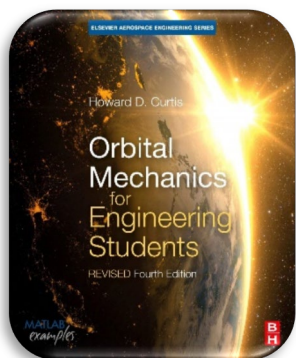# Montogue

Quiz ASQ3
Reviewed Solutions to Curtis's *Orbital Mechanics for Engineering Students*, 4th Ed.

Lucas Monteiro Nogueira

PROBLEM DISTRIBUTION

Chapter	Problems Covered*
2	2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 2.21, 2.22, 2.24, 2.29
3	3.5, 3.6, 3.7, 3.9, 3.10, 3.14, 3.15, 3.16, 3.19, 3.20
4	4.3, 4.5, 4.6, 4.7, 4.14, 4.15, 4.16, 4.18, 4.19
5	5.4, 5.5, 5.6, 5.7, 5.12, 5.13, 5.14, 5.15, 5.16, 5.17, 5.18, 5.22, 5.23, 5.24
6	6.1, 6.8, 6.11, 6.13, 6.14, 6.19, 6.21, 6.23, 6.33, 6.34, 6.35
13	13.1, 13.2, 13.3, 13.4, 13.6

*The corresponding problem numbers for the 3rd edition of the book are also included below.

Download MATLAB files used in the textbook **here**.

PROBLEMS

■ Chapter 2 – The Two-Body Problem

Problem 2.15 (3rd edition: 2.16)

The specific angular momentum of a satellite in circular earth orbit is 60,000 km²/s. Calculate the period.

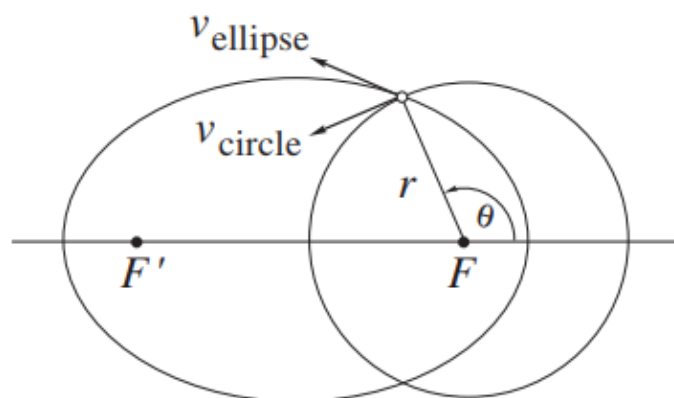### Problem 2.16 (3rd edition: 2.17)

A spacecraft is in a circular orbit of Mars at an altitude of 200 km. Calculate its speed and its period.

Problem 2.17 (3rd edition: 2.18)

Calculate the area A swept out during the time $t = T/4$ since periapsis, where T is the period of the elliptical orbit.

Problem 2.18 (3rd edition: 2.19)

Determine the true anomaly θ of the point(s) on an elliptical orbit at which the speed equals the speed of a circular orbit with the same radius, i.e., $v_{\text{ellipse}} = v_{\text{circle}}$.

Problem 2.19 (3rd edition: 2.20)

Calculate the flight path angle at the locations found in Problem 2.18.

Problem **2.20** (3rd edition: 2.21)

An unmanned satellite orbits the earth with a perigee radius of 10,000 km and an apogee radius of 100,000 km. Calculate
(a) the eccentricity of the orbit;
(b) the semimajor axis of the orbit (km);
(c) the period of the orbit (h);
(d) the specific energy of the orbit (km²/s²);
(e) the true anomaly (degrees) at which the altitude is 10,000 km;
(f) v_r and $v_\perp$ (km/s) at the points found in part (e);
(g) the speed at perigee and apogee (km/s).

Problem **2.21** (3rd edition: 2.22)

A spacecraft is in a 400 km by 600 km low earth orbit. How long (in minutes) does it take to coast from perigee to apogee?

Problem **2.22** (3rd edition: 2.23)

The altitude of a satellite in an elliptical orbit around the earth is 2000 km at apogee and 500 km at perigee. Determine **(a)** the eccentricity of the orbit; **(b)** the orbital speeds at perigee and apogee; **(c)** the period of the orbit.

Problem **2.24** (3rd edition: 2.25)

A satellite is launched into earth orbit at an altitude of 1000 km with a speed of 10 km/s and a flight path angle of 15°. Calculate the true anomaly of the launch point and the period of the orbit.

## Problem **2.29** (3rd edition: 2.30)

For an earth orbiter, the altitude is 1000 km at a true anomaly of 40° and 2000 km at a true anomaly of 150°. Calculate
(a) the eccentricity;
(b) the perigee altitude (km);
(c) the semimajor axis (km).

