Montogue

# Project FM202

# BIOFLUID MECHANICS

## Blood Flow in Arteries – Bifurcations and Elliptical Tubes

Lucas Montogue

# Problems

## → Problem 1: Vascular Metabolic Rate

### Problem 1A

The power *H* required to pump a fluid of viscosity $\mu$ through a tube of radius *R* and length *L*, at a steady flow rate *Q* and under conditions of fully developed Hagen-Poiseuille flow, is given by

$$H = \frac{8\mu L Q^2}{\pi R^4}$$

A well-known implication of this simple formula is the result that almost 94% of this power can be saved by simply doubling the radius of the tube, all else being unchanged. In other words, only 6% of the power is needed to maintain the same fluid through a tube of the same length but double the radius. Taking $\mu$ = 0.03 P for the viscosity of blood, a blood vessel segment of *L* = 10 cm, a flow rate *Q* = 100 cm$^3$/s, and a vessel radius of 1 cm, estimate the power required to maintain the flow in ergs/s. Then, determine the corresponding power in cal/day.

### Problem 1B

For the cardiovascular system as a whole, the pumping power required from the heart can be estimated in a simpler form

$$H = Q\Delta p$$

If one takes $\Delta p$ = 120 mmHg, *Q* = 100 cm$^3$/s, and a (rather generous) cardiac efficiency of 10%, what would be the metabolic rate of the heart for the system with normal radii? In turn, what would be the metabolic rate for the system with double the normal radii? And with half the normal radii? Express these metabolic rates as a percentage of the total metabolic rate of the host organism, which is 2,500 cal/day for an average man at rest.

## → Problem 2: Bifurcations and Murray's Law

### Problem 2A

We know that arteries bifurcate many times before they become capillaries. Can we guess a design pattern of the blood vessel bifurcation? To be more concrete, let us consider three vessels AB, BC, and BD connecting three points A, C, and D in space, as shown in Figure 1. There is a flow $Q_0$ coming through A into AB. The flow is divided into $Q_1$ in BC and $Q_2$ in BD. Let the points A, C and D be fixed, but the location of B and the vessel radii are left for the designer to choose. Is there an optimal position for the point *B*? By asking such a question we are seeking a principle of optimum design, in which a *cost function* is assumed, and the design parameters are chosen so that the cost function is minimized. In 1926, Murray suggested that physiological vascular systems, subjected through evolution to natural selection, must have achieved an optimum arrangement such that, in every segment of the vessel, flow is achieved with the least possible biological work. He and Rosen (1967) thus proposed a cost function made up of two terms, namely, a term representing (a) the energy required to overcome viscous drag in a fluid obeying Poiseuille's law, and (b) another related to the energy metabolically required to maintain the volume of blood and vessel tissue involved in the flow. These energy terms are related to the radius of the vessel, but in opposite ways: the larger the radius, the smaller is the power $P_f$ required for flow, but the larger is the power $P_m$ required for metabolic maintenance of the blood and vessel wall tissue. The vessel can be neither too large nor too small if the cost function, $P_t = P_f + P_m$, is to be minimized.

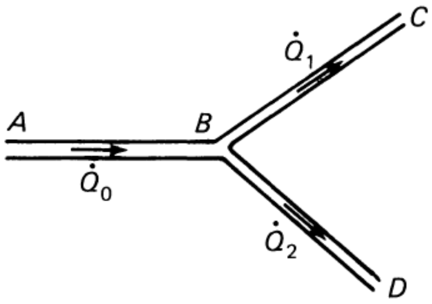


Figure 1 Bifurcation of a blood vessel AB into two branches BC and BD supporting blood at a rate of $Q_0$ (cm³/s) from point $A$ to points $C$ and $D$, with outflow of $Q_1$ at $C$ and $Q_2$ at $D$.

If gravitational and kinetic energy terms can be neglected, a Newtonian fluid exhibits a volumetric flow rate, $Q$, which is linearly proportional to the pressure difference, $\Delta p$, to which it is subjected,

$$Q = c\Delta p$$

where $c$ is a conductance coefficient. In cylindrical conduits, the conductance is proportional (from Poiseuille's law) to $R^4$, the fourth power of the radius of the tube,

$$c = \frac{\pi R^4}{8\mu L}$$

where $\mu$ is the viscosity of the fluid and $L$ is the length of the tube. Letting $a = 8\mu L/\pi$, we have

$$aQ = \Delta p R^4 \rightarrow \Delta p = aQR^{-4}$$

The power required to maintain flow is then

$$P_f = \Delta p Q = aQ^2 R^{-4}$$

Hence, the power required to maintain a given flow is dramatically reduced by small increases in the radius of a vessel. Offsetting this, however, is a metabolic requirement, $P_m$, which increases linearly with the volume of the blood vessel,

$$P_m = k \times \text{Volume} = k\pi L R^2$$

where $k$ is a metabolic constant. Letting $b = k\pi L$, we may write

$$P_m = bR^2$$

The total power required, i.e. the cost function, is then

$$\boxed{P_t = P_f + P_m = aQ^2 R^{-4} + bR^2}$$

Of the two coefficients in this expression, $a$ depends only upon the viscosity of the flowing fluid, whereas $b$ incorporates the metabolic constant $k$ and thus depends upon the metabolism of blood and vessel tissue. For a specified value of $Q$, the cost function $P_i$ depends only upon $R$. With this information, optimize the cost function to obtain its minimum value and the corresponding value of $R$. Show that this optimum value is indeed a minimum $P_t$. Replace $a$ and $b$ with their respective values. What is the proportion between $R$ and $Q$ at optimum conditions?

