Montogue

[AGLA]

Analytic Geometry and Linear Algebra at the AP Level

40+14 Solved Problems

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SECTION I – PART A

No calculators allowed!

1. The equation of the line bisecting perpendicularly the segment joining the points (−4, 6) and (8,8) is:

(A) $x + 2y - 7 = 0$

(B) $2x + y - 3 = 0$

(C) $6x + y - 12 = 0$

(D) $6x + y - 19 = 0$

(E) $6x + y - 23 = 0$

2. Points $A(2,-3)$ and $B(-2,1)$ are vertices of triangle ABC. If its centroid lies on the line $2x + 3y = 1$, then the locus of vertex C is:

(A) $2x + 3y - 9 = 0$

(B) $3x + 2y - 7 = 0$

(C) $3x + 2y - 6 = 0$

(D) $4x - 2y - 9 = 0$

(E) $5x - 2y - 6 = 0$

3. Consider a line whose perpendicular from the origin makes an angle of 30° with the x-axis. Also, the line and the coordinate axes form a triangle of area $50/\sqrt{3}$. Which of the following are possible equations for this line?

(A) $x + \sqrt{3}y \pm 5 = 0$
(B) $x + \sqrt{3}y \pm 10 = 0$
(C) $\sqrt{3}x + y \pm 2\sqrt{5} = 0$
(D) $\sqrt{3}x + y \pm 5 = 0$
(E) $\sqrt{3}x + y \pm 10 = 0$

4. Which of the following is an equation of a line on the xy plane that passes through (2,2) and whose intercepts on the coordinate axes have a sum equal to 9?

(A) $2x + 4y - 8 = 0$
(B) $3x + 4y - 14 = 0$
(C) $3x + 6y - 18 = 0$
(D) $4x + 5y - 18 = 0$
(E) $6x + 3y - 18 = 0$

5. For what values of α and β intercepts cut off on the coordinate axes by the line $\alpha x + \beta y + 8 = 0$ are equal in length but opposite in sign to those cut off by the line $3x - 2y + 12 = 0$?

(A) $\alpha = -2\ ;\ \beta = -\dfrac{4}{3}$

(B) $\alpha = -2\ ;\ \beta = \dfrac{4}{3}$

(C) $\alpha = 2\ ;\ \beta = -\dfrac{4}{3}$

(D) $\alpha = 2\ ;\ \beta = \dfrac{4}{3}$

(E) $\alpha = -2\ ;\ \beta = -4$

6. For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, the equation of the bisector of the angle that contains the origin is:

(A) $7x + 9y + 3 = 0$
(B) $7x - 9y + 3 = 0$
(C) $7x + 9y - 3 = 0$
(D) $11x + 7y - 3 = 0$
(E) $12x + 5y - 3 = 0$

7. The equations of two diameters of a circle are $2x + y - 3 = 0$ and $x - 3y + 2 = 0$. If the circle passes through the point $(-2,5)$, find its equation.
(A) $(x - 1)^2 + (y + 1)^2 = 17$
(B) $(x + 1)^2 + (y - 1)^2 = 17$
(C) $(x - 1)^2 + (y - 1)^2 = 25$
(D) $(x - 1)^2 + (y - 1)^2 = 36$
(E) $(x - 2)^2 + (y + 1)^2 = 42$

8. Point (8,9) lies on the circle $x^2 + y^2 - 10x - 12y + 43 = 0$. Find the other end of the diameter through (8,9).
(A) (1/2,2)
(B) (1,2)
(C) (2,3)
(D) (3,4)
(E) (4,5)

9. What is the equation of the circle that is tangent to the line $y = -1$ and whose center is $(3,-2)$?
(A) $x^2 + y^2 - 6x + 4y + 11 = 0$
(B) $x^2 + y^2 - 6x + 4y + 13 = 0$
(C) $x^2 + y^2 + 6x - 4y + 11 = 0$
(D) $x^2 + y^2 + 6x - 4y + 13 = 0$
(E) $x^2 + y^2 + 6x - 4y + 17 = 0$

10. The equation

$$3x^2 - 7xy + 4y^2 + 5x + 21y - 8 = 0$$

represents:
(A) A parabola, or two parallel straight lines, or no locus at all.
(B) An ellipse.
(C) An ellipse, a point, or no locus at all.
(D) A hyperbola.
(E) A hyperbola or two intersecting straight lines.

11. An ellipse with major axis parallel to the y-axis has center $C(4,-2)$, eccentricity 1/2 and minor axis length equal to 6. What is the equation of this ellipse?
(A) $\frac{x^2}{8} + \frac{y^2}{10} = 1$
(B) $\frac{(x-4)^2}{8} + \frac{(y-2)^2}{10} = 1$
(C) $\frac{(x-4)^2}{10} + \frac{(y-2)^2}{8} = 1$
(D) $\frac{(x-4)^2}{9} + \frac{(y-2)^2}{12} = 1$ **(E)** $\frac{(x-4)^2}{12} + \frac{(y-2)^2}{9} = 1$

12. An ellipse has one vertex at (−4,6) and its focus closest to this vertex is (−4,4). If the eccentricity is 1/2, find its equation.

(A) $\frac{(x+4)^2}{9}+\frac{(y-3)^2}{16}=1$

(B) $\frac{(x+4)^2}{12}+\frac{(y-3)^2}{16}=1$

(C) $\frac{(x+4)^2}{14}+\frac{(y-2)^2}{16}=1$

(D) $\frac{(x+4)^2}{12}+\frac{(y-2)^2}{16}=1$

(E) $\frac{(x+4)^2}{9}+\frac{(y-2)^2}{16}=1$

13. Determine the equation of the parabola whose focus is (−2,3) and whose directrix is $y = -3$.

(A) $x^2 - 4x + 8y - 4 = 0$
(B) $x^2 - 4x + 8y - 6 = 0$
(C) $x^2 + 4x - 8y - 4 = 0$
(D) $x^2 + 4x - 8y - 6 = 0$
(E) $x^2 + 4x - 8y - 8 = 0$

14. Find the equation of the hyperbola with one focus (2,0), eccentricity 2 and directrix $x - y = 0$.

(A) $x^2 + y^2 - 2xy + 2x - 2$
(B) $x^2 + y^2 - 4xy + 2x - 2$
(C) $x^2 + y^2 - 4xy + 2x - 4$
(D) $x^2 + y^2 - 4xy + 4x - 4$
(E) $x^2 + y^2 + 4xy + 4x - 4$

15. What should be the value of k for lines r and s to be orthogonal?

$$r:\begin{cases} y = kx - 3 \\ z = -2x \end{cases} \; ; \; s:\begin{cases} x = -1 + 2t \\ y = 3 - t \\ z = 5t \end{cases}$$

(A) −8
(B) −4
(C) 4
(D) 6
(E) 8

16. What is the point of intersection of lines r_1 and r_2 defined below?

$$r_1 : \begin{cases} y = -3x + 2 \\ z = 3x - 1 \end{cases} \; ; \; r_2 : \begin{cases} x = -t \\ y = 1 + 2t \\ z = -2t \end{cases}$$

(A) (1,−1,2)
(B) (−1,1,2)
(C) (1,2,−1)
(D) (2,1,−1)
(E) The lines do not intersect.

17. Determine the values of α and β for plane π_1 to be parallel to plane π_2.

$$\pi_1 : (2\alpha - 1)x - 2y + \beta z - 3 = 0$$
$$\pi_2 : 4x + 4y - z = 0$$

(A) $\alpha = -1/2$, $\beta = 1/2$
(B) $\alpha = -1/2$, $\beta = 3/2$
(C) $\alpha = -1/2$, $\beta = 1$
(D) $\alpha = -3/2$, $\beta = 1/2$
(E) $\alpha = -3/2$, $\beta = 3/2$

18. Determine the distance from point $P_0(-4,2,5)$ to plane π defined below.

$$\pi : 2x + y + 2z + 8 = 0$$

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

19. Find the inverse of matrix A.

$$A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$$

(A) $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ **(B)** $\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$ **(C)** $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & -1 \end{pmatrix}$ **(D)** $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & -1 & -1 \end{pmatrix}$

(E) Matrix A is singular.

20. In matrix A shown below, c is a real number. Which of the following is true?

$$A = \begin{pmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix}$$

(A) A is invertible if $c = 0$, but not if $c = 1$.

(B) A is invertible if $c = 1$, but not if $c = 0$.

(C) A is invertible only if $c \neq 0$ and $c \neq 1$.

(D) A is invertible only if $c \neq 0$, $c \neq 1$, and $c \neq 2$.

(E) A is invertible regardless of the value of c.

21. Which of the following four statements is true?

(A) It is possible to find a pair of two-dimensional subspaces S and T of $\mathbb{R}^3$ such that $S \cap T = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector.

(B) Let $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, ..., $\boldsymbol{x_k}$ be linearly independent vectors in $\mathbb{R}^n$. If $k < n$ and $\boldsymbol{x_{k+1}}$ is a vector that is not in the span of $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_k}$, then the vectors $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_k}, \boldsymbol{x_{k+1}}$ are linearly independent.

(C) If A is an $m \times n$ matrix and $m \neq n$, then A and A^T have the same nullity.

(D) Let ($\boldsymbol{u_1}$,$\boldsymbol{u_2}$), ($\boldsymbol{v_1}$,$\boldsymbol{v_2}$), and ($\boldsymbol{w_1}$,$\boldsymbol{w_2}$) be bases for $\mathbb{R}^2$. If X is the transition matrix corresponding to a change of basis from ($\boldsymbol{u_1}$,$\boldsymbol{u_2}$) to ($\boldsymbol{v_1}$,$\boldsymbol{v_2}$) and Y is the transition matrix corresponding to a change of basis from ($\boldsymbol{v_1}$,$\boldsymbol{v_2}$) to ($\boldsymbol{w_1}$,$\boldsymbol{w_2}$), then $Z = XY$ is the transition matrix corresponding to a change of basis from ($\boldsymbol{u_1}$,$\boldsymbol{u_2}$) to ($\boldsymbol{w_1}$,$\boldsymbol{w_2}$).

22. Given

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} ; \ \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} ; \ S = \begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix}$$

find two vectors $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$ so that S will be the transition matrix from basis ($\boldsymbol{w_1}$,$\boldsymbol{w_2}$) to ($\boldsymbol{v_1}$,$\boldsymbol{v_2}$). Which of the following is one such vector?

(A) $(-5,3)$

(B) $(-5, 9)$

(C) $(-4, 1)$

(D) $(-4, 2)$

(E) $(1, 4)$

23. A 4×5 matrix has null space with dimension equal to 2. The rank of this matrix is:

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

24. Which of the following four statements is true?
(A) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are unit vectors in $\mathbb{R}^n$ and $|\boldsymbol{x}^T\boldsymbol{y}| = 1$, then $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent.
(B) If U, V and W are subspaces of $\mathbb{R}^3$ and if $U \perp V$ (that is, U is orthogonal to V) and $V \perp W$, then $U \perp W$.
(C) If A is an $m \times n$ matrix, then AA^T and AA^T may not have the same rank.
(D) If Q_1 and Q_2 are orthogonal matrices, then Q_1Q_2 also is an orthogonal matrix.

25. What is the coefficient of power t^1 in the characteristic polynomial of matrix A?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 2 & 7 & 2 \end{pmatrix}$$

(A) -9
(B) -3
(C) 2
(D) 5
(E) 7

26. Which of the following matrices has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 5$?

(A) $\begin{pmatrix} 1 & 1 \\ -1 & 8 \end{pmatrix}$ **(B)** $\begin{pmatrix} 5 & 3 \\ -3 & 6 \end{pmatrix}$ **(C)** $\begin{pmatrix} 4 & 1 \\ 1 & 5 \end{pmatrix}$ **(D)** $\begin{pmatrix} 2 & 2 \\ -3 & 8 \end{pmatrix}$ **(E)** $\begin{pmatrix} 3 & 2 \\ -1 & 6 \end{pmatrix}$

27. Vector $\boldsymbol{v} = (1,0,1)$ is an eigenvector of matrix A and corresponds to an eigenvalue equal to:

$$A = \begin{pmatrix} 6 & 13 & -8 \\ 2 & 5 & -2 \\ 7 & 17 & -9 \end{pmatrix}$$

(A) -3
(B) -2
(C) -1
(D) 1
(E) 2

28. The quadratic form f described below is:

$$f(x, y, z) = 6xy - 4yz + 2xz - 4x^2 - 2y^2 - 4z^2$$

(A) Positive definite.
(B) Negative definite.
(C) Indefinite.
(D) Nothing can be concluded.