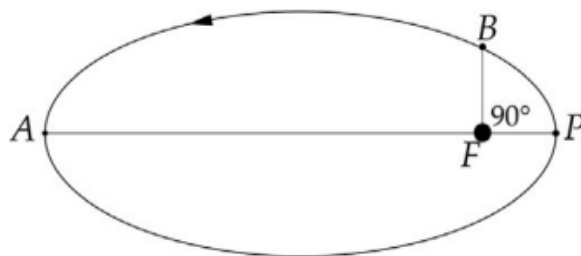## ■ Chapter **3** – Orbital Position as a Function of Time

Problem **3.5** (3rd edition: 3.5)

Calculate the time required to fly from *P* to *B*, in terms of the eccentricity *e* and the period *T*. *B* lies on the minor axis.
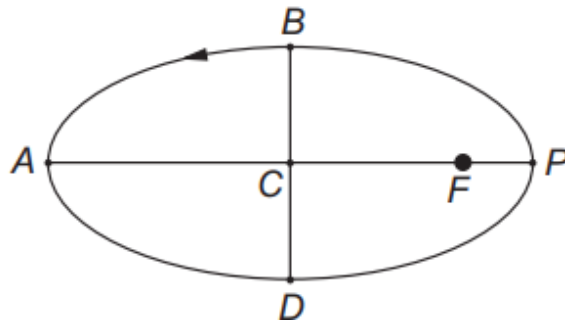
Problem **3.6** (3rd edition: 3.6)

If the eccentricity of the elliptical orbit is 0.3, calculate, in terms of the period *T*, the time required to fly from *P* to *B*.
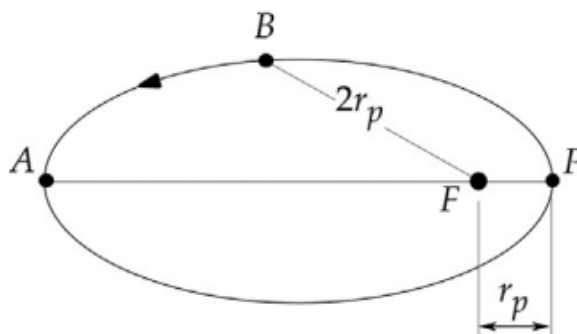

Problem **3.7** (3rd edition: 3.7)

If the eccentricity of the elliptical orbit is 0.5, calculate, in terms of the period *T*, the time required to fly from *P* to *B*.

2

Problem 3.9 (3rd edition: 3.9)

An earth-orbiting satellite has a perigee radius of 7000 km and an apogee radius of 10,000 km.
(a) What true anomaly $\Delta\theta$ is swept out between $t = 0.5$ h and $t = 1.5$ h after perigee passage?
(b) What area is swept out by the position vector during that time interval?

Problem 3.10 (3rd edition: 3.10)

An earth-orbiting satellite has a period of 14 hours and a perigee radius of 10,000 km. At time $t = 10$ hours after perigee passage, determine:
(a) The radial position.
(b) The speed.
(c) The radial component of the velocity.

Problem 3.14 (3rd edition: 3.14)

Calculate the time required for a spacecraft launched into a parabolic trajectory at a perigee altitude of 200 km to leave the earth's sphere of influence (see Table A.2).

Problem 3.15 (3rd edition: 3.15)

A spacecraft on a parabolic trajectory around the earth has a perigee radius of 6600 km.
(a) How long does it take to coast from $\theta = -90$ degrees to $\theta = +90$ degrees?
(b) How far is the spacecraft from the center of the earth 36 hours after passing through perigee?

Problem 3.16 (3rd edition: 3.16)

A spacecraft on a hyperbolic trajectory around the earth has a perigee radius of 6600 km and a perigee speed of $1.2v_{esc}$.
(a) How long does it take to coast from $\theta = -90$ degrees to $\theta = +90$ degrees?
(b) How far is the spacecraft from the center of the earth 24 hours after passing through perigee?

Problem 3.19 (3rd edition: 3.19)

At a given instant the radial position of an earth-orbiting satellite is 7200 km, its radial speed is 1 km/s. If the semimajor axis is 10,000 km, use Algorithm 3.3 to find the universal anomaly 60 minutes later. Check your result using Equation 3.58.

Problem 3.20 (3rd edition: 3.20)

At a given instant a space object has the following position and velocity vectors relative to an earth-centered inertial frame of reference:

$$\mathbf{r}_0 = 20,000\mathbf{i} - 105,000\mathbf{j} - 19,000\mathbf{k}\ \text{km}$$
$$\mathbf{v}_0 = 0.9000\mathbf{i} - 3.4000\mathbf{j} - 1.5000\mathbf{k}\ \text{km/s}$$

Use algorithm 3.4 to find $\mathbf{r}$ and $\mathbf{v}$ 2 h later.