## Problem 2B

Let us return to the design problem of a vessel bifurcation. Conservation of mass from the main vessel segment $AB$ to the daughter vessels BC and CD allows us to state that

$$Q_0 = Q_1 + Q_2$$

Use this result and the equation you have obtained for the optimum radius $R$ to derive Murray's law, according to which the cube of the radius of the main vessel is equal to the sum of the cubed radii of the daughter vessels. Mathematically,

$$\boxed{R_o^3 = R_1^3 + R_2^3}$$

## Problem 2C

Suppose that $R^*$ denotes the optimum radius obtained in Problem 2A. Use the equation for $R^*$ to obtain an expression for the metabolic constant $k$. Then, note that the pressure drop $\Delta p$ is given by

$$\Delta p = \frac{8\mu LQ}{\pi R^4}$$

which is the basic Poiseuille flow equation with the following interpretation. With a fixed flow in a blood vessel of fixed length $L$, if the vessel radius $R$ is changed the pressure drop $\Delta p$ required to maintain the flow will change correspondingly. In particular, when the radius has an optimum value $R^*$, the pressure drop will have an optimum value $\Delta p^*$ and we thus write the previous equation as

$$\Delta p^* = \frac{8\mu LQ}{\pi R^{*4}}$$

Use this relation and the equation you have obtained for $R$ to obtain a relationship for the metabolic constant $k$. Use $V^* = \pi R^2 L$ as the volume of a hypothetical vessel of length $L$ and radius $R^*$. Then, apply the equation to the entire cardiovascular system with the approximate data $Q$ = 100 cm$^3$/s, $\Delta p^*$ = 120 mmHg, and $V^*$ = 4500 cm$^3$ to obtain a value of $k$. What are the dimensions of $k$ in the CGS system?

Much of earlier data on cardiovascular system measurements originated from Green (1950), who presents his information with the following comments: "*accurate quantitative data are not available for the capacity of the circulatory system. I have, however, made a rough estimate of the relative capacities of the component parts of the circulatory system based in part on calculations by Mall from detailed microscopic examinations of the mesenteric vascular bed.*" The data relate to the entire cardiovascular system of a 13-kg dog and are shown graphically in Figure 2. The solid triangles refer to arteries, whereas the inverse triangles pertain to veins. Also shown in this figure is the optimum relation between $Q$ and $R^*$ as given by the equation discussed in Problem 2A. For the purpose of comparison, this relation has been written in the form

$$Q = K\left(R^*\right)^3$$

in which $K$ is a constant given by

$$K = \frac{\pi}{4}\left(\frac{k}{\mu}\right)^{\frac{1}{2}}$$

so that, applying logarithms to both sides of the equation,

$$\log Q = \log K + \log\left(R^*\right)^3$$

In the graph shown, the vertical axis is the logarithm of flow rate, whereas the horizontal axis is the logarithm of $R^3$. What are the units of constant $K$? Use the values provided earlier and $\mu$ = 0.03 poise to obtain a representative value of $K$.

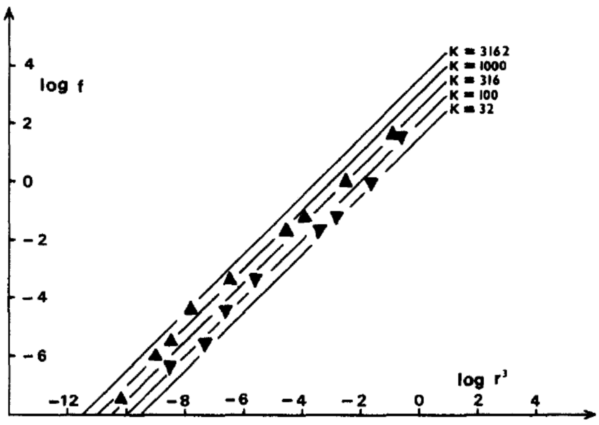


Figure 2 Relation between blood flow $Q$ (cm³ /s) and vessel radius $R$ (cm). The solid triangles are based on data by Green (1950) for a 13-kg dog where arteries are represented by upright triangles and veins by inverse triangles. The straight lines are based on the logarithmic relationship presented in Problem 2C.

## Problem **2D**

Once again, let us consider the bifurcation problem. Since the cost functions of all vessels are additive, we see at once that the vessel connecting A, C, and D in Figure 1 should be straight and lie in a plane (because this minimizes the length, $L$, when other things are fixed). To find out the details, let the geometric parameters be specified as shown in Figure 3. The three branches will be denoted by subscripts 0, 1, and 2. The total cost function will be denoted by $P$.

$$P = \frac{3\pi K}{2}\left(R_o^2 L_o + R_1^2 L_1 + R_2^2 L_2\right)$$

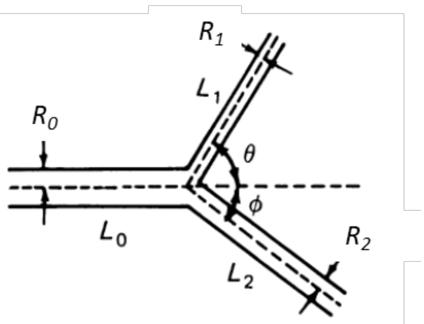


Figure 3 Geometric parameters of the branching pattern. Theory shows that $B$ should lie in the plane of ACD.