29. Consider square matrix A; all elements not shown are zeros. What is the characteristic polynomial of A?

$$A = \left[\begin{array}{ccc|cc|cc|c}
6 & 1 & 0 & & & & & \\
0 & 6 & 1 & & & & & \\
0 & 0 & 6 & & & & & \\
\hline
 & & & 6 & 1 & & & \\
 & & & 0 & 6 & & & \\
\hline
 & & & & & 2 & 1 & \\
 & & & & & 0 & 2 & \\
\hline
 & & & & & & & 2
\end{array}\right]$$

(A) $(t-2)^2(t-6)^5$
(B) $(t-2)^2(t-6)^5(t-1)$
(C) $(t-2)^3(t-6)^5$
(D) $(t-2)^4(t-6)^4$
(E) $(t-2)^5(t-6)^3$

30. Which of the following four statements is true? In the following statements, $i = \sqrt{-1}$.
(A) If A is a real matrix, then $A + iI$ is invertible.
(B) If A is a Hermitian matrix, then $A + iI$ is invertible.
(C) If U is a unitary matrix, then $A + iI$ is invertible.
(D) If A is a Hermitian matrix and c is a real number, cA is not Hermitian.

SECTION I – PART B

Calculator-free zone!

31. Consider the following system of linear equations,

$$\begin{cases} x + y + 7z = -7 \\ 2x + 3y + 17z = -16 \\ x + 2y + \left(a^2 + 1\right)z = 3a \end{cases}$$

Which of the following is true?
(A) The system is inconsistent for $a = -3$.
(B) The system has infinitely many solutions for $a = 3$.
(C) The system has exactly one solution for $a = 0$.
(D) The system is inconsistent for any $a \neq -1$.
(E) The system has infinitely many solutions for all $a \in \mathbb{R}$.

32. Which of the following vectors is simultaneously orthogonal to $\boldsymbol{v_1} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{v_2} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$?

(A) $\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

(B) $-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

(C) $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

(D) $2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

(E) $-\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

33. Determine the general equation of the plane determined by points $A(2,1,-1)$, $B(0,-1,1)$, and $C(1,2,1)$.

(A) $-2x + y - 3z + 3 = 0$

(B) $-2x + 2y - 3z + 3 = 0$

(C) $-3x + y - 2z + 3 = 0$

(D) $-3x + 2y - 3z + 3 = 0$

(E) $-3x + 3y - 2z + 3 = 0$

34. The angle between planes π_1 and π_2 is, most nearly:

$$\pi_1 : 2x - 3y + 5z = 0$$
$$\pi_2 : 3x + 2y + 5z - 4 = 0$$

(A) 23°

(B) 36°

(C) 49°

(D) 61°

(E) 73°

35. The transition matrix S corresponding to the change of basis from $(\boldsymbol{u_1},\boldsymbol{u_2})$ to the basis formed by Cartesian unit vectors $(\boldsymbol{e_1},\boldsymbol{e_2})$ is:

$$\mathbf{u}_1 = (1,2)^T \ ; \ \mathbf{u}_2 = (2,5)^T$$
$$\mathbf{e}_1 = (1,0)^T \ ; \ \mathbf{e}_2 = (0,1)^T$$

(A) $S = \begin{pmatrix} 1 & 2 \\ 5 & 2 \end{pmatrix}$ **(B)** $S = \begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix}$ **(C)** $S = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ **(D)** $S = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$ **(E)** $S = \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}$

36. Let $\boldsymbol{v_1} = (3,2)^T$ and $\boldsymbol{v_2} = (4,3)^T$. What is the transition matrix from $(\boldsymbol{v_1},\boldsymbol{v_2})$ to $(\boldsymbol{u_1},\boldsymbol{u_2})$, where $(\boldsymbol{u_1},\boldsymbol{u_2})$ are the vectors given in the previous problem?

(A) $S = \begin{pmatrix} 11 & -14 \\ 4 & -5 \end{pmatrix}$ **(B)** $S = \begin{pmatrix} 11 & 14 \\ -4 & -5 \end{pmatrix}$

(C) $S = \begin{pmatrix} 7 & -11 \\ -8 & 5 \end{pmatrix}$ **(D)** $S = \begin{pmatrix} -7 & 2 \\ 8 & -5 \end{pmatrix}$

(E) $S = \begin{pmatrix} 7 & 2 \\ 8 & -5 \end{pmatrix}$

37. Which of the following four statements is true?
(A) If A is an $n \times n$ matrix whose eigenvalues are all nonzero, then A is nonsingular.
(B) If A has eigenvalues of multiplicity greater than 1, then A must be defective.
(C) If A is a 4×4 matrix of rank 1 and $\lambda = 0$ is an eigenvalue of multiplicity 3, then A is defective.
(D) If A is symmetric and $\det A = 0$, then A is positive definite.

38. Find the least-squares straight-line fit $y = a + bx$ to the given points.

x	2	3	5	6
y	1	2	3	4

(A) $y = -\frac{4}{10} + \frac{5x}{10}$
(B) $y = -\frac{4}{10} + \frac{7x}{10}$
(C) $y = -\frac{3}{10} + \frac{5x}{10}$
(D) $y = -\frac{3}{10} + \frac{7x}{10}$
(E) $y = -\frac{3}{10} + \frac{9x}{10}$

39. Given two vectors $\boldsymbol{u} = (u_1,u_2)$ and $\boldsymbol{v} = (v_1,v_2)$ in $V \in \mathbb{R}^2$, suppose we define an inner product by the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$$

Determine the inner product of vectors $\boldsymbol{u} = (1,2)$ and $\boldsymbol{v} = (-2,1)$.
(A) 0
(B) 1
(C) 2
(D) 3
(E) 4

40. Matrix P below is:

$$P = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{pmatrix}$$

(A) Hermitian.
(B) Skew-Hermitian.
(C) Invertible.
(D) Invertible and unitary.

SECTION II

Instructions: Solve any 7 of the following problems.
Use of calculators is permitted.

Problem 1. Find the equation of the circle that passes through points $(-2,4)$ and $(1,3)$ and has center on the line $x - 2y + 5 = 0$.

Problem 2. Determine the general equation of the plane that contains lines r_1 and r_2.

$$r_1 : \begin{cases} y = 2x + 1 \\ z = -3x - 2 \end{cases} \; ; \; r_2 : \begin{cases} x = -1 + 2t \\ y = 4t \\ z = 3 - 6t \end{cases}$$

Problem 3. Find the point at which the line defined by the two planes π_1 and π_2 intersects the line determined by the two points $(3,-4,3)$ and $(0,-1,9)$.

$$\pi_1 : 2x - 3y + 2z - 23 = 0$$
$$\pi_2 : 4x + 4y - z + 9 = 0$$

Problem 4. Regarding the ellipse given by the following equation,

$$4x^2 + 9y^2 - 8x - 36y + 4 = 0$$

determine the coordinates of:
A. The center of the ellipse.
B. The vertices and co-vertices of the ellipse.
C. The foci of the ellipse.
Finally,
D. Determine the eccentricity of the ellipse.

Problem 5. Find the equation of the hyperbola with one vertex at $(-1,2)$, eccentricity $\sqrt{5}$, and asymptotes $2x - y + 8 = 0$ and $2x + y + 4 = 0$.

Problem 6. Consider matrix A.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix}$$

A. Find the determinant of A using the Laplace expansion theorem. Show your work.
B. Find the classical adjoint of A.
C. Use $\operatorname{adj} A$ to find the inverse matrix A^{-1}.

Problem 7. Consider the linear systems

$$\begin{cases} x+y+z=1 \\ 2x+2y+2z=4 \end{cases} ; \begin{cases} x+y+z=0 \\ 2x+2y+2z=0 \end{cases}$$

A. Show that the system to the left has no solution, and state what this tells you about the planes represented by these equations.
B. Show that the system to the right has infinitely many solutions, and state what this tells you about the planes represented by these equations.

Problem 8. Let A be a 8 × 5 matrix with rank equal to 4 and let $\boldsymbol{\zeta}$ be a vector in $\mathbb{R}^8$. The four fundamental subspaces associated with A are the row spaces $R(A)$ and $R(A^T)$ and the null spaces $N(A)$ and $N(A^T)$.
A. What is the dimension of $N(A^T)$, and which of the other fundamental subspaces is the orthogonal complement of $N(A^T)$?
B. If A is a vector in $R(A)$ and $A^T\boldsymbol{x} = 0$, then what can you conclude about the value of the norm $\|\boldsymbol{x}\|$?
C. What is the dimension of $N(A^TA)$? How many solutions will the least squares system $A\boldsymbol{x} = \boldsymbol{\zeta}$ have? Explain.

Problem 9. Use the Gram-Schmidt process to transform the given basis into an orthonormal basis in $\mathbb{R}^3$.

$$\mathbf{w}_1 = (1,1,1) \ ; \ \mathbf{w}_2 = (-1,1,0) \ ; \ \mathbf{w}_3 = (1,2,1)$$

Problem 10. Consider matrix A.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

A. Determine the characteristic polynomial of A.
B. Determine the eigenvalues of A.
C. Find two eigenvectors for A, one belonging to each of the eigenvalues you determined in Part (B).
D. Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

Problem 11. Let A be a 4 × 4 real symmetric matrix with eigenvalues

$$\lambda_1 = 1, \ \lambda_2 = \lambda_3 = \lambda_4 = 0$$

A. Explain why the multiple eigenvalue $\lambda = 0$ must have three linearly independent eigenvectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$.
B. Let $\boldsymbol{x}_1$ be an eigenvector belonging to λ_1. How is $\boldsymbol{x}_1$ related to $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, and $\boldsymbol{x}_4$? Explain.
C. Explain how to use $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, and $\boldsymbol{x}_4$ to construct an orthogonal matrix U that diagonalizes A.
D. What type of matrix is e^A? Is it symmetric? Is it positive definite? Explain your answers.

Problem 12. Consider the inner product of two vectors $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^2$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y} \quad \text{where } A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Here, the 1×1 matrix $\boldsymbol{x}^T A \boldsymbol{y}$ is interpreted as the real number which is its only entry. Consider vectors $\boldsymbol{v}$ = (1, 1) and $\boldsymbol{w}$ = (−1,2).

A. Find $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ under this inner product.

B. Find the length of vector $\boldsymbol{v}$ in the norm defined by this inner product.

C. Find the set of all vectors that are orthogonal to vector $\boldsymbol{v}$ under the inner product defined above. That is, if $S = \mathrm{Lin}(\mathbf{v})$ is the linear span of $\boldsymbol{v}$, find set $S^{\perp}$ such that, for a vector $\boldsymbol{\zeta}$,

$$S^{\perp} = \left\{ \zeta \in \mathbb{R}^2 \mid \zeta \perp \mathbf{v} \right\}$$

D. Express the vector $\boldsymbol{w}$ above as $\boldsymbol{w} = \boldsymbol{w_1} + \boldsymbol{w_2}$, where $\boldsymbol{w}_1 \in S$ and $\boldsymbol{w_2} \in S^{\perp}$.

E. Write down an orthonormal basis of $\mathbb{R}^2$ with respect to this inner product.

Problem 13. Consider matrices A and B.

$$A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \; ; \; B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

A. Show that A is positive definite and that $\boldsymbol{x}^T A \boldsymbol{x} = \boldsymbol{x}^{\boldsymbol{T}} B \boldsymbol{x}$ for all vectors $\boldsymbol{x} \in \mathbb{R}^2$.