■ Chapter 4 – Orbits in Three Dimensions

Problem 4.3 (3rd edition: 4.3)

Find the orbital elements of a geocentric satellite whose inertial position and velocity vectors in a geocentric equatorial frame are

$$\mathbf{r} = 2500\mathbf{I} + 16,000\mathbf{J} + 4000\mathbf{K}\ \text{km}$$
$$\mathbf{v} = -3\mathbf{I} - \mathbf{J} + 5\mathbf{K}\ \text{km/s}$$

Problem 4.5 (3rd edition: 4.5)

At time t_0 (relative to perigee passage) the position $\mathbf{r}$ and velocity $\mathbf{v}$ of a satellite in the geocentric equatorial frame are

$$\mathbf{r} = 6500\mathbf{I} - 7500\mathbf{J} - 2500\mathbf{K}\ \text{km}$$
$$\mathbf{v} = 4\mathbf{I} + 3\mathbf{J} - 3\mathbf{K}\ \text{km/s}$$

Find the orbital elements.

Problem 4.6 (3rd edition: 4.6)

With respect to the geocentric equatorial frame, the position vector of a spacecraft is $\mathbf{r} = -6000\mathbf{I} - 1000\mathbf{J} - 5000\mathbf{K}$ (km) and the orbit's eccentricity vector is $\mathbf{e} = -0.4\mathbf{I} - 0.5\mathbf{J} - 0.6\mathbf{K}$. Calculate the true anomaly θ if the satellite is approaching perigee.

3

Problem **4.7** (3rd edition: 4.7)

Given that, relative to the geocentric equatorial frame, **r** = −6600**I** − 1300**J** − 5200**K** (km), the eccentricity vector is **e** = −0.4**I** − 0.5**J** − 0.6**K**, and the satellite is flying toward perigee, calculate the inclination of the orbit.

Problem **4.14** (3rd edition: 4.14)

At time t_0 the position **r** and velocity **v** of a satellite in the geocentric equatorial frame are

$$\mathbf{r} = -5000\mathbf{I} - 8000\mathbf{J} - 2100\mathbf{K} \ \ \text{km}$$

$$\mathbf{v} = -4\mathbf{I} + 3.5\mathbf{J} - 3\mathbf{K} \ \ \text{km/s}$$

Find **r** and **v** at time t_0 = +50 min.

Problem **4.15** (3rd edition: 4.15)

At time t_0 (relative to perigee passage), a spacecraft has the following orbital parameters: e = 1.5; perigee altitude = 300 km. i = 35°; Ω = 130°; and ω = 115°. Calculate **r** and **v** at perigee relative to
(a) The perifocal reference frame.
(b) The geocentric equatorial frame.

Problem **4.16** (3rd edition: 4.16)

For the spacecraft of Problem 4.15 calculate **r** and **v** at two hours past perigee relative to
(a) The perifocal reference frame.
(b) The geocentric equatorial frame.

Problem **4.18** (3rd edition: 4.18)

For a spacecraft, the following orbital parameters are given: e = 1.2; perigee altitude = 200 km; i = 50°; Ω = 75°; and ω = 80°. Calculate **r** and **v** at perigee relative to
(a) The perifocal reference frame.
(b) The geocentric equatorial frame.

Problem **4.19** (3rd edition: 4.19)

For the spacecraft of Problem 4.18 calculate **r** and **v** 2 h past perigee relative to
(a) The perifocal reference frame.
(b) The geocentric equatorial frame.

■ Chapter **5** – Preliminary Orbit Determination

Problem **5.4** (3rd edition: 5.4)

At a given instant, the geocentric equatorial position vector of an earth satellite is

$$\mathbf{r}_1 = 3600\mathbf{I} + 4600\mathbf{J} + 3600\mathbf{K} \ \ (\text{km})$$

Thirty minutes later the position is

$$\mathbf{r}_2 = -5500\mathbf{I} + 6240\mathbf{J} - 5200\mathbf{K} \ \ (\text{km})$$

Find the specific energy of the orbit.

Problem **5.5** (3rd edition: 5.5)

Compute the perigee altitude and the inclination of the orbit in the previous problem.

Problem **5.6** (3rd edition: 5.6)

At a given instant, the geocentric equatorial position vector of an earth satellite is

$$\mathbf{r}_1 = 5644\mathbf{I} - 2830\mathbf{J} - 4170\mathbf{K} \ \ \text{km}$$

Some 20 min later, the position is

$$\mathbf{r}_2 = -2240\mathbf{I} + 7320\mathbf{J} + 4980\mathbf{K} \ \ \text{km}$$

Calculate $\mathbf{v_1}$ and $\mathbf{v_2}$.