The lengths $L_o$, $L_1$, and $L_2$ are affected by the location of point $B$, and the radii $R_O$, $R_1$, and $R_2$ are related to the flows $Q_o$, $Q_1$, and $Q_2$ through the equation developed in Problem 2A. You must minimize $P$ by properly choosing the location of the bifurcation point $B$. Since a small movement of $B$ changes $P$ by

$$\delta P = \frac{3\pi K}{2}\left(R_o^2 \delta L_o + R_1^2 \delta L_1 + R_2^2 \delta L_2\right)$$

an optimal location of $B$ would make $\delta P = 0$ for an arbitrary small movement of $B$. Consider three special movements of $B$. First, let B move to B' in the direction of AB, as shown in Figure 4. In this case,

$$\delta L_o = \delta \ ; \ \delta L_1 = -\delta \cos\theta \ ; \ \delta L_2 = -\delta \cos\phi$$

$$\therefore \delta P = \frac{3\pi k}{2}\delta\left(R_o^2 - R_1^2 \cos\theta - R_2^2 \cos\phi\right)$$

The optimum is obtained when

$$R_o^2 = R_1^2 \cos\theta + R_2^2 \cos\phi$$

Next, let $B$ move to $B'$ in the direction of $CB$, as shown in Figure 5. Finally, let $B$ move a short distance $\delta$ in the direction of $DB$, as shown in Figure 6. Use the three equations derived from these small displacements to show the following relationships,

$$\cos\theta = \frac{R_o^4 + R_1^4 - R_2^4}{2R_o^2 R_1^2}$$

$$\cos\phi = \frac{R_o^4 - R_1^4 + R_2^4}{2R_o^2 R_2^2}$$

$$\cos(\theta+\phi) = \frac{R_o^4 - R_1^4 - R_2^4}{2R_1^2 R_2^2}$$

Then, use Murray's Law to simplify the equations and obtain the following results,

$$\boxed{\begin{aligned} \cos\theta &= \frac{R_0^4 + R_1^4 - \left(R_0^3 - R_1^3\right)^{\frac{4}{3}}}{2R_0^2 R_1^2} \\ \cos\phi &= \frac{R_0^4 + R_2^4 - \left(R_0^3 - R_2^3\right)^{\frac{4}{3}}}{2R_0^2 R_2^2} \\ \cos(\theta+\phi) &= \frac{\left(R_1^3 + R_2^3\right)^{\frac{4}{3}} - R_1^4 - R_2^4}{2R_1^2 R_2^2} \end{aligned}}$$

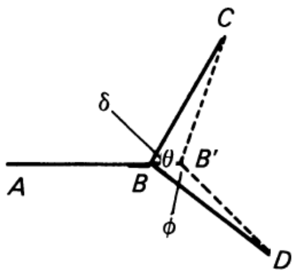


Figure 4 A particular variation of $\delta L_0, \delta L_1, \delta L_2$ by a small displacement of *B* in the direction of AB.

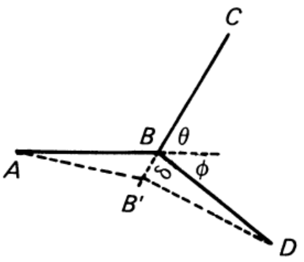


Figure 5 Another particular variation of $\delta L_0, \delta L_1, \delta L_2$ by a displacement of *B* to *B'* along BC.

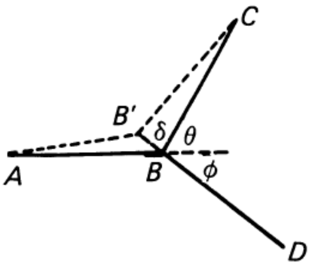


Figure 6 A third variation caused by a displacement of *B* to *B'* along BD.

## Problem 2E

Show that, according to Murray's cost function, if $R_1 = R_2$, then $\theta = \phi$. That is, if the radii of the daughter branches are equal, the bifurcating angles are equal. Then, demonstrate that if $R_2 \gg R_1$, then $R_2 \doteq R_o$ and $\theta \doteq \pi/2$.

## Problem 2F

In 1808, Thomas Young, in his Croonian Lecture, established that when he wished to estimate the resistance of an arterial system: "*in order to calculate the magnitude of the resistance, it is necessary to determine the dimensions of the arterial system, and the velocity of the blood which flows through it.*" Starting with assumed dimensions for the aorta and for the capillaries, Young had to decide upon a probable branching pattern which would connect one with the other. He chose a symmetrical, dichotomous system in which the diameter of each branch was "*about 4/5 of that of the trunk, or more accurately 1:$2^{1/3}$*". By assuming this geometric ratio between the diameters of daughter and parent vessels, Young calculated that 29 bifurcations were necessary to diminish the aorta to the size of the capillaries. From estimates of the lengths of the aorta and capillaries, he constructed another geometric series for lengths of the thirty generations of vessels and went on to calculate blood volumes, velocities of flow, and resistances in the different stages of the system. Young does not say why he chose a ratio of $2^{1/3}$:1, but it seems certain that he was familiar with a rule – either empirical or theoretical – that favored this choice.

Support Young's results by demonstrating that when $R_1 = R_2$, we have $R_1/R_0 = 2^{-1/3} = 0.794$, and $\cos\theta = 0.794$, so that $\theta = 37.5°$. Let $R_o$ denote the radius of the aorta, and assume equal bifurcation in all generations. Accordingly, the radius of the first generation is $0.794R_0$, the radius of the second generation is $0.794^2 R_0$, and, generally, that of the $n$-th generation is

$$R_n = (0.794)^n R_0$$

If a capillary blood vessel has a radius of 5×10$^{-4}$ cm and the radius of the aorta is $R_o = 1.5$ cm, is the number of generations consistent with Young's results?

## → Problem 3: Flow in Elliptical Tubes

### Problem 3A

Explain the need to consider flow in noncircular vessels in some regions of the human circulation.

