alternate in sign.

Solution

Part A: If A is a symmetric negative definite matrix, then its eigenvalues are all negative. Since the determinant of A is the product of the eigenvalues, it follows that $\det A$ will be positive if n is even and negative if n is odd.

Part B: Let A_k denote the leading principal subminor of A of order k and let $\boldsymbol{x}_1$ be a nonzero vector in $\mathbb{R}^k$. If we set

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad \mathbf{x} \in \mathbb{R}^n$$

then

$$\mathbf{x}_1^T A_k \mathbf{x}_1 = \mathbf{x}^T A \mathbf{x} < 0$$

Accordingly, the leading principal minors are all negative definite.

Part C: The fact that the determinants of the leading principal minors of A alternate in sign is a consequence of the results of parts A and B.

9.3. Jordan Canonical Form

Finally, the student must be able to effortlessly extract the characteristic polynomial from Jordan canonical form matrices.

Solved Problem 40

Consider square matrix B; all elements not shown are zeros. What is the characteristic polynomial of B?

$$\boldsymbol{B} = \begin{bmatrix} 7 & 1 & & & & & & \\ 0 & 7 & & & & & & \\ & & 7 & 1 & & & & \\ & & 0 & 7 & & & & \\ & & & & 7 & & & \\ & & & & & 4 & 1 & 0 \\ & & & & & 0 & 4 & 1 \\ & & & & & 0 & 0 & 4 \end{bmatrix}$$

(A) $(t - 7)^3(t - 4)^5$

(B) $(t-7)^4(t-4)^3(t-1)$
(C) $(t-7)^4(t-4)^4$
(D) $(t-7)^5(t-4)^3$
(E) $(t-7)^6(t-4)^2$

Solution

This matrix consists of Jordan blocks, i.e. square blocks in which the diagonal elements are repetitions of the same eigenvalue λ, the superdiagonal has only 1s, and all other elements are zeros. The characteristic polynomial of the matrix is then

$$\boxed{\Delta(t)=(t-7)^5(t-4)^3}$$

The exponent 5 comes from the fact that eigenvalue $\lambda_1 = 7$ occurs five times, while exponent 3 is attributable to the fact that eigenvalue $\lambda_2 = 4$ occurs three times.

▶ **The correct answer is D.**

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