Problem **5.7** (3rd edition: 5.7)

Compute the orbital elements and perigee altitude for the previous problem.

4

Problem 5.12 (3rd edition: 5.12)

A sea level tracking station whose local sidereal time is 40° and latitude is 35° makes the following observations of a space object:

Azimuth: 36.0°
Azimuth rate: 0.590°/s
Elevation: 36.6°
Elevation rate: −0.263°/s
Range: 988 km
Range rate: 4.86 km/s

What is the state vector of the object?

Problem 5.13 (3rd edition: 5.13)

Calculate the orbital elements of the satellite in the previous problem.

Problem 5.14 (3rd edition: 5.14)

A tracking station at latitude −20° and elevation 500 m makes the following observations of a satellite at the given times.

Time (min)	Local sidereal time (°)	Azimuth (°)	Elevation angle (°)	Range (km)
0	60.0	165.931	9.53549	1214.89
2	60.5014	145.967	45.7711	421.441
4	61.0027	2.40962	21.8825	732.079

Use the Gibbs method to calculate the state vector of the satellite at the central observation time.

Problem 5.15 (3rd edition: 5.15)

Calculate the orbital elements of the satellite in the previous problem.

Problem 5.16 (3rd edition: 5.16)

A sea level tracking station at latitude +29° makes the following observations of a satellite at the given times.

Time (min)	Local sidereal time (°)	Topographic right ascension (°)	Topocentric declination (°)
0.0	0	0	51.5110
1.0	0.250684	65.9279	27.9911
2.0	0.501369	79.8500	14.6609

Use the Gauss method without iterative improvement to estimate the state vector of the satellite at the middle observation time.

Problem 5.17 (3rd edition: 5.17): Refine the estimate in the previous problem using iterative improvement.

Problem 5.18 (3rd edition: 5.18): Calculate the orbital elements from the state vector obtained in the previous problem.

Problem 5.22 (3rd edition: 5.22)

The position vector **R** of a tracking station and the direction cosine vector ρ of a satellite relative to the tracking station at three times are as follows:

$$for\ t_1 = 0\ min$$

$$\mathbf{R}_1 = -1825.96\mathbf{I} + 3583.66\mathbf{J} + 4933.54\mathbf{K}\ \ (km)$$

$$\boldsymbol{\rho}_1 = -0.301687\mathbf{I} + 0.200673\mathbf{J} + 0.932049\mathbf{K}$$

$$for\ t_2 = 1\ min$$

$$\mathbf{R}_2 = -1841.63\mathbf{I} + 3575.63\mathbf{J} + 4933.54\mathbf{K}\ \ (km)$$

$$\boldsymbol{\rho}_2 = -0.793090\mathbf{I} - 0.210324\mathbf{J} + 0.571640\mathbf{K}$$

$$for\ t_3 = 2\ min$$

$$\mathbf{R}_3 = -1857.25\mathbf{I} + 3567.54\mathbf{J} + 4933.54\mathbf{K}\ \ (km)$$

$$\boldsymbol{\rho}_3 = -0.873085\mathbf{I} - 0.362969\mathbf{J} + 0.325539\mathbf{K}$$

Use the Gauss method without iterative improvement to estimate the state vector of the satellite at the central observation time.

> **Note: In some printings of the book, the vector component highlighted in red appears as −1816.30 instead of −1841.63.**

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Problem 5.23 (3rd edition: 5.23): Refine the estimate in the previous problem using iterative improvement.

Problem 5.24 (3rd edition: 5.24): Calculate the orbital elements from the state vector obtained in the previous problem.

■ Chapter 6 – Orbital Maneuvers
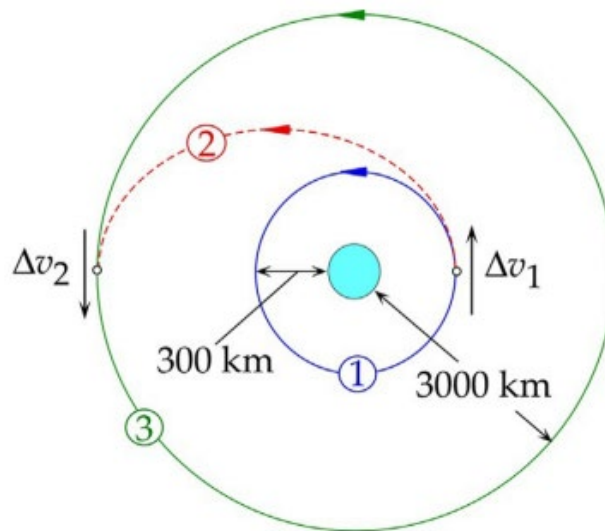
Problem 6.1 (3rd edition: 6.1)

A large spacecraft has a mass of 125,000 kg. Its orbital maneuvering engines produce a thrust of 50 kN. The spacecraft is in a 400 km circular earth orbit. A delta-*v* maneuver transfers the spacecraft to a coplanar 300 km by 400 km elliptical orbit. Neglecting propellant loss and using elementary physics (linear impulse equals change in linear momentum, distance equals speed times time), estimate

(a) the time required for the Δ*v* burn and

(b) the distance travelled by the spacecraft during the burn.