### Problem 3B

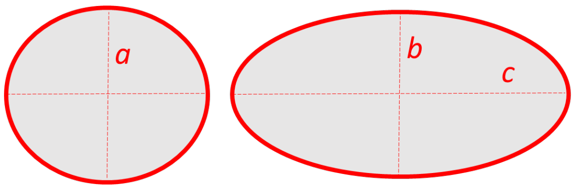


Figure 7 Circular and elliptical cross-sections.

Consider steady flow of blood (assumed as a Newtonian fluid) in a rigid tube of elliptical cross-section, as illustrated in Figure 7. Flow occurs along the $x$-plane, and the ellipse is defined in the $yz$-plane by the expression

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The velocity profile has the form

$$u(y,z) = \frac{kb^2c^2}{2\mu\left(b^2 + c^2\right)}\left(\frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

where $k$ is the pressure gradient in the $x$-axis, assumed to be a constant ($dp/dx = \Delta p/L = k$), and $\mu$ is the viscosity of blood (also a constant). Obtain an expression for the flow rate $q$. Use factor $\delta$ to simplify your result,

$$\delta = \left(\frac{2b^3c^3}{b^2 + c^2}\right)^{1/4}$$

Next, obtain an expression for the pumping power, $H$ (such that $H = \Delta p \times q$). Also, show that the flow rate reduces to the expression for a tube of circular cross-section when $b = c = a$, where $a$ is the radius of a circle described by the equation

$$y^2 + z^2 = a^2$$

### Problem 3C

Flow in a tube of circular cross-section is more efficient than flow in a conduit of any other cross-section geometry, including an elliptical section. Flow inefficiency in a tube of noncircular cross-section manifests itself in terms of lower flow rate for a given pumping power or higher pumping power for a given flow rate. For a meaningful comparison between the two cases, either the *areas* or *perimeters* of the circular and the noncircular cross-sections must be made equal. In the case of the elliptic cross-section, this establishes a relation between the radius of the circular cross-section and the axes of the elliptic cross-section. The perimeter of an ellipse of semi minor and major axes $b$ and $c$ is equal to that of a circle of radius $a$ if

$$a^2 \approx \frac{1}{2}\left(b^2 + c^2\right)$$

Consider an elliptical section for which the ratio of the major axis to the minor axis is $\lambda$ = 2:1. Suppose that the pumping power driving the flow is left unchanged. Show that, in this case, the flow rate of the elliptical blood vessel will be a fraction $q_e = (64/125)q_c$ of the flow rate $q_c$ expected for the circular cross-section. Next, suppose that the flow rate is kept unchanged. Show that the pumping power required to drive the flow in the elliptical blood vessel will be a portion $H_e = (125/64)H_c$ of the pumping power $H_c$ expected for the circular cross-section.

# Solutions

## ▸ Problem 1

### P.1A ➡ *Solution*

All we have to do is substitute the pertaining variables in the expression for $H$,

$$H = \frac{8\mu L Q^2}{\pi R^4} = \frac{8\times 0.03\times 10\times 100^2}{\pi\times 1^4} = 7,640 \text{ ergs/s}$$

Knowing that 1 erg = $10^{-7}$ joules, 1 J = 0.239 cal, and 1 day = 86,400 s, the energy consumption for a full day is such that

$$H_{\text{daily}} = 7,640\ \frac{\cancel{\text{erg}}}{\cancel{\text{s}}}\times 86,400\ \frac{\cancel{\text{s}}}{\text{day}}\times 10^{-7}\ \frac{\cancel{\text{J}}}{\cancel{\text{erg}}}\times 0.239\ \frac{\text{cal}}{\cancel{\text{J}}} \approx \boxed{16 \text{ cal/day}}$$

### P.1B ➡ *Solution*

Substituting $Q$ = 100 cm$^3$/s and $\Delta p$ = 120 mmHg and performing the necessary unit conversions ($Q$ = $10^{-4}$ m$^3$/s, $\Delta p$ = 16 kPa), we obtain $H$ = 33 cal/day. If all blood vessels in the cardiovascular system were of double their normal radii while still conveying the same flow, the pumping power required from the heart would be approximately 2 cal/day (since Poiseuille's law implies that it would drop 16-fold). Similarly, if all blood vessels had their radii reduced by half, the new pumping power would be 528 cal/day. For a cardiac efficiency of 10% (i.e., only a tenth of power received is converted to effective pumping power), the corresponding metabolic rates for the heart would be 330 cal/day for the system with normal radii, 20 cal/day for the configuration with double radii, and 5280 cal/day for the setting with half the normal radii. Expressed in terms of the total metabolic rate of the host organism, which was given as approximately 2500 cal/day for an average man at rest, these metabolic rates correspond to 330/2500 = 13% for normal conditions, 0.8% for dilated radii conditions, and 211% for contracted radii conditions; this means that adjustments in the radius of the vessel alone are capable of changing the percentage of cardiac pumping power, relative to overall metabolism, by more than two orders of magnitude.