B. Show that B is positive definite, but B^2 is not positive definite.

Problem 14. Consider the following matrix and vector.

$$A = \begin{pmatrix} 5 & 5 & -5 \\ 3 & 3 & -5 \\ 4 & 0 & -2 \end{pmatrix} \quad ; \; \zeta = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

A. Show that $\boldsymbol{\zeta}$ is an eigenvector of A and find the corresponding eigenvalue.

B. Show that $\lambda = 4 + 2i$ is an eigenvalue of A and find a corresponding eigenvector.

C. Deduce a third eigenvalue and corresponding eigenvector of A. Write down an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solutions

→ Section I

1.D. The equation of the line passing through (–4,6) and (8,8) is

$$y-6=\left(\frac{8-6}{8+4}\right)(x+4)\rightarrow y-6=\frac{1}{6}(x+4)$$

The line has slope equal to 1/6; a perpendicular line should have slope –1/(1/6) = –6 and can be stated as

$$y=mx+n\rightarrow y=-6x+n\ \text{(I)}$$

This line should pass through the midpoint P of the segment that joins (–4,6) and (8,8), namely

$$P\left(\frac{-4+8}{2},\frac{6+8}{2}\right)=(2,7)$$

Substituting (2,7) in (I) gives

$$y=-6x+n\rightarrow 7=-6\times 2+n$$

$$\therefore 7=-12+n$$

$$\therefore n=19$$

Finally,

$$y=-6x+n\rightarrow y=-6x+19$$

$$\therefore \boxed{6x+y-19=0}$$

2.A. Suppose $C(x,y)$ denotes the coordinates of vertex C and $G(\alpha,\beta)$ denotes the centroid. The values of α and β are, respectively,

$$\alpha=\frac{2-2+x}{3}=\frac{x}{3}$$

and

$$\beta=\frac{-3+1+y}{3}=\frac{y-2}{3}$$

Since G lies on $2x + 3y = 1$, it follows that $2\alpha + 3\beta = 1$, or

$$2\alpha+3\beta=1\rightarrow 2\times\left(\frac{x}{3}\right)+3\times\left(\frac{y-2}{3}\right)=1$$

Multiplying through by 3,

$$2\times\left(\frac{x}{3}\right)+3\times\left(\frac{y-2}{3}\right)=1\rightarrow 2x+3(y-2)=3$$

$$\therefore 2x+3y-6=3$$

$$\therefore \boxed{2x+3y-9=0}$$

The equation above gives the locus of vertex C.

3.E. Let p denote the length of the perpendicular from the origin to the line in question. Since the perpendicular makes an angle of 30° with the horizontal, we may write

$$x\cos\alpha + y\sin\alpha = p \rightarrow x\cos 30^{\circ} + y\sin 30^{\circ} = p$$

$$\therefore \frac{\sqrt{3}x}{2} + \frac{y}{2} = p$$

$$\therefore \sqrt{3}x + y = 2p \ \ \text{(I)}$$

or, equivalently,

$$\sqrt{3}x + y = 2p \rightarrow \frac{x}{\frac{2p}{\sqrt{3}}} + \frac{y}{2p} = 1$$

As the reader should notice, the line equation has been written in the intercept form. The x-intercept is $2p/\sqrt{3}$ and the y-intercept is $2p$. These intercepts are the sides of a rectangle triangle whose area is $50/\sqrt{3}$, as indicated in the problem statement. Thus,

$$A_{\Delta} = \frac{1}{2} \times \frac{2p}{\sqrt{3}} \times 2p = \frac{50}{\sqrt{3}} \rightarrow \frac{2p^2}{\sqrt{3}} = \frac{50}{\sqrt{3}}$$

$$\therefore p = \sqrt{\frac{50}{2}} = \pm 5$$

Substituting p in (I) gives

$$\sqrt{3}x + y = 2p \rightarrow \sqrt{3}x + y = 2 \times (\pm 5)$$

$$\sqrt{3}x + y = \pm 10$$

$$\boxed{\sqrt{3}x + y - 10 = 0} \text{ or } \boxed{\sqrt{3}x + y + 10 = 0}$$

4.E. If the intercepts of the line on the x- and y-axes are a and $9 - a$, the line can be written in the intercept form

$$\frac{x}{a} + \frac{y}{9-a} = 1$$

Since the line passes through point (2,2), we may write

$$\frac{x}{a} + \frac{y}{9-a} = 1 \rightarrow \frac{2}{a} + \frac{2}{9-a} = 1$$

$$\therefore \frac{2(9-a) + 2a}{a(9-a)} = 1$$

$$\therefore 18 - 2a + 2a = 9a - a^2$$

$$\therefore 18 = 9a - a^2$$

$$\therefore a^2 - 9a + 18 = 0$$

Solving the quadratic equation above,

$$a = \frac{-(-9) \pm \sqrt{81 - 4 \times 1 \times 18}}{2} = \frac{9 \pm \sqrt{9}}{2} = 3 \text{ or } 6$$

Substituting a = 3 in the line equation, we obtain

$$\frac{x}{a} + \frac{y}{9-a} = 1 \to \frac{x}{3} + \frac{y}{6} = 1$$

$$\therefore 6x + 3y = 18$$

$$\therefore \boxed{6x + 3y - 18 = 0}$$

5.A. Line $\alpha x + \beta y + 8 = 0$ can be written in the intercept form

$$\alpha x + \beta y + 8 \to \frac{x}{\left(\frac{-8}{\alpha}\right)} + \frac{y}{\left(\frac{-8}{\beta}\right)} = 1$$

Likewise, line $3x - 2y + 12 = 0$ can be restated as

$$3x - 2y + 12 = 0 \to 3x - 2y = -12$$

$$\therefore \frac{3x}{-12} - \frac{2y}{-12} = 1$$

$$\therefore \frac{x}{-4} + \frac{y}{-6} = 1$$

The intercepts of the first line must be equal in length but opposite in sign to those cut off by the second line; it follows that

$$-\frac{8}{\alpha} = -(-4) \to \boxed{\alpha = -2}$$

and

$$-\frac{8}{\beta} = -(-6) \to \boxed{\beta = -\frac{4}{3}}$$

6.C. Firstly, multiply the first line equation by −1 so that coefficient c becomes positive,

$$4x + 3y - 6 = 0 \to -4x - 3y + 6 = 0$$

Then, find $a_1a_2 + b_1b_2 = (-4) \times 5 + (-3) \times 12 = -56 < 0$. The negative result indicates that the origin lies in the acute angle. The equation of the bisector is then

$$\frac{-4x - 3y + 6}{\sqrt{(-4)^2 + (-3)^2}} = +\frac{5x + 12y + 9}{\sqrt{(-12)^2 + 5^2}}$$

$$\therefore \frac{-4x - 3y + 6}{5} = +\frac{5x + 12y + 9}{13}$$

$$\therefore 13(-4x - 3y + 6) = +5(5x + 12y + 9)$$

$$\therefore -52x - 39y + 78 = 25x + 60y + 45$$

$$\therefore -52x - 25x - 39y - 60y + 78 - 45 = 0$$

$$\therefore -77x - 99y + 33 = 0$$

Dividing through by −11 gives

$$-77x - 99y + 33 = 0 \rightarrow \boxed{7x + 9y - 3 = 0}$$

7.C. The center of the circle is simply the point of intersection of the two lines we were given,

$$\begin{cases} 2x + y = 3 \\ x - 3y = -2 \end{cases}$$

Multiplying the second equation by 2 and subtracting it from the first, we get

$$2x + y - 2(x - 3y) = 3 - 2(-2) \rightarrow \cancel{2x} + y - \cancel{2x} + 6y = 7$$

$$\therefore 7y = 7$$

$$\therefore y = 1$$

Substituting $y = 1$ into the second equation,

$$x - 3 \times (1) = -2 \rightarrow x = 1$$

Thus, the center of the circle is $C(1,1)$. Since the circle passes through point $P(-2,5)$, segment $\overline{CP}$ should equal the radius of the circle,

$$r = \overline{CP} = \sqrt{[1-(-2)]^2 + (1-5)^2} = \sqrt{3^2 + (-4)^2}$$

$$\therefore r = 5$$

Finally, the equation of the circle is

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \rightarrow (x-1)^2 + (y-1)^2 = 5^2$$

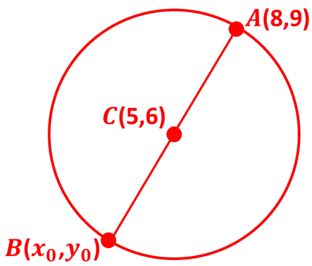
$$\therefore \boxed{(x-1)^2 + (y-1)^2 = 25}$$

8.C. We first complete squares to obtain the reduced equation that describes the circle,

$$(x^2 - 10x + \_\_) + (y^2 - 12y + \_\_) = -43$$

$$\therefore (x-5)^2 + (y-6)^2 = -43 + 25 + 36 = 18$$

Clearly, the circle is centered at $C(5,6)$. This center should be the point that bisects the segment joining $A(8,9)$ to point $B(x_0,y_0)$ at the opposite end of the diameter, as illustrated below.



The coordinates of B are

$$\left(\frac{x_0+8}{2},\frac{y_0+9}{2}\right)=(5,6)\rightarrow\frac{x_0+8}{2}=5\ ;\ \frac{y_0+9}{2}=6$$

$$\therefore x_0+8=10\ ;\ y_0+9=12$$

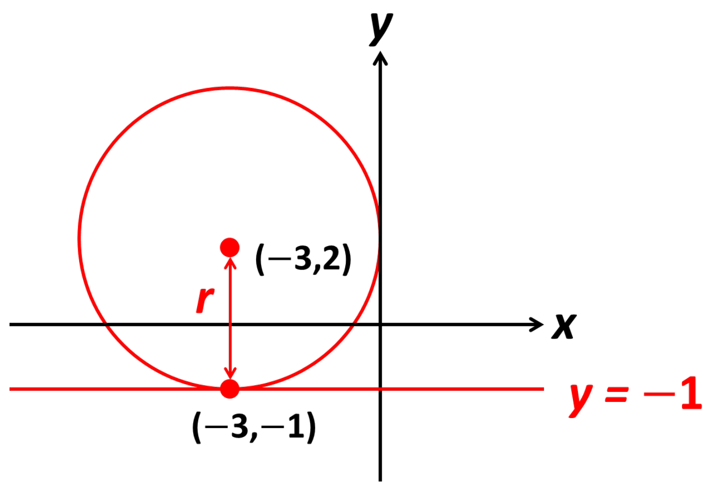
$$\therefore x_0=2\ ;\ y_0=3$$

Thus, the other end of the diameter is $B(x_0,y_0)$ = (2,3).

9.D. We already have the center of the circle, so all we need to determine is the radius. Note that tangent y = −1 is parallel to the x-axis; hence, the radius drawn to the point of contact of this tangent is parallel to the y-axis and thus lies on the line x = −3, as illustrated below. Therefore the point of tangency is (−3,−1) and the radius is 3. The required circle equation may be written as

$$(x+3)^2+(y-2)^2=9\rightarrow x^2+6x+9+y^2-4y+4=0$$

$$\therefore\boxed{x^2+y^2+6x-4y+13=0}$$



10.E. A conic section equation has the generalized form

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0\ \left(B\neq 0\right)$$

Such an equation represents:

1. A parabola (or two parallel straight lines, or no locus), if B^2-4AC = 0;
2. An ellipse (or a point, or no locus) if B^2-4AC < 0;
3. A hyperbola (or two intersecting straight lines) if B^2-4AC > 0.

We proceed to determine B^2-4AC for the present case,

$$B^2-4AC=(-7)^2-4\times3\times4=49-48=1>0$$

Since B^2-4AC > 0, the equation represents a hyperbola or two intersecting straight lines.