(c) Calculate the ratio of your answer for (b) to the circumference of the initial circular orbit.

(d) What percent of the initial mass was expelled as combustion products?
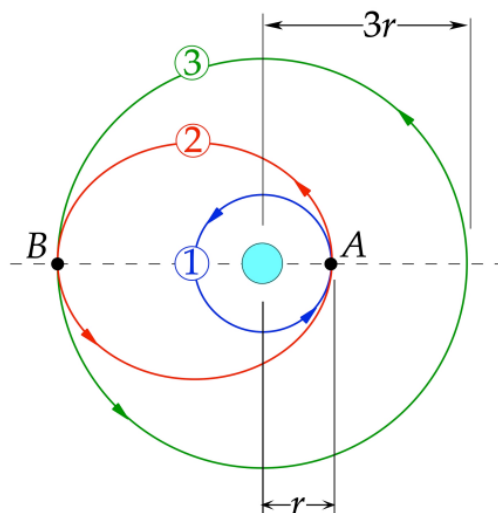
Problem 6.8 (3rd edition: 6.8)

A spacecraft is in a 300-km circular earth orbit. Calculate

(a) The total delta-*v* required for a Hohmann transfer to a 3000-km coplanar circular earth orbit.

(b) The transfer orbit time.

Problem 6.11 (3rd edition: 6.11)

Calculate the total delta-*v* required for a Hohmann transfer from the smaller circular orbit to the larger one.


6

Problem 6.13 (3rd edition: 6.13)

Two geocentric elliptical orbits have common apse lines and their perigees are on the same side of the earth. The first orbit has a perigee radius of $r_p = 7000$ km and $e = 0.3$, whereas for the second orbit $r_p = 32,000$ km and $e = 0.5$.

(a) Find the minimum total delta-v and the time of flight for a transfer from the perigee of the inner orbit to the apogee of the outer orbit.

(b) Do part (a) for a transfer from the apogee of the inner orbit to the perigee of the outer orbit.
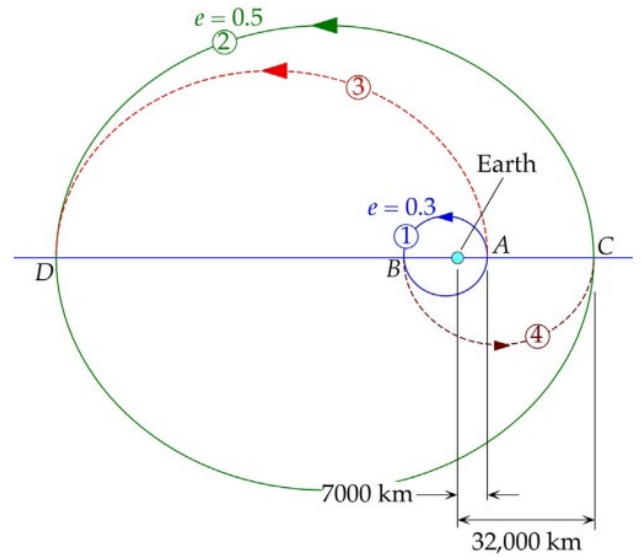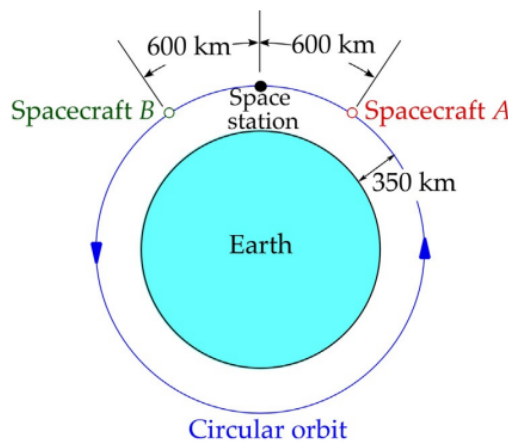

Problem 6.14 (3rd edition: 6.14)

The space shuttle was launched on a 15-day mission. There were four orbits after injection, all of them at 39° inclination.

→ Orbit 1: 302 km by 296 km
→ Orbit 2 (day 11): 291 km by 259 km
→ Orbit 3: (day 12): 259 km circular
→ Orbit 4: (day 13): 255 km by 194 km

Calculate the total delta-v, which should be as small as possible, assuming Hohmann transfers.