## ▸ Problem 2

### P.2A ➡ *Solution*

$P_t$ as a function of $R$ will be minimized by that value of $R$ for which $dP_t/dR = 0$. Thus, differentiating the cost function, we have

$$\frac{dP_t}{dR} = -4aQ^2R^{-5} + 2bR = 0$$

$$\therefore 2bR = 4aQ^2R^{-5}$$

$$\therefore 2b = 4aQ^2R^{-6}$$

$$\therefore R^6 = \frac{4a}{2b}Q^2$$

$$\therefore R^6 = \left(\frac{2a}{b}\right)Q^2$$

$$\therefore R = \left(\frac{2a}{b}\right)^{\frac{1}{6}} Q^{\frac{1}{3}}$$

$$\therefore \boxed{R = \left(\frac{16\mu}{\pi^2 k}\right)^{\frac{1}{6}} Q^{\frac{1}{3}}}$$

Hence, we have $R \propto Q^{\frac{1}{3}}$, that is, the optimal radius of a blood vessel is proportional to $Q$ to the 1/3 power. Substituting this result in the cost function, the minimum power is determined to be

$$(P_t)_{\min} = \frac{8\mu}{\pi} Q^2 \left[\left(\frac{16\mu}{\pi^2 k}\right)^{\frac{1}{6}} Q^{\frac{1}{3}}\right]^{-4} + \pi k \left[\left(\frac{16\mu}{\pi^2 k}\right)^{\frac{1}{6}} Q^{\frac{1}{3}}\right]^{2} = \boxed{\frac{3\pi}{2} kLR^2}$$

To prove that this is indeed a minimum, we must have $d^2 P_t / dR^2 > 0$. Accordingly, differentiating $P_t$ a second time gives

$$\frac{d^2 P_t}{dR^2} = 2b + 20aQ^2 R^{-6}$$

The result is greater than zero; therefore, the result obtained above is indeed a minimum.

## P.2B ➡ *Solution*

Note that the result we have obtained for the optimum radius $R$ can be manipulated to yield

$$R = \left(\frac{16\mu}{\pi^2 k}\right)^{\frac{1}{6}} Q^{\frac{1}{3}}$$

$$\therefore R^3 = \left(\frac{16\mu}{\pi^2 k}\right)^{\frac{1}{2}} Q$$

$$\therefore Q = \left(\frac{\pi^2 k}{16\mu}\right)^{\frac{1}{2}} R^3$$

Accordingly, the flow rates in the main vessel and the daughter vessels are given by

$$Q_0 = \left(\frac{\pi^2 k}{16\mu}\right)^{\frac{1}{2}} R_0^{\,3} \; ; \; Q_1 = \left(\frac{\pi^2 k}{16\mu}\right)^{\frac{1}{2}} R_1^{\,3} \; ; \; Q_2 = \left(\frac{\pi^2 k}{16\mu}\right)^{\frac{1}{2}} R_2^{\,3}$$

Substituting these results in the continuity equation, we obtain

$$Q_0 = Q_1 + Q_2$$

$$\therefore \cancel{\left(\frac{\pi^2 k}{16\mu}\right)^{\frac{1}{2}}} R_0^{\,3} = \cancel{\left(\frac{\pi^2 k}{16\mu}\right)^{\frac{1}{2}}} R_1^{\,3} + \cancel{\left(\frac{\pi^2 k}{16\mu}\right)^{\frac{1}{2}}} R_2^{\,3}$$

$$\therefore \boxed{R_0^3 = R_1^3 + R_2^3}$$

This result is known as Murray's Law.

## P.2C ➡ *Solution*

Taking the pressure drop as defined by Poiseuille's law and performing some manipulations, it follows that

$$\Delta p^* = \frac{8\mu LQ}{\pi R^4} = \frac{16\mu LQ^2 R^2}{2\pi QR^6} = \frac{\overbrace{\pi R^2 L}^{V^*}}{2Q} \times \underbrace{\frac{16\mu Q^2}{\pi^2 R^6}}_{=k}$$

$$\therefore \Delta p^* = \frac{V^*}{2Q} \times k$$

$$\therefore \boxed{k = \frac{2Q\Delta p^*}{V^*}}$$

To identify the units of $k$, we write

$$[k] = \frac{[Q][\Delta p]}{[V^*]} = \frac{\frac{\cancel{\text{cm}^3}}{\text{s}} \times \frac{\text{dyn}}{\text{cm}^2}}{\cancel{\text{cm}^3}} = \frac{\text{dyn}}{\text{s} \times \text{cm}^2}$$

Substituting $Q$ = 100 cm$^3$/s, $\Delta p^*$ = 120 mmHg = 160,000 dyn/cm$^2$, and $V^*$ = 4500 cm$^3$, we obtain

$$k = \frac{2Q\Delta p^*}{V^*} = \frac{2 \times 100 \times 160{,}000}{4500} = \boxed{7111 \text{ dynes/cm}^2\text{/s}}$$

Note that this can be converted to 0.015 cal/day/cm$^3$, which is comparable to the metabolic rate of approximately 0.025 cal/day/g. The agreement in orders of magnitude is certainly hard to dismiss.

We proceed to obtain the units of constant $K$, which is given by

$$K = \frac{\pi}{4}\left(\frac{k}{\mu}\right)^{\frac{1}{2}}$$

In view of $[k]$ = dyn/cm²/s and $[\mu]$ = dyn-s/cm², the units of $K$ are determined to be

$$[K] = \left(\frac{[k]}{[\mu]}\right)^{\frac{1}{2}} = \left(\frac{\frac{\cancel{\text{dyn}}}{\cancel{\text{cm}^2} \times \text{s}}}{\frac{\cancel{\text{dyn}} \times \text{s}}{\cancel{\text{cm}^2}}}\right)^{\frac{1}{2}} = \left(\frac{1}{\text{s}^2}\right)^{\frac{1}{2}} = \text{s}^{-1}$$

Substituting $k$ = 7111 dyn/cm²/s and $\mu$ = 0.03 P gives

$$K = \frac{\pi}{4} \times \left(\frac{7111}{0.03}\right)^{\frac{1}{2}} = 382 \text{ s}^{-1}$$