11.D. Since the center of the ellipse is $C(x_0$ = 4,y_0 = 2) and the major axis is parallel to y-axis, the equation of the ellipse has the form

$$\frac{(x-x_0)^2}{b^2}+\frac{(y-y_0)^2}{a^2}=0$$

We were given the minor axis length $2b = 6$, so $b = 3$. Further, from the definition of eccentricity, we have

$$e = \frac{c}{a} = \frac{1}{2} \to c = \frac{a}{2}$$

so that

$$b^2 + c^2 = a^2 \to 3^2 + \left(\frac{a}{2}\right)^2 = a^2$$

$$\therefore 9 + \frac{a^2}{4} = a^2$$

$$\therefore \frac{3a^2}{4} = 9$$

$$\therefore a^2 = 12$$

Substituting in the initial equation yields

$$\frac{(x-x_0)^2}{b^2} + \frac{(y-y_0)^2}{a^2} = 0 \to \frac{(x-4)^2}{3^2} + \frac{(y-2)^2}{12} = 0$$

$$\therefore \boxed{\frac{(x-4)^2}{9} + \frac{(y-2)^2}{12} = 0}$$

12.D. The given vertex V and focus F lie on a line parallel to the y-axis, and their distance equals 2; in mathematical terms,

$$a - c = 2 \text{ (I)}$$

Since the eccentricity equals 1/2, we write $c = a/2$ and substitute in (I), giving

$$a - c = 2 \to a - \frac{a}{2} = 2$$

$$\therefore \frac{a}{2} = 2$$

$$\therefore a = 4$$

Substituting a in (I) gives $c = 2$. Applying the Pythagorean theorem yields the semi-minor axis length b,

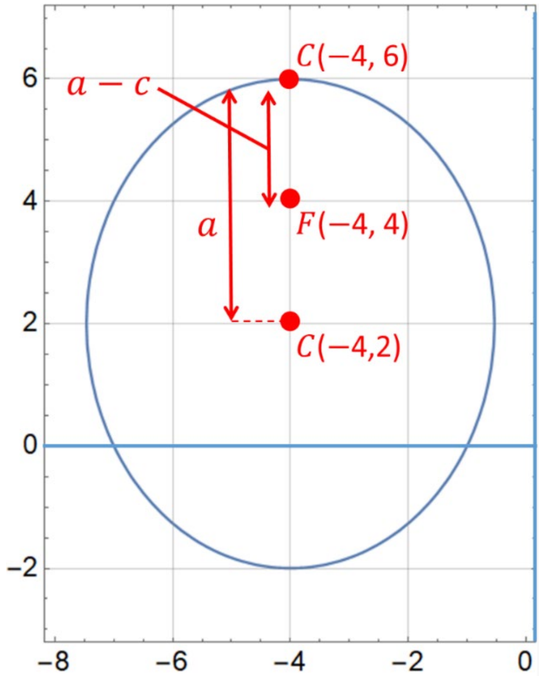
$$a^2 = b^2 + c^2 \to b^2 = a^2 - c^2$$

$$\therefore b^2 = 4^2 - 2^2 = 12$$

$$\therefore b = \sqrt{12}$$

The last step is to locate the center of the ellipse. We know it lies on the line $x = -4$ and at a distance of 4 units from the vertex, in the direction of the focus. It follows that the center is at $C(-4,2)$ and the required equation is

$$\boxed{\frac{(x+4)^2}{12}+\frac{(y-2)^2}{16}=1}$$



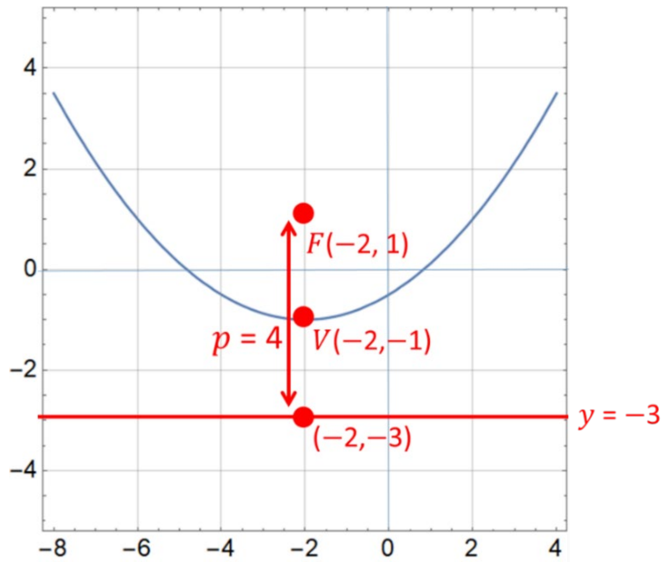
13.C. Since the vertex is midway between (−2,3) and y = −3, its coordinates are those of the midpoint of the segment whose endpoints are (−2,1) and (−2,−3); hence the vertex is at (−2,−1). The distance from the directrix to the focus is p = 4. Thus the required equation is

$$(x-h)^2=2p(y-k)\to\left[x-(-2)\right]^2=2\times4\left[y-(-1)\right]$$

$$\therefore(x+2)^2=8(y+1)$$

$$\therefore x^2+4x+4=8y+8$$

$$\therefore\boxed{x^2+4x-8y-4=0}$$



14.C. Let $P(x,y)$ be a point on the hyperbola, $F(2,0)$ be the focus, and M be the point of intersection of a perpendicular to the directrix and the hyperbola. By definition,

$$\overline{FP}^2 = e^2\overline{PM}^2 \to (x-2)^2 + (y-0)^2 = 2^2 \times \left(\frac{x-y}{\sqrt{2}}\right)^2$$

$$\therefore (x-2)^2 + y^2 = 4\times\left(\frac{x-y}{\sqrt{2}}\right)^2$$

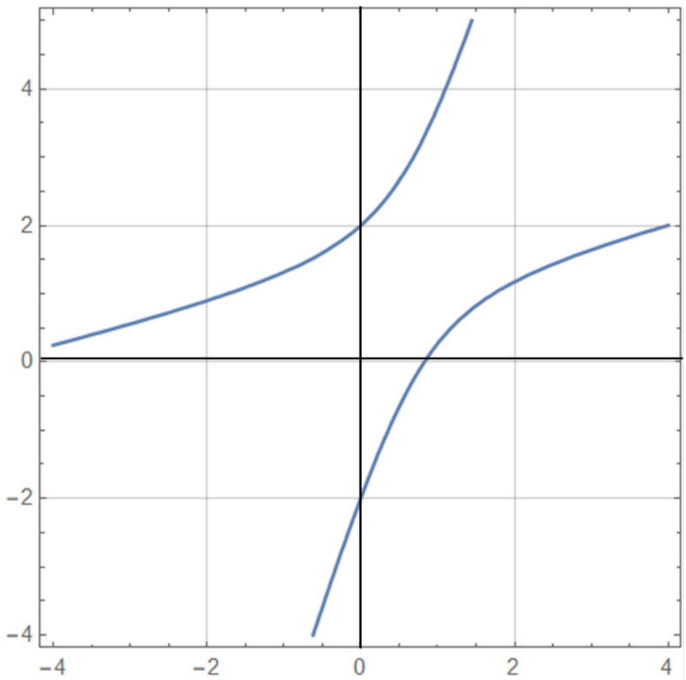
$$\therefore x^2 + y^2 - 4x + 4 = 4\left(\frac{x^2-2xy+y^2}{2}\right)$$

$$\therefore x^2 + y^2 - 4x + 4 = 2x^2 - 4xy + 2y^2$$

$$\therefore 2x^2 - x^2 + 2y^2 - y^2 - 4xy + 4x - 4 = 0$$

$$\therefore \boxed{x^2 + y^2 - 4xy + 4x - 4 = 0}$$

The hyperbola is plotted below.



15.A. The direction vectors of r is $\boldsymbol{u}$ = (1,k,−2), while the direction vector of s is $\boldsymbol{v}$ = (2,−1,5). The two lines will be orthogonal if the dot product of their direction vectors equals zero.

$$\mathbf{u}\bullet\mathbf{v} = (1,k,-2)\bullet(2,-1,5) = 1\times2 - 1\times k - 2\times5 = 0$$

$$\therefore 2 - k - 10 = 0$$

$$\therefore \boxed{k=-8}$$

16.A. Equating $y = -3x + 2$ and $y = 1 + 2t$, we have

$$-3x + 2 = 1 + 2t \to -3\times(-t) + 2 = 1 + 2t$$

$$\therefore 3t + 2 = 1 + 2t$$

$$\therefore t=-1$$

Substituting t in the equations that define r_2, we get

$$\begin{cases} x=-t \\ y=1+2t \\ z=-2t \end{cases} \rightarrow \begin{cases} x=-(-1)=1 \\ y=1+2\times(-1)=-1 \\ z=-2\times(-1)=2 \end{cases}$$

Thus, the lines intersect at (1,−1,2). Substituting these coordinates in the equations of r_2, we can confirm that the line also passes through this point,

$$\begin{cases} y=-3x+2 \\ z=3x-1 \end{cases} \rightarrow \begin{cases} -1=-3\times(1)+2=-1 \\ 2=3\times1-1=2 \end{cases}$$

The equalities are true, as expected.

17.A. Two planes are parallel if their normal vectors are parallel. In the present case, the normal vectors are $\boldsymbol{v_1}$ = $(2\alpha-1, -2, \beta)$ and $\boldsymbol{v_2}$ = (4,4,−1). From the condition of parallelism, we write

$$\frac{2\alpha-1}{4}=\frac{-2}{4}=\frac{\beta}{-1}$$

so that

$$\begin{cases} 2\alpha-1=-2 \\ -\dfrac{2}{4}=-\beta \end{cases} \rightarrow \boxed{\alpha=-\frac{1}{2},\beta=\frac{1}{2}}$$

18.D. The distance from a point $P_0(x_0,y_0,z_0)$ to a plane $ax+by+cz+d$ = 0 is given by the general formula

$$d(P_0,\pi)=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

For the line and point given,

$$d(P_0,\pi)=\frac{|2\times(-4)+1\times2+2\times5+8|}{\sqrt{2^2+1^2+2^2}}=\boxed{4}$$

19.E. Let us apply the inversion algorithm.

$$A=\left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right)$$

Add −2 times the first row to the second row, then add the first row to the third row.

$$\left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2-2\times 1 & 4-2\times 6 & -1-2\times 4 & 0-2\times 1 & 1 & 0 \\ -1+1 & 2+6 & 5+4 & 0+1 & 0 & 1 \end{array}\right)$$

$$\therefore \left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right)$$

Add the second row to the third row,

$$\therefore \left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right)$$

A row of zeros has been found on the left-hand side; accordingly, matrix A is singular.

20.C. It is easy to see that, if $c = 0$, the first row will be a row of zeros and the matrix will not be invertible. If $c \neq 0$, then after multiplying the first row by $1/c$, we have

$$\begin{pmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix} \to \begin{pmatrix} \frac{c}{c} & \frac{c}{c} & \frac{c}{c} \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 1 & 1 \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix}$$

Then, add −1 times the first row to the second row and to the third row.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & c & c \\ 1 & 1 & c \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & c-1 & c-1 \\ 0 & 0 & c-1 \end{pmatrix}$$

If $c = 1$, then the second and third rows are rows of zeros, and so the matrix is not invertible. If $c \neq 1$, we can divide the second and third rows by $c - 1$, obtaining

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & c-1 & c-1 \\ 0 & 0 & c-1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and from this it is clear that the reduced row echelon form is the identity matrix. Thus, we conclude that the matrix is invertible if and only if $c \neq 0$ and $c \neq 1$. The matrix is invertible for $c = 2$.

21.B. Statement A is false. A two-dimensional subspace of $\mathbb{R}^3$ is a plane through the origin in 3-space. If S and T are two different two-dimensional subspaces of $\mathbb{R}^3$ then both correspond to planes through the origin, and their intersection must be a line through the origin. Thus the intersection cannot consist of just the zero vector.

Statement B is true. If

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_k\mathbf{x}_k + c_{k+1}\mathbf{x}_{k+1} = 0$$

then since $\boldsymbol{x}_{k+1}$ is not in $\mathrm{Span}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k)$, the scalar c_{k+1} must equal zero, otherwise we could solve for $\boldsymbol{x}_{k+1}$ as a linear combination of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$. As a result, the equation above simplifies to

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_k\mathbf{x}_k = 0$$

and the linear independence of $x_1,\ldots,x_k$ implies that

$$c_1 = c_2 = \ldots = c_k = 0$$

So all the scalars c_1, c_2, ..., c_{k+1} must be 0 and hence $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k, \boldsymbol{x}_{k+1}$ must be linearly independent.

Statement C is false. If $m \neq n$, the statement is false since

$$\dim N(A) = n - r$$

$$\dim N\left(A^T\right) = m - r$$

where r is the rank of A.

Statement D is false. To determine the transition matrix, let $\boldsymbol{x}$ be any vector in $\mathbb{R}^2$. If

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 = e_1\mathbf{w}_1 + e_2\mathbf{w}_2$$

then since X is the transition matrix corresponding to the change of basis from ($\boldsymbol{u}_1$,$\boldsymbol{u}_2$) to ($\boldsymbol{v}_1$,$\boldsymbol{v}_2$) and Y is the transition matrix corresponding to the change of basis from ($\boldsymbol{w}_1$,$\boldsymbol{w}_2$) to ($\boldsymbol{v}_1$,$\boldsymbol{v}_2$), we have

$$\mathbf{d} = X\mathbf{c}$$

and

$$\mathbf{e} = Y\mathbf{d}$$

It follows that

$$\mathbf{e} = Y\mathbf{d} = Y\left(X\mathbf{c}\right) = \left(YX\right)\mathbf{c}$$

and hence YX is the transition matrix for the change of basis from ($\boldsymbol{u}_1$,$\boldsymbol{u}_2$) to ($\boldsymbol{w}_1$,$\boldsymbol{w}_2$). In general YX is not equal to XY, which means that if X and Y do not commute, then $Z = XY$ will not be the desired transition matrix.