Problem 6.19 (3rd edition: 6.19)

The space station and spacecraft A and B are all in the same circular earth orbit of 350 km altitude. Spacecraft A is 600 km behind the space station and spacecraft B is 600 km ahead of the space station. At the same instant, both spacecraft apply a $\Delta v_\perp$ so as to arrive at the space station in one revolution of their phasing orbits.

(a) Calculate the time required for each spacecraft to reach the space station.
(b) Calculate the total delta-v requirement for each spacecraft.

Problem 6.21 (3rd edition: 6.21)

Two spacecraft are in the same elliptical earth orbit with perigee radius 8000 km and apogee radius 13,000 km. Spacecraft 1 is at perigee and spacecraft 2 is 30° ahead. Calculate the total delta-v required for spacecraft 1 to intercept and rendezvous with spacecraft 2 when spacecraft 2 has traveled 60°.

7
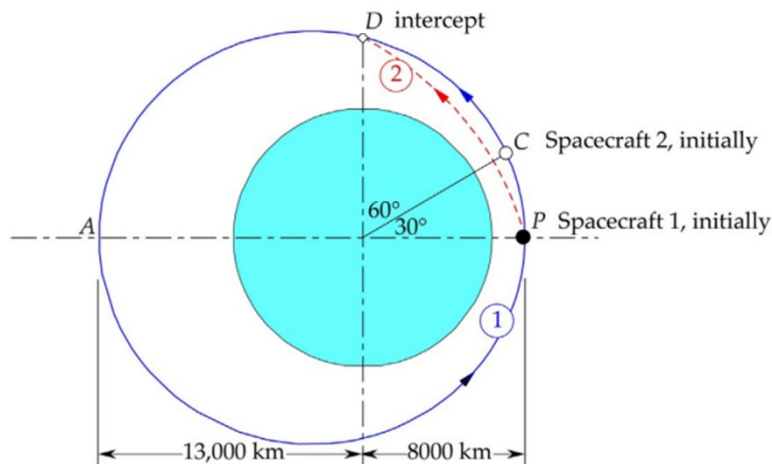
Spacecraft *B* and *C*, which are in the same elliptical earth orbit 1, are located in the true anomalies shown. At this instant, spacecraft *B* executes a phasing maneuver so as to rendezvous with spacecraft *C* after one revolution of its phasing orbit 2. Calculate the total delta-*v* required. Note that the apse line of orbit 2 is at 45° to that of orbit 1.

Spacecraft *A* and *B* are in concentric, coplanar circular orbits 1 and 2, respectively. At the instant shown, spacecraft *A* executes an impulsive delta-*v* maneuver to embark on orbit 3 to intercept and rendezvous with spacecraft *B* in a time equal to the period of orbit 1. Calculate the total delta-*v* required.
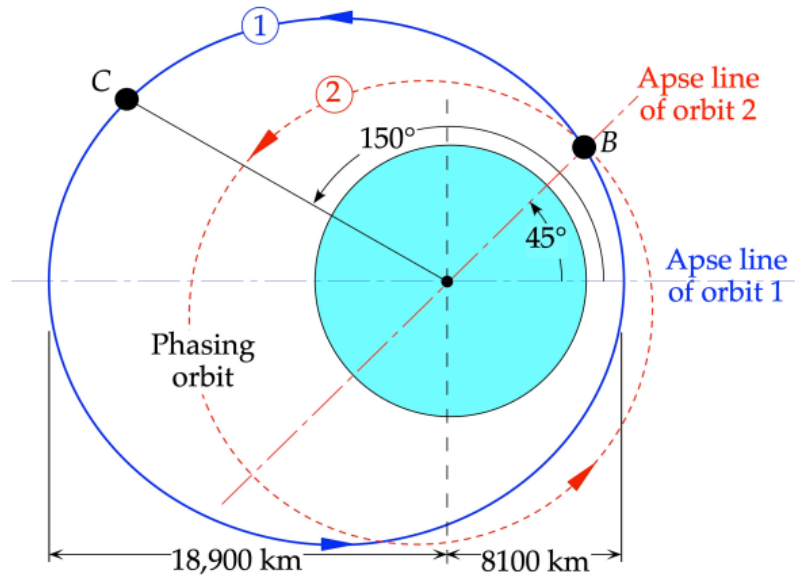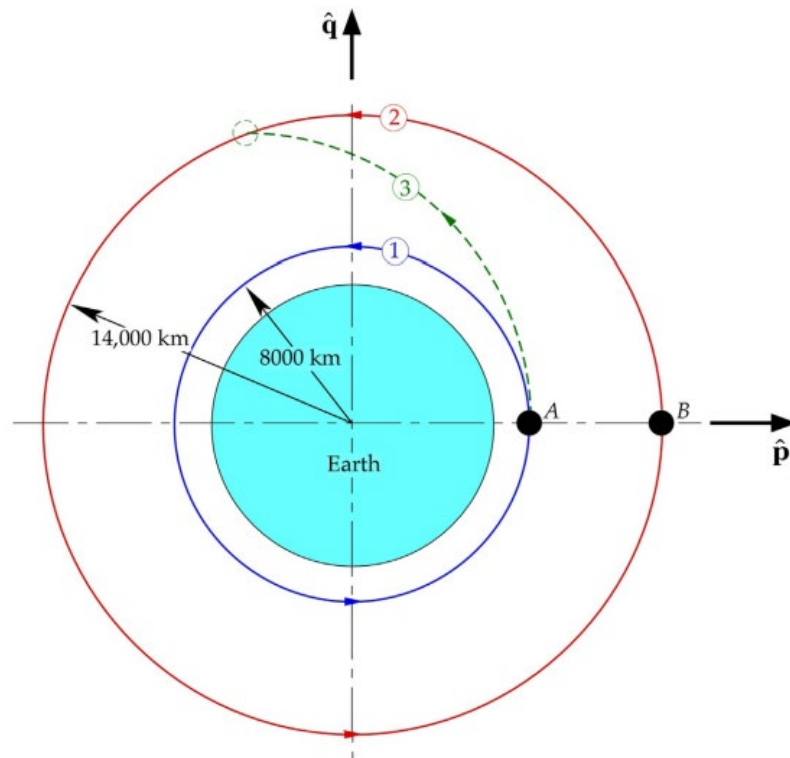
Spacecraft *A* is in orbit 1, a 10,000 km radius equatorial earth orbit. Spacecraft *B* is in elliptical polar orbit 2, having eccentricity 0.5 and perigee radius 16,000 km. At the instant shown, both spacecraft are in the equatorial plane and *B* is at its perigee. At that instant, spacecraft *A* executes an impulsive delta-*v* maneuver to intercept *B* one hour later at point *C*. Calculate the delta-*v* required for *A* to switch to the intercept trajectory 3.