## P.2D ➡ *Solution*

We already have the expression that results from a displacement of $B$ in the direction of $AB$, which is repeated here for convenience,

$$R_0^2 = R_1^2 \cos\theta + R_2^2 \cos\phi$$

Next, let us move $B$ to $B'$ in the direction of CB, as shown in Figure 5. In this case, we can identify the displacements

$$\delta L_0 = -\delta\cos\theta \;\; ; \;\; \delta L_1 = \delta \;\; ; \;\; \delta L_2 = \delta\cos(\theta + \phi)$$

so that $\delta P$ becomes

$$\delta P = \frac{3\pi k}{2}\delta\left[-R_0^2\cos\theta + R_1^2 + R_2^2\cos(\theta+\phi)\right]$$

and the optimal condition is

$$-R_0^2\cos\theta + R_1^2 + R_2^2\cos(\theta+\phi) = 0$$

Finally, let $B$ move a short distance $\delta$ in the direction of $DB$, as outlined in Figure 6. The optimal condition, in this case, is found to be

$$-R_0^2\cos\phi + R_1^2\cos(\theta+\phi) + R_2^2 = 0$$

Let us glean the 3 equations that we have obtained,

$$\begin{cases} R_o^2 = R_1^2\cos\theta + R_2^2\cos\phi \\ -R_0^2\cos\theta + R_1^2 + R_2^2\cos(\theta+\phi) = 0 \\ -R_0^2\cos\phi + R_1^2\cos(\theta+\phi) + R_2^2 = 0 \end{cases}$$

We want to solve these equations for the trigonometric terms $\cos\theta$, $\cos\phi$, and $\cos(\theta+\phi)$. This can be done in Mathematica via the command *Solve*; we use symbols $\theta$, $\phi$ and $\theta\phi$ to identify the trigonometric functions $\cos\theta$, $\cos\phi$, and $\cos(\theta+\phi)$, respectively. In addition, we omit the square power when typing the radii terms. The adequate syntax should look as follows.

```
Solve[{R0 == R1 * θ + R2 * ϕ, -R0 * θ + R1 + R2 * θϕ == 0, -R0 * ϕ + R1 * θϕ + R2 == 0}, {θ, ϕ, θϕ}]
```

$$\left\{\left\{\theta \to -\frac{-R_0^2 - R_1^2 + R_2^2}{2\,R_0\,R_1},\ \phi \to -\frac{-R_0^2 + R_1^2 - R_2^2}{2\,R_0\,R_2},\ \theta\phi \to -\frac{-R_0^2 + R_1^2 + R_2^2}{2\,R_1\,R_2}\right\}\right\}$$

Notice that each trigonometric term is expressed in terms of the 3 radii involved in our problem. Using Murray's Law, we can express these functions in terms of 2 radii only; that is, we can write

$$R_0^3 = R_1^3 + R_2^2$$

$$R_2^3 = R_0^3 - R_1^3$$

$$\left(R_2^3\right)^{\frac{4}{3}} = R_2^4 = \left(R_0^3 - R_1^3\right)^{\frac{4}{3}}$$

For instance, substituting in the expression for $\cos\theta$, we obtain

$$\cos\theta = \frac{R_0^4 + R_1^4 - \left(R_0^3 - R_1^3\right)^{\frac{4}{3}}}{2R_0^2R_1^2}$$

Thus, we have expressed $\cos\theta$ as a function of radii $R_0$ and $R_1$ only. Similarly, $\cos\phi$ can become a function of radii $R_0$ and $R_2$ only,

$$R_0^3 = R_1^3 + R_2^3$$

$$\therefore R_1^3 = R_0^3 - R_2^3$$

$$\therefore \left(R_1^3\right)^{\frac{4}{3}} = R_1^4 = \left(R_0^3 - R_2^3\right)^{\frac{4}{3}}$$

$$\therefore \cos\phi = \frac{R_o^4 + R_2^4 - R_1^4}{2R_o^2R_2^2} = \frac{R_o^4 + R_2^4 - \left(R_0^3 - R_2^3\right)^{\frac{4}{3}}}{2R_o^2R_2^2}$$

Moreover, $\cos(\theta+\phi)$ can be expressed as a function of $R_1$ and $R_2$ only, as shown.

$$\cos(\theta+\phi) = \frac{\left(R_1^3 + R_2^3\right)^{\frac{4}{3}} - R_1^4 - R_2^4}{2R_1^2R_2^2}$$

In summary, we have

$$\boxed{\begin{aligned} \cos\theta &= \frac{R_0^4 + R_1^4 - \left(R_0^3 - R_1^3\right)^{\frac{4}{3}}}{2R_0^2R_1^2} \\ \cos\phi &= \frac{R_0^4 + R_2^4 - \left(R_0^3 - R_2^3\right)^{\frac{4}{3}}}{2R_o^2R_2^2} \\ \cos(\theta+\phi) &= \frac{\left(R_1^3 + R_2^3\right)^{\frac{4}{3}} - R_1^4 - R_2^4}{2R_1^2R_2^2} \end{aligned}}$$

## P.2E ➡ *Solution*

Let the radii of the daughter vessels be the same; mathematically, $R_1 = R_2$. We know that $\cos\theta$ is given by

$$\cos\theta = \frac{R_0^4 + R_1^4 - \left(R_0^3 - R_1^3\right)^{4/3}}{2R_0^2 R_1^2}$$

In addition, substituting $R_2 = R_1$ in the expression for $\cos\phi$ gives

$$\cos\phi = \frac{R_0^4 + R_1^4 - \left(R_0^3 - R_1^3\right)^{4/3}}{2R_0^2 R_1^2}$$