22.E. Suppose V denotes the matrix that has ($\boldsymbol{v}_1$,$\boldsymbol{v}_2$) as columns and W denotes the matrix that has ($\boldsymbol{w}_1$,$\boldsymbol{w}_2$) as columns. The transition matrix S from ($\boldsymbol{w}_1$,$\boldsymbol{w}_2$) to ($\boldsymbol{v}_1$,$\boldsymbol{v}_2$) is such that

$$S = V^{-1}W$$

This can be adjusted to yield

$$S = V^{-1}W \to W = VS$$

so that

$$W = VS \to W = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix}$$

$$\therefore W = \begin{pmatrix} 1\times3+2\times1 & 1\times5-2\times2 \\ 2\times3+3\times1 & 2\times5-2\times3 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 9 & 4 \end{pmatrix}$$

Thus, $\boldsymbol{w_1}$ = (5,9) and $\boldsymbol{w_2}$ = (1,4).

23.C. According to the dimension theorem for matrices, if A is a $m \times n$ matrix, then $rank(A)$ + $nullity(A)$ = n. In the present case, $nullity(A)$ = 2, n = 5, and

$$\text{rank}(A) + \text{nullity}(A) = n \to \text{rank}(A) = n - \text{nullity}(A)$$

$$\therefore \text{rank}(A) = 5 - 2 = \boxed{3}$$

24.D. Statement A is false. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are unit vectors and θ is the angle between the two vectors, then the condition $|\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{y}|$ = 1 implies that $\cos\theta$ = 1, giving $\boldsymbol{y} = \boldsymbol{x}$ or $\boldsymbol{y} = -\boldsymbol{x}$. It follows that x and y are linearly dependent.

Statement B is false. For the basis ($\boldsymbol{e_1}$,$\boldsymbol{e_2}$,$\boldsymbol{e_3}$) of Cartesian unit vectors, for example, we may write

$$U = \text{Span}(\mathbf{e}_1) \ ; \ V = \text{Span}(\mathbf{e}_3) \ ; \ W = \text{Span}(\mathbf{e}_1 + \mathbf{e}_2)$$

Since $\boldsymbol{e_1} \perp \boldsymbol{e_3}$ and $\boldsymbol{e_3} \perp (\boldsymbol{e_1} + \boldsymbol{e_2})$, it follows that $U \perp V$ and $V \perp W$, but $\boldsymbol{e_1}$ is not orthogonal to $\boldsymbol{e_1} + \boldsymbol{e_2}$, so U and W are not orthogonal subspaces.

Statement C is false. AA^T and A^TA necessarily have the same rank.

Statement D is true. In general an $n \times n$ matrix Q is orthogonal if and only if $Q^TQ = I$. If Q_1 and Q_2 are both $n \times n$ orthogonal matrices, then

$$(Q_1Q_2)^T(Q_1Q_2) = Q_1^T\underbrace{Q_2^TQ_1}_{=I}Q_2 = Q_1^TIQ_2 = Q_2^TQ_2 = I$$

Therefore Q_1Q_2 is an orthogonal matrix.

25.A. The characteristic polynomial of a third-order square matrix is given by

$$\Delta(t) = t^3 - (a_{11} + a_{22} + a_{33})t^2 + \left(\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right)t - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Thus, the term of power t is given by

$$\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} = (8-7) + (2-6) + (4-10) = 1 - 4 - 6 = \boxed{-9}$$

26.E. For a matrix to have $\lambda_1 = 4$ and $\lambda_2 = 5$ as eigenvalues, its trace must equal $\lambda_1 + \lambda_2 = 9$ and its determinant must equal $\lambda_1\lambda_2 = 4 \times 5 = 20$. Matrix (E) obeys both conditions.

27.B. For $\boldsymbol{v}$ to be an eigenvector of A, we must have $A\boldsymbol{v} = \lambda\boldsymbol{v}$, where λ is an eigenvalue of A; that is,

$$A\mathbf{v} = \begin{pmatrix} 6 & 13 & -8 \\ 2 & 5 & -2 \\ 7 & 17 & -9 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6\times1+13\times0-8\times1 \\ 2\times1+5\times0-2\times1 \\ 7\times1+17\times0-9\times1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Thus, $\lambda = -2$ is an eigenvalue of A.

28.C. The equation for the quadratic form at hand is repeated below.

$$f(x,y,z) = 6xy - 4yz + 2xz - 4x^2 - 2y^2 - 4z^2$$

The matrix F that represents this quadratic form is given in continuation. The entries are shown with colors corresponding to the equation above.

$$F = \begin{pmatrix} -4 & 3 & 1 \\ 3 & -2 & -2 \\ 1 & -2 & -4 \end{pmatrix}$$

The first two principal minors are

$$a_{11} = -4 \ ; \ \begin{vmatrix} -4 & 3 \\ 3 & -2 \end{vmatrix} = -4\times(-2) - 3\times3 = -1$$

The first and second principal minors are both negative. For the matrix (and, by extension, the quadratic form) to be positive definite, both should be positive. This requirement is not fulfilled. If the matrix and quadratic form were negative definite, the first PM should be negative and the second positive. Likewise, this requirement is not fulfilled. So the quadratic form is neither. If F is neither positive nor negative definite and $|A| \neq 0$ (the determinant is $|A| = 10$), then A has both positive and negative eigenvalues and is therefore indefinite.

29.C. This matrix consists of Jordan blocks, i.e. square blocks in which the diagonal elements are repetitions of the same eigenvalue λ, the superdiagonal has only 1s, and all other elements are zeros. The characteristic polynomial of the matrix is then

$$\boxed{\Delta(t) = (t-2)^3(t-6)^5}$$

The exponent 3 comes from the fact that eigenvalue $\lambda_1 = 2$ occurs three times, while exponent 5 is attributable to the fact that eigenvalue $\lambda_2 = 6$ occurs five times.

30.B. Statement B is true. Strang's *Linear Algebra* approaches the other statements using counterexamples. For instance, matrix A below is both real and unitary, but $(A - iI)$ is not invertible.

$$A = U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Further, Strang mentions that cA is still Hermitian for *real* c; if $c = i$, then

$$(iA)^H = -iA^H = -iA$$

and the ensuing matrix is skew-Hermitian.

31.C. The augmented matrix of the system is

$$\begin{pmatrix} 1 & 1 & 7 & -7 \\ 2 & 3 & 17 & -16 \\ 1 & 2 & a^2+1 & 3a \end{pmatrix}$$

This reduces to

$$\begin{pmatrix} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & a^2-9 & 3a+9 \end{pmatrix}$$

The last row corresponds to $(a^2 - 9)z = 3a + 9$. If $a = -3$ this becomes 0 = 0 and the system will have infinitely many solutions. If $a = 3$, then the last row corresponds to 0 = 18 and the system will be inconsistent. If $a \neq \pm 3$, then

$$\left(a^2 - 9\right)z = 3a + 9 \rightarrow z = \frac{3(a+3)}{a^2 - 9}$$

$$\therefore z = \frac{3\cancel{(a+3)}}{\cancel{(a+3)}(a-3)} = \frac{3}{a-3}$$

and, from back substitution, y and z will be uniquely determined as well; the system has exactly one solution.

32.B. Simply compute the cross product with your calculator; you should find $\boldsymbol{v_1} \times \boldsymbol{v_2}$ = (1,1,2) × (1,2,3) = (−1,−1,1).

33.C. The base vectors of the plane are $\boldsymbol{AB}$ = (0−2, −1−1,1−(−1)) = (−2,−2,2) and $\boldsymbol{AC}$ = (1−2,2−1,1−(−1)) = (−1,1,2). The normal vector that defines the plane is then

$$\mathbf{n} = \mathbf{AB} \times \mathbf{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -2 & 2 \\ -1 & 1 & 2 \end{vmatrix} = (-6, 2, -4)$$

Further, the plane should pass through point $C(1,2,1)$. The equation that defines the plane is determined to be

$$-6\times(x-1)+2\times(y-2)-4\times(z-1)=0$$
$$\therefore -6x+6+2y-4-4z+4=0$$
$$\therefore -6x+2y-4z+6=0$$

Dividing through by 2,

$$\boxed{-3x+y-2z+3=0}$$

34.C. The normal vectors to planes π_1 and π_2 are $\boldsymbol{n_1}$ = (2,−3,5) and $\boldsymbol{n_2}$ = (3,2,5), respectively. The angle between the planes is then

$$\cos\theta=\frac{\langle \mathbf{n_1},\mathbf{n}_2\rangle}{\|\mathbf{n_1}\|\|\mathbf{n}_2\|}\rightarrow\cos\theta=\frac{2\times3-3\times2+5\times5}{\sqrt{2^2+(-3)^2+5^2}\times\sqrt{3^2+2^2+5^2}}$$
$$\therefore\cos\theta=\frac{25}{\sqrt{38}\times\sqrt{38}}=\frac{25}{38}$$
$$\therefore\theta=\cos^{-1}\frac{25}{38}=\boxed{48.9^\circ}$$

35.C. The solution is started by expressing $\boldsymbol{u_1}$ as a linear combination of $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$,

$$\begin{pmatrix}1\\2\end{pmatrix}=a_1\begin{pmatrix}1\\0\end{pmatrix}+b_1\begin{pmatrix}0\\1\end{pmatrix}\rightarrow\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}a_1\\b_1\end{pmatrix}$$

so a_1 = 1, b_1 = 2, and the vector in question when expressed in the ($\boldsymbol{e_1}$,$\boldsymbol{e_2}$) basis is

$$(\mathbf{u}_1)_{(\mathbf{e}_1,\mathbf{e}_2)}=1\mathbf{e}_1+2\mathbf{e}_2$$

Proceeding similarly with $\boldsymbol{u_2}$, we have

$$\begin{pmatrix}2\\5\end{pmatrix}=a_2\begin{pmatrix}1\\0\end{pmatrix}+b_2\begin{pmatrix}0\\1\end{pmatrix}\rightarrow\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}a_2\\b_2\end{pmatrix}$$

so a_2 = 2, b_2 = 5, and the vector in question when expressed in the ($\boldsymbol{e_1}$,$\boldsymbol{e_2}$) basis is

$$(\mathbf{u}_2)_{(\mathbf{e}_1,\mathbf{e}_2)}=2\mathbf{e}_1+5\mathbf{e}_2$$

Next, we expand the sum $m_1\boldsymbol{u_1}+m_2\boldsymbol{u_2}$,

$$m_1\mathbf{u}_1+m_2\mathbf{u}_2\rightarrow m_1(1\mathbf{e}_1+2\mathbf{e}_2)+m_2(2\mathbf{e}_1+5\mathbf{e}_2)$$
$$\therefore m_1\mathbf{e}_1+2m_1\mathbf{e}_2+2m_2\mathbf{e}_1+5m_2\mathbf{e}_2$$
$$\therefore(m_1+2m_2)\mathbf{e}_1+(2m_1+5m_2)\mathbf{e}_2$$

Thus, the coordinate vector of $m_1\boldsymbol{u_1}+m_2\boldsymbol{u_2}$ with respect to ($\boldsymbol{e_1}$,$\boldsymbol{e_2}$) is

$$\mathbf{x}=\begin{pmatrix}m_1+2m_2\\2m_1+5m_2\end{pmatrix}\rightarrow\mathbf{x}=\begin{pmatrix}1&2\\2&5\end{pmatrix}\begin{pmatrix}m_1\\m_2\end{pmatrix}$$

where projection matrix P is

$$\boxed{S=\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}}$$

36.B. Let S denote the projection matrix we are looking for. If U is the matrix assembled by vectors ($\boldsymbol{u_1}$,$\boldsymbol{u_2}$) and V is the matrix assembled by vectors ($\boldsymbol{v_1}$,$\boldsymbol{v_2}$), we may write

$$S=U^{-1}V=\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}^{-1}\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}$$

To obtain U^{-1} without a calculator, simply apply the relation

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

In the present case,

$$U^{-1}=\begin{pmatrix}5 & -2\\ -2 & 1\end{pmatrix}$$

and, finally,

$$S=\begin{pmatrix}5 & -2\\ -2 & 1\end{pmatrix}\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}=\begin{pmatrix}5\times3-2\times2 & 5\times4-2\times3\\ -2\times3+1\times2 & -2\times4+1\times3\end{pmatrix}=\boxed{\begin{pmatrix}11 & 14\\ -4 & -5\end{pmatrix}}$$

37.A. Statement A is true. If A were singular then we'd have

$$\det\left(A-0I\right)=\det A=0$$

so λ = 0 would have to be an eigenvalue. Therefore if all of the eigenvalues are nonzero, then A cannot be singular. One could also show that the statement is true by noting that if the eigenvalues of A are nonzero, then

$$\det A=\lambda_1\lambda_2\ldots\lambda_n\neq 0$$

and therefore A must be nonsingular.