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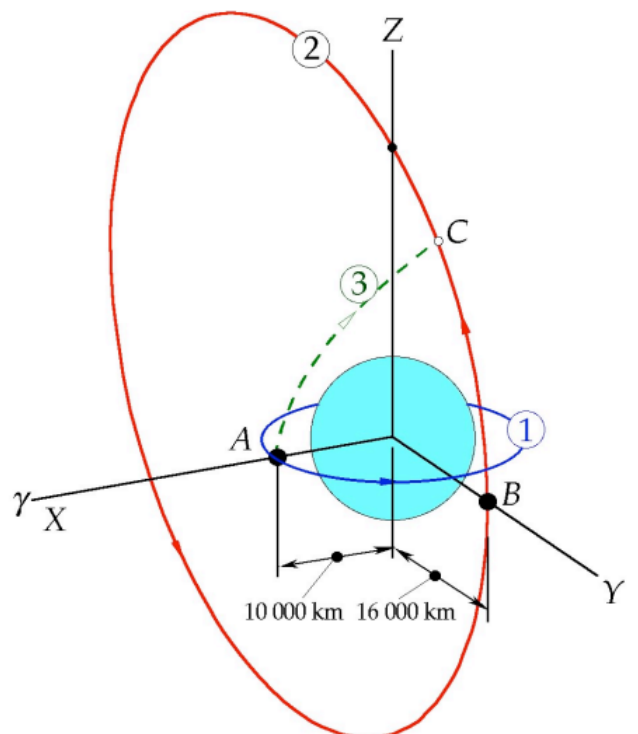
Problem 6.35 (3rd edition: 6.35)

Spacecraft B and C are in the same elliptical orbit 1, characterized by a perigee radius of 7000 km and an apogee radius of 10,000 km. The spacecraft are in the position shown when B executes an impulsive transfer to orbit 2 to catch and rendezvous with C when C arrives at apogee A. Find the total delta-v requirement.
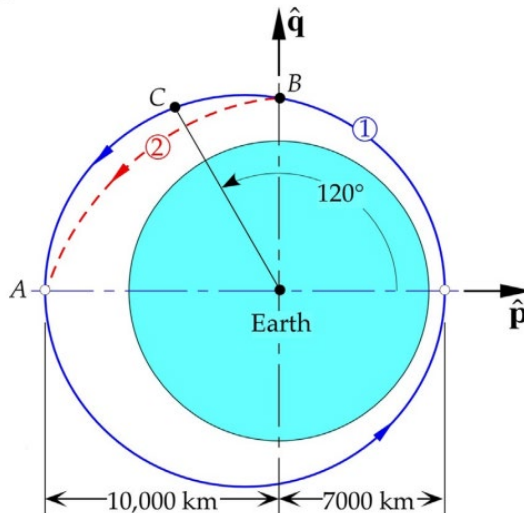
■ Chapter 13 – Rocket Vehicle Dynamics (Chapter 11 in 3rd edition)

Problem 13.1 (3rd edition: 11.1) [Answer does not match textbook]

A two stage, solid-propellant sounding rocket has the following properties.