We observe that the expressions for $\cos\theta$ and $\cos\phi$ have become identical. Thus,

$$\cos\theta = \cos\phi$$

and consequently,

$$\theta = \phi$$

This shows that if the radii of the daughter branches are equal, the bifurcating angles are equal. Now, consider the hypothesis that $R_2 \gg R_1$, that is, the radius of one of the daughter branches is much larger than the radius of the other. The equation for $\cos(\theta + \phi)$ can be adjusted as follows,

$$\cos\left(\theta+\phi\right) = \frac{\left(R_1^3 + R_2^3\right)^{\frac{4}{3}} - R_1^4 - R_2^4}{2R_1^2 R_2^2} = \frac{\left(R_1^3 + R_2^3\right)^{\frac{4}{3}} - \left(R_1^4 + R_2^4\right)}{2R_1^2 R_2^2}$$

However, given the approximation we are using, the terms on the right-hand side are such that $R_1^3 + R_2^3 \approx R_2^3$ and $R_1^4 + R_2^4 \approx R_2^4$, which reduces the expression above to

$$\cos\left(\theta+\phi\right) = \frac{\underbrace{\left(R_1^3 + R_2^3\right)}_{\approx R_2^3}{}^{\frac{4}{3}} - \underbrace{\left(R_1^4 + R_2^4\right)}_{\approx R_2^4}}{2R_1^2 R_2^2} = \frac{\left(R_2^3\right)^{\frac{4}{3}} - R_2^4}{2R_1^2 R_2^2} = \frac{R_2^4 - R_2^4}{2R_1^2 R_2^2} = 0$$

Then, from elementary trigonometry,

$$\theta + \phi = \arccos\left(0\right) \to \theta + \phi = \frac{\pi}{2}$$

Similarly, consider the equation obtained for $\cos\theta$,

$$\cos\theta = \frac{R_0^4 + R_1^4 - \left(R_0^3 - R_1^3\right)^{\frac{4}{3}}}{2R_0^2 R_1^2}$$

Here, noting that $R_0^4 + R_1^4 \approx R_0^4$ and $R_0^3 - R_1^3 \approx R_0^3$, the right-hand side becomes

$$\cos\theta = \frac{R_0^4 - \left(R_0^3\right)^{\frac{4}{3}}}{2R_0^2 R_0^2} = 0 \to \theta = \arccos 0 = \frac{\pi}{2}$$

or 90°. Finally, noting that $\theta + \phi = \pi/2$, we have

$$\theta + \phi = \frac{\pi}{2} \to \frac{\pi}{2} + \phi = \frac{\pi}{2}$$

$$\therefore \phi = 0$$

which is to say that, when the radius of one of the daughter vessels is much greater than that of the other, the bifurcation approaches a T-shape.

## P.2F ➡ *Solution*

As before, suppose that $R_1 = R_2$. We substitute this result in the expression for $\cos(\theta+\phi)$ to give

$$\cos(\theta+\phi)=\frac{\left(R_1^3+R_1^3\right)^{\frac{4}{3}}-R_1^4-R_1^4}{2R_1^2R_1^2}=\frac{(2R_1^3)^{\frac{4}{3}}-2R_1^4}{2R_1^4}=\frac{2^{\frac{4}{3}}R_1^4-2R_1^4}{2R_1^4}=\frac{\cancel{R_1^4}\left(2^{\frac{4}{3}}-2\right)}{2\cancel{R_1^4}}$$

$$\therefore \cos(\theta+\phi)=\frac{2^{\frac{4}{3}}}{2}-1=2^{\frac{1}{3}}-1=0.2599$$

However, we established in the previous problem that $\theta=\phi$ when $R_1 = R_2$. Thus, the result above becomes

$$\cos 2\theta = 0.2599 \rightarrow \theta = 37.5^{\circ}$$

and, consequently, $\phi = 37.5^{\circ}$ as well.

Note that, when $R_1 = R_2$, we have $R_1/R_0 = 2^{-1/3} = 0.794$, and $\cos\theta = 0.794$, thus $\theta = 37.5^{\circ}$. Let $R_o$ denote the radius of the aorta, and assume equal bifurcation in all generations. Then the radius of the first generation is $0.794R_0$, the radius of the second generation is $0.794^2R_0$, and, in general terms, that of the $n$-th generation is

$$R_n=(0.794)^n R_0$$

Substituting $R_n = 5\times10^{-4}$ cm and $R_0 = 1.5$ cm in the foregoing expression, the number of generations $n$ will be given by

$$5\times10^{-4}=(0.794)^n\times1.5$$

This equation can be easily solved with logarithms, yielding $n = 34.7 \approx 35$. That is, 35 generations of equal bifurcations are needed to reduce the blood vessel from the aorta to the smallest capillary. The number of bifurcations is somewhat larger than that predicted by Young's model ($n = 29$). Since each generation multiplies the number of vessels by 2, the total number of blood vessels is $2^{35} \approx 34.4\times10^9$, or over thirty-four billion conduits. However, the symmetry we assumed cannot be taken too seriously, because arteries seldom bifurcate symmetrically (there is only one symmetric bifurcation in humans, and none in the dog).

### ▶ Problem 3

## P.3A ➡ *Solution*

Many blood vessels are embedded within a particular soft tissue. Examples include the arteries within muscular organs such as the diaphragm, heart, uterus, and skeletal muscle. It is easy to imagine that, as the surrounding tissue deforms, the cross-section of the embedded vessel can likewise change. For example, blood vessels in the heart are compressed by the contracting muscle – indeed, vessels in the left heart are substantially compressed during systole, which is why the left heart is perfused during diastole. In short, there is a need to consider flows in noncircular geometries in certain sites of the human circulatory system.