Statement B is false. The 2 × 2 identity matrix has eigenvalues λ_1 = λ_2 = 1, but it is not defective.

Statement C is false. If A is a 4 × 4 matrix of rank 1, then the nullity of A is 3. Since λ = 3 is an eigenvalue of multiplicity 3 and the dimension of the eigenspace is also 3, the matrix is diagonalizable.

Statement D is false. For example, let

$$A=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\ ;\ \mathbf{x}=\begin{pmatrix}1\\ 1\end{pmatrix}$$

Although $\det A$ > 0, the matrix is not positive definite since $\boldsymbol{x}^T A\boldsymbol{x}$ = −2.

38.D. The linear model for the given data is $M\boldsymbol{v}=\boldsymbol{y}$, where

$$M = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} \; ; \; y = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

The least squares solution is obtained by solving the system $M^T M\boldsymbol{v} = M^T\boldsymbol{y}$, which is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 4 & 16 \\ 16 & 74 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 47 \end{pmatrix}$$

Since the matrix on the left-hand side is nonsingular, the system has a unique solution given by

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4 & 16 \\ 16 & 74 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 47 \end{pmatrix} = \begin{pmatrix} \frac{37}{20} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 10 \\ 47 \end{pmatrix} = \begin{pmatrix} -\frac{3}{10} \\ \frac{7}{10} \end{pmatrix}$$

Thus the least-squares line fit to the given data is $y = -3/10 + 7x/10$.

39.C. All we have to do is substitute $\boldsymbol{u}$ and $\boldsymbol{v}$ into the expression that defines the IP,

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2 \times 1 \times (-2) + 3 \times 2 \times 1 = -4 + 6 = \boxed{2}$$

40.D. Since $a_{iJ} \neq \bar{a}_{ji}$ for at least one entry, the matrix is not Hermitian. To establish if it is invertible or not, simply type the matrix in your calculator and use its built-in inversion function; you should find that there is a P^H such that

$$P^H = \begin{pmatrix} 0 & 0 & -i \\ -i & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}$$

This leaves us with either option (C) or option (D). To check whether P is unitary, compute PP^H,

$$PP^H = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -i \\ -i & 0 & 0 \\ 0 & -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Since $PP^H = I$, P is unitary. Option D is correct.

Answer Summary

1	D	21	B
2	A	22	E
3	E	23	C
4	E	24	D
5	A	25	A
6	C	26	E
7	C	27	B
8	C	28	C
9	D	29	C
10	E	30	B
11	D	31	C
12	D	32	B
13	C	33	C
14	D	34	C
15	A	35	C
16	A	36	B
17	A	37	A
18	D	38	D
19	E	39	C
20	C	40	D

→ Section II

Problem 1

Let the center be denoted by coordinates $C(h,k)$. For the center to be on the given line, we must have

$$x-2y+5=0 \rightarrow h-2k+5=0$$

$$\therefore h-2k=-5 \text{ (I)}$$

Further, the center is equally distant from each of the given points,

$$\sqrt{(h+2)^2+(k-4)^2}=\sqrt{(h-1)^2+(k-3)^2}$$

$$\therefore (h+2)^2+(k-4)^2=(h-1)^2+(k-3)^2$$

Expanding the squares and collecting terms,

$$\cancel{h^2}+4h+4+\cancel{k^2}-8k+16=\cancel{h^2}-2h+1+\cancel{k^2}-6k+9$$

$$\therefore 4h-8k+20=-2h-6k+10$$

$$\therefore 6h-2k=-10 \text{ (II)}$$

(I) and (II) are a system of linear equations,

$$\begin{cases} h-2k=-5 \text{ (I)} \\ 6h-2k=-10 \text{ (II)} \end{cases}$$

Subtracting (II) from (I) gives

$$h-2k-(6h-2k)=-5-(-10)$$

$$\therefore h-\cancel{2k}-6h+\cancel{2k}=-5+10$$

$$\therefore -5h=5$$

$$\therefore h=-1$$

Then, substituting h in (I),

$$h-2k=-5 \to -1-2k=-5$$

$$\therefore -2k=-4$$

$$\therefore k=2$$

Thus, the center of the circle is $C(-1,2)$. To find the radius, we simply determine the distance from C to either of the two points we were given,

$$r=\sqrt{\left[-1-(-2)\right]^2+(2-4)^2}=\sqrt{5}$$

Finally, the equation of the circle is determined to be

$$(x+1)^2+(y-2)^2=\left(\sqrt{5}\right)^2 \to x^2+2x+1+y^2-4y+4=5$$

$$\therefore \boxed{x^2+y^2+2x-4y=0}$$

Problem 2

Line r_1 passes through point $A_1(0,1,-2)$ and has the direction of vector $\boldsymbol{v_1}$ = (1,2,−3), while line r_2 passes through point $A_2(-1,0,3)$ and has the direction of vector $\boldsymbol{v_2}$ = (2,4,−6). Since $\boldsymbol{v_2}$ = 2$\boldsymbol{v_1}$, the lines are parallel. Let the base vectors be $\boldsymbol{v_1}$ and $\boldsymbol{A_1A_2}$ = (−1−0, 0−1, 3−(−2)) = (−1,−1,5). Given fixed point A_1 and generic point $P(x,y,z)$, the following mixed product must equal zero,

$$\left(\mathbf{A_1P},\mathbf{v_1},\mathbf{A_1A_2}\right)=0 \to \begin{vmatrix} x-0 & y-1 & z-(-2) \\ 1 & 2 & -3 \\ -1 & -1 & 5 \end{vmatrix}=0$$

$$\therefore \begin{vmatrix} x & y-1 & z+2 \\ 1 & 2 & -3 \\ -1 & -1 & 5 \end{vmatrix}=0$$

$$\therefore \boxed{7x-2y+z+4=0}$$

Problem 3

To begin, the line that joins the two points is given by the set of parametric equations

$$\begin{cases} x = 3 + 3t \\ y = -4 - 3t \quad \text{(a)} \\ z = 3 - 6t \end{cases}$$

To find the line that defines the intersection of the two planes, we eliminate x from the equation that defines π_1,

$$2x - 3y + 2z - 23 = 0 \rightarrow x = \frac{3y - 2z + 23}{2}$$

and substitute in the equation for π_2,

$$4x + 4y - z + 9 = 0 \rightarrow 4\left(\frac{3y - 2z + 23}{2}\right) + 4y - z + 9 = 0$$

$$\therefore 6y - 4z + 46 + 4y - z + 9 = 0$$

$$\therefore 10y - 5z + 55 = 0$$

$$\therefore 2y - z + 11 = 0 \quad \text{(I)}$$

Next, let us isolate y in the equation for π_1,

$$2x - 3y + 2z - 23 = 0 \rightarrow y = \frac{2x + 2z - 23}{3}$$

and substitute in the equation for π_2,

$$4x + 4y - z + 9 = 0 \rightarrow 4x + 4\left(\frac{2x + 2z - 23}{3}\right) - z + 9 = 0$$

$$\therefore \frac{12x}{3} + 4\left(\frac{2x + 2z - 23}{3}\right) - \frac{3z}{3} + \frac{27}{3} = 0$$

$$\therefore 12x + 8x + 8z - 92 - 3z + 27 = 0$$

$$\therefore 20x + 5z - 65 = 0$$

$$\therefore 4x + z - 13 = 0 \text{ (II)}$$

Solving (I) and (II) for z, we get

$$2y - z + 11 = 0 \rightarrow z = 2y + 11$$

and

$$4x + z - 13 = 0 \rightarrow z = -4x + 13$$

Then, we may write

$$2y + 11 = -4x + 13 = z$$

Subtracting 11 from the three expressions,

$$2y = -4x + 2 = z - 11$$

Dividing the three expressions by 4,

$$\frac{y}{2} = -x + \frac{1}{2} = \frac{z - 11}{4}$$

$$\therefore \frac{y}{2}=\frac{x-\frac{1}{2}}{-1}=\frac{z-11}{4}$$

It would be more convenient to have the line represented parametrically, and the symmetric form equations devised above provide the details needed for this purpose. The parametric equations are

$$\begin{cases} x=\frac{1}{2}-t \\ y=2t \qquad \text{(b)} \\ z=11+4t \end{cases}$$

For the two lines to intersect, there should be some t_1 in equations (a) that corresponds to a t_2 in equations (b); that is,

$$\begin{cases} 3+3t_1=\frac{1}{2}-t_2 \\ -4-3t_1=2t_2 \\ 3-6t_1=11+4t_2 \end{cases}$$

Adding the first equation to the second gives

$$3+3t_1+\left(-4-3t_1\right)=\frac{1}{2}-t_2+\left(2t_2\right)\rightarrow -1=\frac{1}{2}+t_2$$

$$\therefore t_2=-\frac{3}{2}$$

Substituting t_2 in the second equation,

$$-4-3t_1=2\times\left(-\frac{3}{2}\right)\rightarrow -4-3t_1=-3$$

$$\therefore -3t_1=1$$

$$\therefore t_1=-\frac{1}{3}$$

These values of t_1 and t_2 must satisfy the third equation, too; indeed,

$$3-6t_1=11+4t_2\rightarrow 3-6\times\left(-\frac{1}{3}\right)=11+4\times\left(-\frac{3}{2}\right)$$

$$\therefore 3+2=11-6$$

$$\therefore 5=5$$

The equality checks. Substituting t_1 into equations (a) yields

$$x=3+3\times\left(-\frac{1}{3}\right)=2$$

$$y = -4 - 3 \times \left(-\frac{1}{3}\right) = -3$$

$$z = 3 - 6 \times \left(-\frac{1}{3}\right) = 5$$

The two lines intersect at point (2,−3,5).

Problem 4

First, we group the terms containing x and y and complete the squares,

$$(4x^2 - 8x + \_\_) + (9y^2 - 36y + \_\_) = -4$$

$$\therefore 4(x^2 - 2x + \_\_) + 9(y^2 - 4y + \_\_) = -4$$

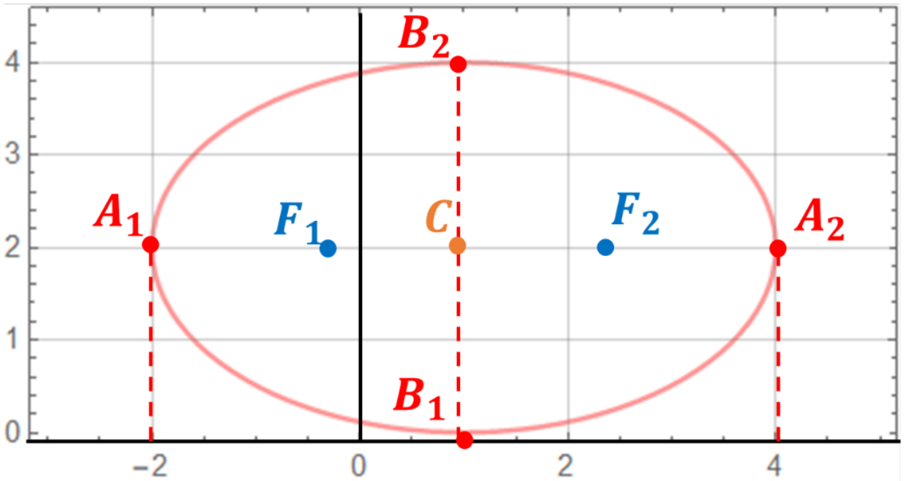
$$\therefore 4(x^2 - 2x + 1) + 9(y^2 - 4y + 4) = -4 + 4 + 36$$

$$\therefore 4(x-1)^2 + 9(y-2)^2 = 36$$

Then, we divide through by 36,

$$4(x-1)^2 + 9(y-2)^2 = 36 \rightarrow \frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1$$

Clearly, the center of the ellipse is $C(1,2)$. The major and minor semiaxes have lengths $a = \sqrt{9} = 3$ and $b = \sqrt{4} = 2$, respectively. A plot of the ellipse should help in finding the vertices and co-vertices.



The coordinates of vertices A_1 and A_2, as well as the co-vertices B_1 and B_2, are listed below.

A_1	(−2,2)
A_2	(4,2)
B_1	(1,0)
B_2	(1,4)

We proceed to determine focal distance c,

$$a^2 = b^2 + c^2 \rightarrow 3^2 = 2^2 + c^2$$

$$\therefore 9 = 4 + c^2$$

$$\therefore c = \sqrt{5}$$

Thus, the foci are $F_1(1-\sqrt{5}, 2)$ and $F_2(1+\sqrt{5}, 2)$. The final step is to determine eccentricity e,

$$e = \frac{c}{a} = \boxed{\frac{\sqrt{5}}{3}}$$

Problem 5

Let $P(x,y)$ be a point on the hyperbola, $F(1,2)$ be the focus, and M be the point of intersection of a perpendicular to the directrix and the hyperbola. By definition,

$$\overline{FP}^2 = e^2\overline{PM}^2 \rightarrow (x-1)^2 + (y-2)^2 = \left(\sqrt{3}\right)^2 \times \left(\frac{2x+y-1}{\sqrt{1^2+2^2}}\right)^2$$

$$\therefore x^2 - 2x + 1 + y^2 - 4y + 4 = 3 \times \left(\frac{2x+y-1}{\sqrt{5}}\right)^2$$

$$\therefore x^2 - 2x + 1 + y^2 - 4y + 4 = \frac{3}{5} \times \left(4x^2 + y^2 + 4xy - 4x - 2y + 1\right)$$

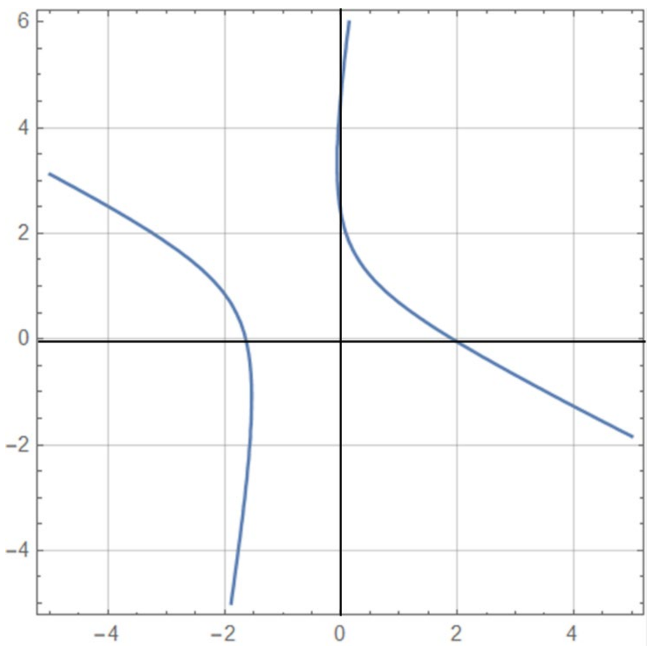
$$\therefore 5\left(x^2 - 2x + 1 + y^2 - 4y + 4\right) = 3\left(4x^2 + y^2 + 4xy - 4x - 2y + 1\right)$$

$$\therefore 5x^2 - 10x + 5 + 5y^2 - 20y + 20 = 12x^2 + 3y^2 + 12xy - 12x - 6y + 3$$

$$\therefore 12x^2 - 5x^2 + 3y^2 - 5y^2 + 12xy - 12x + 10x - 6y + 20y + 3 - 25$$

$$\therefore \boxed{7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0}$$

The hyperbola is plotted below.



Problem 6

(A) Let us expand along the third column.

$$\det A = (-1)^{1+3} a_{13} \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} + (-1)^{2+3} a_{23} \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} + (-1)^{3+3} a_{33} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$\therefore \det A = 1\times 3\times(10-3) - 1\times 4\times(5-2) + 1\times 7\times(3-4)$$

$$\det A = 21 - 12 - 7 = \boxed{2}$$

(B) The classical adjoint of A is the transpose of the matrix of cofactors.

$$\text{adj } A = \begin{pmatrix} \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{pmatrix}^T$$

$$\therefore \text{adj } A = \begin{pmatrix} 1 & -10 & 7 \\ -(-1) & 4 & -3 \\ -1 & -(-2) & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & -10 & 7 \\ 1 & 4 & -3 \\ -1 & 2 & -1 \end{pmatrix}^T$$

$$\therefore \boxed{\text{adj } A = \begin{pmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{pmatrix}}$$

(C) To establish the inverse of A, we multiply all elements of adj A by the reciprocal of $\det A$, giving

$$A^{-1} = \frac{1}{|A|}\text{adj}\, A = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -5 & 2 & 1 \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}}$$

Problem 7

(A) If $x + y + z = 1$, then, substituting in the second equation,

$$2x + 2y + 2z = 1 \rightarrow 2\underbrace{(x+y+z)}_{=1} = 4$$

$$\therefore 2\times 1 = 4$$

$$\therefore 2 = 4$$

which is false. Accordingly, the system has no solution. In geometrical terms, the two equations represent planes in $\mathbb{R}^3$ that do not intersect (i.e., they are parallel).

(B) If $x + y + z = 0$, then, substituting in the second equation,

$$2x + 2y + 2z = 0 \rightarrow 2\underbrace{(x+y+z)}_{=0} = 0$$

$$\therefore 2\times 0 = 0$$

$$\therefore 0 = 0$$

which is true. Accordingly, the system is redundant and has infinitely many solutions. In geometrical terms, the two equations represent planes in $\mathbb{R}^3$ that coincide. Any set of values of the form $x = -s - t$, $y = s$ and $z = t$ will satisfy both equations.

Problem 8

(A) If A has rank 4 then A^T must also have rank 4. The matrix A^T has 7 columns, so by the dimension theorem for matrices rank and nullity must add up to 7. Since the rank is 4, the nullity must be 3 and $\dim N(A^T) = 3$. The orthogonal complement of $N(A^T)$ is row space $R(A)$.
(B) If $\boldsymbol{x}$ is in $R(A)$ and $A^T\boldsymbol{x} = \mathbf{0}$, then $\boldsymbol{x}$ is also in $N(A^T)$. Since $R(A)$ and $N(A^T)$ are orthogonal subspaces their intersection is $\{\mathbf{0}\}$. It follows that $\boldsymbol{x}$ is the zero vector and $\|\boldsymbol{x}\| = 0$.
(C) Appealing to the rank-nullity theorem a second time, we find that

$$\text{Rank}(A) + \text{Nullity}(A) = 5 \to \text{Nullity}(A) = 5 - \text{Rank}(A)$$
$$\therefore \text{Nullity}(A) = 5 - 4 = 1$$

Accordingly, $\dim N(A^TA) = \dim N(A) = 1$. Therefore, the normal equations will involve a free variable and hence the least squares problem will have infinitely many solutions.

Problem 9

Let $\boldsymbol{v_1} = \boldsymbol{w_1}$. To determine $\boldsymbol{v_2}$, we calculate

$$\therefore \mathbf{v}_2 = \mathbf{w}_2 - \frac{\mathbf{w}_2 \bullet \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (-1,1,0) - \frac{[-1\times 1 + 1\times 1 + 0\times 1]}{1^2+1^2+1^2}(1,1,1)$$
$$\therefore \mathbf{v}_2 = (-1,1,0)$$

Then, we determine $\boldsymbol{v_3}$,

$$\mathbf{v}_3 = \mathbf{w}_3 - \frac{\mathbf{w}_3 \bullet \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{w}_3 \bullet \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

$$\therefore \mathbf{v}_3 = (1,2,1) - \frac{1\times 1 + 2\times 1 + 1\times 1}{1^2+1^2+1^2}(1,1,1) - \frac{1\times(-1) + 2\times 1 + 1\times 0}{(-1)^2+1^2}(-1,1,0)$$
$$\therefore \mathbf{v}_3 = \left(1 - \frac{4}{3} + \frac{1}{2},\ 2 - \frac{4}{3} - \frac{1}{2},\ 1 - \frac{4}{3} - 0\right)$$
$$\therefore \mathbf{v}_3 = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$$

The penultimate step is to determine the norms of vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$,

$$\|\mathbf{v}_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\|\mathbf{v}_2\| = \sqrt{(-1)^2 + 1^2 + 0^2} = \sqrt{2}$$

$$\|\mathbf{v}_3\| = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{1}{\sqrt{6}}$$

Lastly, we conclude that $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is an orthogonal basis in $\mathbb{R}^3$, and the vectors $\{\boldsymbol{q_1}, \boldsymbol{q_2}, \boldsymbol{q_3}\}$, such that

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(-1,1,0)}{\sqrt{2}} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)}{\frac{1}{\sqrt{6}}} = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}\right) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$$

constitute an orthonormal basis for $\mathbb{R}^3$.

Problem 10

(A) To establish the characteristic polynomial, we compute $tI - A$, where I is the identity matrix, giving

$$tI - A = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

$$\therefore tI - A = \begin{pmatrix} t-1 & -4 \\ -2 & t-3 \end{pmatrix}$$

The characteristic polynomial is given by the determinant of the matrix above,

$$|tI - A| = \begin{vmatrix} t-1 & -4 \\ -2 & t-3 \end{vmatrix} = (t-1)\times(t-3) - (-4)\times(-2)$$

$$\therefore |tI - A| = t^2 - 4t + 3 - 8$$

$$\therefore |tI - A| = \boxed{t^2 - 4t - 5}$$

Another way to arrive at the same result is to apply the general formula

$$\Delta(t) = t^2 - \mathrm{tr}(A) + \det A = t^2 - (1+3)t + (1\times 3 - 2\times 4)$$

$$\therefore \Delta(t) = t^2 - 4t - 5$$

(B) The eigenvectors are the solutions of the equation $\Delta(t) = 0$, namely

$$\Delta(t) = 0 \to t^2 - 4t - 5 = 0$$

$$t = \frac{-(-4) \pm \sqrt{16 - 4 \times 1 \times (-5)}}{2} = \frac{4 \pm 6}{2} = -1,\ 5$$

(C) Substitute $t = -1$ in the matrix $tI - A$ to obtain

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -1-1 & -4 \\ -2 & -1-3 \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ -2 & -4 \end{pmatrix}$$

The eigenvectors belonging to $\lambda_1 = -1$ form the solution of the homogeneous system $MX = 0$, that is,

$$MX = 0 \to \begin{pmatrix} -2 & -4 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{cases} -2x - 4y = 0 \\ -2x - 4y = 0 \end{cases} \to x = -2y$$

One solution to the homogeneous system above is $x = 2$, $y = -1$. Thus, $\boldsymbol{u} = (2, -1)$ is an eigenvector that spans the eigenspace of $\lambda_1 = -1$.

Moving on to eigenvalue $\lambda_2 = 5$, we proceed as we did above,

$$tI - A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -2 & 2 \end{pmatrix}$$

The eigenvectors belonging to $\lambda_2 = 5$ form the solution of the homogeneous system $MX = 0$, that is,

$$MX = 0 \to \begin{pmatrix} 4 & -4 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{cases} 4x - 4y = 0 \\ -2x - 2y = 0 \end{cases} \to x = y$$

One solution to this homogeneous system is $x = 1$, $y = 1$. Thus, $\boldsymbol{v} = (1,1)$ is an eigenvector that spans the eigenspace of $\lambda_2 = 5$.

(D) Let P be the matrix whose columns are the above eigenvectors,

$$P = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

Then, $B = P^{-1}AP$ is the diagonal matrix whose diagonal entries are the respective eigenvalues,

$$B = P^{-1}AP = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

Problem 11

(A) Since A is symmetric there is an orthogonal matrix that diagonalizes A. So A cannot be defective and hence the eigenspace corresponding to the triple eigenvalue $\lambda = 0$ (that is, the null space of A) must have dimension 3.