## P.3B ➡ *Solution*

The flow rate $q$ is obtained by integrating the velocity profile along the $y$- and $z$-axes sequentially. In mathematical terms,

$$q=4\int_0^c\int_0^{b\sqrt{1-z^2/c^2}}u(y,z)\,dydz=4\int_0^c\int_0^{b\sqrt{1-z^2/c^2}}\frac{kb^2c^2}{2\mu\left(b^2+c^2\right)}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)dydz$$

where factor 4 has been included so as to encompass all four quadrants. This integral can be evaluated by hand or using a CAS such as Mathematica. In this software, we apply the command *Integrate* twice along with *Simplify*, namely,

$$\text{Simplify}\left[4 * \text{Integrate}\left[\text{Integrate}\left[\frac{k * b^2 * c^2}{2 * \mu * (b^2 + c^2)}\left(\frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right), \left\{y, 0, b * \sqrt{1 - z^2/c^2}\right\}\right], \{z, 0, c\}\right]\right]$$

The result is

$$q = -\frac{k\pi b^3 c^3}{4\mu\left(b^2 + c^2\right)}$$

which, upon introducing the quantity $\delta$, becomes

$$\boxed{q = -\frac{k\pi}{8\mu}\delta^4}$$

The pumping power $H = \Delta p \times |q|$, where $\Delta p = k \times L$. Accordingly,

$$H = \Delta p \times |q| = kL \times \left(\frac{k\pi}{8\mu}\delta^4\right) = \frac{k^2 \pi L \delta^4}{8\mu} = \boxed{\frac{8\mu L}{\pi}\frac{q^2}{\delta^4}}$$

Now, if the cross-section of the tube were a circle, we'd have $b = c = a$, where $a$ is the radius of the circle, causing the flow rate to become

$$q = -\frac{k\pi}{8\mu}\left(\frac{2b^3c^3}{b^2+c^2}\right)^{\frac{4}{4}} = -\frac{k\pi}{8\mu}\left(\frac{2a^3 \times a^3}{a^2 + a^2}\right) = -\frac{k\pi}{8\mu}\left(\frac{2a^6}{2a^2}\right) = -\frac{k\pi a^4}{8\mu} = \frac{\Delta p \pi a^4}{8\mu L}$$

which is identical to Poiseuille's law for steady flow in a circular tube.

## P.3C ➡ *Solution*

In the first scenario, the pumping power driving the flow is kept unchanged,

$$H_e = H_c = H$$

Since the elliptical section is less efficient than the circular section, the flow rate through the elliptical tube will be reduced. For direct comparison we consider the ratio $\bar{q}$ of flow in the elliptic tube divided by the corresponding flow in the circular tube,

$$\bar{q} = \frac{q_e}{q_c} = \frac{\left(-\frac{k\pi}{8\mu}\delta^4\right)}{\left(-\frac{k\pi}{8\mu}a^4\right)} = \left(\frac{\delta}{a}\right)^4$$

Using the pertaining equations and simplifying yields

$$\bar{q} = \frac{q_e}{q_c} = \frac{\left(-\frac{k\pi}{8\mu}\delta^4\right)}{\left(-\frac{k\pi}{8\mu}a^4\right)} = \left(\frac{\delta}{a}\right)^4 = \frac{\frac{2b^3c^3}{b^2+c^2}}{a^4} = \frac{\frac{2b^3c^3}{b^2+c^2}}{\left[\frac{1}{2}\left(b^2+c^2\right)\right]^2}$$

$$= \frac{\frac{2b^3c^3}{b^2+c^2}}{\frac{1}{4}\left(b^2+c^2\right)^2} = \frac{8b^3c^3}{\left(b^2+c^2\right)^3} = \left(\frac{2bc}{b^2+c^2}\right)^3$$

or, alternatively,

$$\bar{q} = \left(\frac{2\lambda}{1+\lambda^2}\right)^3$$

where $\lambda$ is the ratio of the major axis to the minor axis, $\lambda = c/b$, which in this case is $\lambda = 2$. Substituting this quantity into the relation for $q$, we obtain

$$\bar{q} = \left(\frac{2\times 2}{1+2^2}\right)^3 = \frac{64}{125}$$

That is to say, if the pumping power is held constant, the flow rate for the elliptical cross-section will be a fraction of 64/125 = 0.512 times the flow rate of the circular cross-section. This result is a testament of the circular cross-section's inherently superior efficiency.

Suppose now that the flow rate is kept unchanged, so we may write

$$q_e = q_c = q$$

In view of the circular tube's greater efficiency, the pumping power required to drive flow in the elliptical tube will be higher than that required for the circular tube. For direct comparison we consider the ratio $\bar{H}$ of the power required to drive flow in a tube of elliptic cross-section divided by the corresponding driving power of its circular counterpart,

$$\bar{H} = \frac{\left(\dfrac{8\mu L q^2}{\pi\delta^4}\right)}{\left(\dfrac{8\mu L q^2}{\pi a^4}\right)} = \left(\frac{a}{\delta}\right)^4$$

Substituting these variables and manipulating as we did previously, the result is

$$\bar{H} = \left(\frac{b^2+c^2}{2bc}\right)^3$$

or, introducing the ratio of axes $\lambda$,

$$\bar{H} = \left(\frac{1+\lambda^2}{2\lambda}\right)^3$$

We have $\lambda = 2$; accordingly,

$$\bar{H} = \left(\frac{1+2^2}{2\times 2}\right)^3 = \frac{125}{64}$$

This demonstrates that, for the same flow rate in both tubes, the pumping power required to drive the flow in the elliptic tube is higher by a factor of 125/64 $\approx$ 1.95.

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