(B) Since λ_1 is distinct from the other eigenvalues, the eigenvector $\boldsymbol{x_1}$will be orthogonal to $\boldsymbol{x_2}$, $\boldsymbol{x_3}$, and $\boldsymbol{x_4}$.

(C) To construct an orthogonal matrix A that diagonalizes A, set $\boldsymbol{u_1} = \boldsymbol{x_1}/\|\boldsymbol{x_1}\|$. Vectors x_2, x_3, x_4 form a basis for nullity $N(A)$. Use the Gram-Schmidt algorithm to convert this basis into an orthonormal basis ($\boldsymbol{u_2}$,$\boldsymbol{u_3}$,$\boldsymbol{u_4}$). Since the vector u_1 is in $N(A)^{\perp}$, it follows that U = ($\boldsymbol{u_1}$,$\boldsymbol{u_2}$,$\boldsymbol{u_3}$,$\boldsymbol{u_4}$) is an orthogonal matrix and U diagonalizes A.

(D) Since A is symmetric it can be factored into a product $A = QDQ^T$, where Q is orthogonal and D is diagonal. It follows that $e^A = Qe^AQ^T$. Matrix $\boldsymbol{e^A}$ is symmetric since

$$\left(e^A\right)^T = Q\left(e^D\right)^T Q^T = Qe^DQ^T = e^A$$

The eigenvalues of e^A are $\lambda_1 = e$ and $\lambda_2 = \lambda_3 = \lambda_4 = 1$. Since e^A is symmetric and its eigenvalues are all positive, it follows that e^A is positive definite.

Problem 12

(A) Substituting $\boldsymbol{v}$, $\boldsymbol{w}$ and A in the definition of the inner product proposed, we have

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T A\,\mathbf{w} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1\times5+1\times2 & 1\times2+1\times1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\therefore \langle \mathbf{v}, \mathbf{w} \rangle = \begin{pmatrix} 7 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 7\times(-1)+3\times2 = \boxed{-1}$$

(B) The norm of $\boldsymbol{v}$ satisfies the relation

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle \to \|\mathbf{v}\|^2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \|\mathbf{v}\|^2 = \begin{pmatrix} 1\times5+2\times1 & 1\times2+1\times1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 10$$

so that

$$\|\mathbf{v}\|^2 = 10 \to \boxed{\|\mathbf{v}\| = \sqrt{10}}$$

(C) For a vector $\boldsymbol{\zeta} = (x, y)$ to be orthogonal to $\boldsymbol{v}$ under a certain inner product, we must have $\langle \boldsymbol{v}, \boldsymbol{\zeta} \rangle = 0$; that is,

$$\langle \mathbf{v}, \zeta \rangle = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1\times5+1\times2 & 1\times2+1\times1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\therefore \langle \mathbf{v}, \zeta \rangle = \begin{pmatrix} 7 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 7x+3y = 0$$

Thus, set $S^{\perp}$ is defined as

$$\boxed{S^{\perp}=\left\{\begin{pmatrix}x\\y\end{pmatrix}\middle|\ 7x+3y=0\right\}}$$

One base of $S^{\perp}$, obtained by taking, say, x = 3 and substituting in the relationship above, which brings to

$$7x+3y=0\rightarrow 7\times 3+3y=0$$
$$\therefore y=\frac{-21}{3}=-7$$

Thus, $\boldsymbol{n}$ = $(3,-7)^{\mathrm{T}}$ is a basis of $S^{\perp}$.

(D) We were essentially asked to express $\boldsymbol{w}$ as a linear combination of $\boldsymbol{w_1}$, a vector from the linear span of $\boldsymbol{v}$, and $\boldsymbol{w_2}$, a vector from $S^{\perp}$, the space obtained in the previous part. We can use $\boldsymbol{v}$ itself as $\boldsymbol{w_1}$. Further, we can use $\boldsymbol{n}$ to represent $S^{\perp}$. Accordingly,

$$\mathbf{w}=\mathbf{w}_1+\mathbf{w}_2\rightarrow\begin{pmatrix}-1\\2\end{pmatrix}=\alpha\begin{pmatrix}1\\1\end{pmatrix}+\beta\begin{pmatrix}3\\-7\end{pmatrix}$$
$$\therefore\begin{cases}\alpha+3\beta=-1\ \text{(I)}\\\alpha-7\beta=2\ \text{(II)}\end{cases}$$

Subtracting (II) from (I) yields

$$\alpha+3\beta-(\alpha-7\beta)=-1-2\rightarrow\cancel{\alpha}+3\beta-\cancel{\alpha}+7\beta=-3$$
$$\therefore 10\beta=-3$$
$$\therefore\beta=-\frac{3}{10}$$

Substituting β = $-3/10$ into (I) yields

$$\alpha+3\beta=-1\rightarrow\alpha+3\times\left(-\frac{3}{10}\right)=-1$$
$$\therefore\alpha-\frac{9}{10}=-1$$
$$\therefore\alpha=-\frac{1}{10}$$

Finally, we conclude that w can be expressed by the linear combination

$$\mathbf{w}=\alpha\begin{pmatrix}1\\1\end{pmatrix}+\beta\begin{pmatrix}3\\-7\end{pmatrix}=\boxed{-\frac{1}{10}\begin{pmatrix}1\\1\end{pmatrix}-\frac{3}{10}\begin{pmatrix}3\\-7\end{pmatrix}}$$

(E) The linearly independent vectors $\boldsymbol{v}$ and $\boldsymbol{n}$ are orthogonal under the inner product at hand, so all we need to do is normalize them. We computed the norm of $\boldsymbol{v}$ in Part C. It remains to normalize $\boldsymbol{n}$,

$$\|\mathbf{n}\|^2=\langle\mathbf{n},\mathbf{n}\rangle\rightarrow\|\mathbf{n}\|^2=(3\ \ -7)\begin{pmatrix}5&2\\2&1\end{pmatrix}\begin{pmatrix}3\\-7\end{pmatrix}$$

$$\therefore \|\mathbf{n}\|^2 = (3\times5-7\times2 \quad 3\times2-7\times1)\begin{pmatrix} 3 \\ -7 \end{pmatrix} = (1 \quad -1)\begin{pmatrix} 3 \\ -7 \end{pmatrix} = 10$$

$$\therefore \|\mathbf{n}\|^2 = 10 \rightarrow \boxed{\|\mathbf{n}\| = \sqrt{10}}$$

Therefore,

$$\mathbf{u}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \mathbf{u}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

is an orthonormal basis of $\mathbb{R}^2$ with the given inner product.

Problem 13

(A) We begin by computing the eigenvalues of A. The characteristic polynomial of A is

$$\begin{vmatrix} 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)^2 - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = 0$$

$$\therefore 1 - 2\lambda + \lambda^2 - \frac{1}{4} = 0$$

$$\therefore \lambda^2 - 2\lambda + \frac{3}{4} = 0$$

$$\lambda = \frac{-(-2) \pm \sqrt{4 - 4\times\frac{3}{4}}}{2} = \frac{2\pm1}{2} = \frac{1}{2}, \frac{3}{2}$$

Matrix A is symmetric and both of its eigenvalues are positive; thus, A is positive definite. If $\boldsymbol{x} = (x_1, x_2)^{\mathsf{T}}$ is some vector in $\mathbb{R}^2$, we can establish the product $\boldsymbol{x}^T A\boldsymbol{x}$ as

$$\mathbf{x}^T A\mathbf{x} = (x_1 \ \ x_2)\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 - x_1x_2 + x_2^2$$

As for $\boldsymbol{x}^T B\boldsymbol{x}$, we have

$$\mathbf{x}^T B\mathbf{x} = (x_1 \ \ x_2)\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 - x_1x_2 + x_2^2$$

Thus, $\boldsymbol{x}^T A\boldsymbol{x} = \boldsymbol{x}^T B\boldsymbol{x}$ as we were supposed to show.

(B) Using the equality studied in the previous part, it is clear that, for any $\boldsymbol{x} \neq \mathbf{0}$, then

$$\mathbf{x}^T B\mathbf{x} = \mathbf{x}^T A\mathbf{x} > 0$$

since A is positive definite. Therefore, B is also positive definite. However,

$$B^2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

is not positive definite. Indeed, if we take, say, $\boldsymbol{x}$ = $(1,1)^T$, then

$$\mathbf{x}^T B^2 \mathbf{x} = \begin{pmatrix} 1\,1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \times 1 - 1 \times 1 = 0$$

This violates the definition of a positive definite matrix.

Problem 14

(A) For $\boldsymbol{\zeta}$ to be an eigenvector of $\boldsymbol{A}$, we must have

$$A\boldsymbol{\zeta} = \lambda\boldsymbol{\zeta} \to \begin{pmatrix} 5 & 5 & -5 \\ 3 & 3 & -5 \\ 4 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 5 & 5 & -5 \\ 3 & 3 & -5 \\ 4 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \times 0 + 5 \times 1 - 5 \times 1 \\ 3 \times 0 + 3 \times 1 - 5 \times 1 \\ 4 \times 0 + 0 \times 1 - 2 \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Thus, $\lambda = -2$ is an eigenvalue belonging to the given eigenvector.

(B) To establish the eigenvector $\boldsymbol{w} = (x,y,z)$ that corresponds to $\lambda = 4 + 2i$, we solve $(A - \lambda I)\boldsymbol{w} = 0$, namely

$$A - \lambda I = \begin{pmatrix} 5 - (4 + 2i) & 5 & -5 \\ 3 & 3 - (4 + 2i) & -5 \\ 4 & 0 & -2 - (4 + 2i) \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 - 2i & 5 & -5 \\ 3 & -1 - 2i & -5 \\ 4 & 0 & -6 - 2i \end{pmatrix} = 0$$

Row-reducing the matrix brings to

$$\therefore \begin{pmatrix} 1 - 2i & 5 & -5 \\ 3 & -1 - 2i & -5 \\ 4 & 0 & -6 - 2i \end{pmatrix} = 0 \to \begin{pmatrix} 1 & 0 & -\frac{3}{2} - \frac{1}{2}i \\ 0 & 5 & -\frac{5}{2} - \frac{5}{2}i \\ 0 & 0 & 0 \end{pmatrix}$$

With $z = 2$, we find, in the first equation,

$$x + 0y + \left(-\frac{3}{2} - \frac{1}{2}i \right) z = 0 \to x + \left(-\frac{3}{2} - \frac{1}{2}i \right) \times 2 = 0$$

$$\therefore x = 3 + i$$

and $y = 1 + i$ in the second. Thus, $\boldsymbol{w} = (3 + i, 1 + i, 2)$ is an eigenvector belonging to eigenvalue $\lambda_2 = 4 + 2i$.

(C) Since matrix A is real, complex eigenvalues occur in conjugate pairs, and $4-2i$ must also be an eigenvalue of the matrix in question. Further, the corresponding eigenvector is $\bar{\boldsymbol{w}} = (3-i, 1-i, 2)$. Gleaning the three eigenvectors $\boldsymbol{\zeta} = (0, 1, 1)^{\mathrm{T}}$, $\boldsymbol{w} = (3+i, 1+i, 2)^{\mathrm{T}}$ and $\bar{\boldsymbol{w}} = (3-i, 1-i, 2)^{\mathrm{T}}$, we construct the desired matrix P,

$$P = \begin{pmatrix} 0 & 3+i & 3-i \\ 1 & 1+i & 1-i \\ 1 & 2 & 2 \end{pmatrix}$$

and, gathering the eigenvalues $\lambda_1 = -2$, $\lambda_2 = 4+2i$, and $\lambda_3 = 4-2i$, we produce the diagonal matrix D,

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4+2i & 0 \\ 0 & 0 & 4-2i \end{pmatrix}$$

Along with the starting matrix A, matrices D and P are interrelated by the expression $P^{-1}AP = D$.

$$\begin{pmatrix} 0 & 3+i & 3-i \\ 1 & 1+i & 1-i \\ 1 & 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 5 & -5 \\ 3 & 3 & -5 \\ 4 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 3+i & 3-i \\ 1 & 1+i & 1-i \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4+2i & 0 \\ 0 & 0 & 4-2i \end{pmatrix}$$

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