First stage: m_0 = 249.5 kg, m_f = 170.1 kg, $\dot{m}_e$ = 10.61 kg/s, I_{sp} = 235 s

Second stage: m_0 = 113.4 kg, m_f = 58.97 kg, $\dot{m}_e$ = 4.053 kg/s, I_{sp} = 235 s

The delay time between burnout of first stage and ignition of second stage is 3 s. As a preliminary estimate, neglect drag and the variation of earth's gravity with altitude to calculate the maximum height reached by the second stage after burnout.

Problem 13.2 (3rd edition: 11.2)

A two-stage launch vehicle has the following properties:

First stage: Two solid propellant rockets. Each one has a total mass of 525,000 kg, 450,000 kg of which is propellant. I_{sp} = 290 s.

Second stage: Two liquid rockets with I_{sp} = 450 s. Dry mass = 30,000 kg, propellant mass = 600,000 kg.

Calculate the payload mass to a 300 km orbit if launched due east from KSC. Let the total gravity and drag loss be 2 km/s.

Problem 13.3 (3rd edition: 11.3)

Suppose a spacecraft in permanent orbit around the earth is to be used for delivering payloads from low earth orbit (LEO) to geostationary equatorial orbit (GEO). Before each flight from LEO, the spacecraft is refueled with propellant, which it uses up in its round trip to GEO. The outbound leg requires four times as much propellant as the inbound return leg. The delta-v for transfer from LEO to GEO is 4.22 km/s. The specific impulse of the propulsion system is 430 s. If the payload mass is 3500 kg, calculate the empty mass of the vehicle.

> **Note: The number highlighted in red appears as 450 s in some printings of the 4th edition.**

Problem 13.4 (3rd edition: 11.4)

Consider a rocket comprising similar stages (i.e., each stage has the same specific impulse, structural ratio, and payload ratio). The common specific impulse is 310 s. The total mass of the vehicle is 150,000 kg, the total structural mass (empty mass) is 20,000 kg, and the payload mass is 10,000 kg. Calculate

(a) The mass ratio n and the total Δv for the three-stage rocket.

(b) $m_{p,1}$, $m_{p,2}$, and $m_{p,3}$

(c) $m_{E,1}$, $m_{E,2}$, and $m_{E,3}$

(d) $m_{0,1}$, $m_{0,2}$, and $m_{0,3}$

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A small two-stage vehicle is to propel a 10-kg payload to a speed of 6.2 km/s. The properties of the stages are as follows. For the first stage, $I_{sp} = 300$ s and $\varepsilon = 0.2$. For the second stage, $I_{sp} = 235$ s and $\varepsilon = 0.3$. Estimate the optimum mass of the vehicle.

SOLUTIONS

■ P2.15 (Precision: Minimum)

We first compute the radial distance of the orbit:

$$r = \frac{h^2}{\mu} = \frac{60,000^2}{398,600} = 9030 \text{ km}$$

Then, the period follows:

$$T = \frac{2\pi}{\sqrt{\mu}} r^{3/2} = \frac{2\pi}{\sqrt{398,600}} \times 9030^{3/2} = \boxed{8540 \text{ s}}$$

or 2 h 22 min.

■ P2.16 (Precision: Minimum)

For Mars, $\mu = 42,828$ km^3/s^2 and $r = 3396$ km. The speed of the spacecraft is then

$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{42,828}{3396 + 200}} = \boxed{3.45 \text{ km/s}}$$

The period is, in turn,

$$T = \frac{2\pi}{\sqrt{\mu}} r^{3/2} = \frac{2\pi}{\sqrt{42,828}} \times (3396 + 200)^{3/2} = \boxed{6550 \text{ s}}$$

or 1 h 49 min.

■ P2.17 (Precision: Minimum)

The ratio of area swept to the corresponding period must remain constant. Accordingly,

$$\frac{\Delta A}{\Delta t} = \frac{\pi ab}{T} = \frac{A}{T/4}$$

$$\therefore \pi ab = 4A$$

$$\therefore A = \frac{\pi ab}{4} = \boxed{0.785ab}$$

■ P2.18

For the ellipse, per Problem 2.9,

$$v_{\text{ellipse}}^2 = \frac{\mu^2}{h^2}\left(e^2 + 2e\cos\theta + 1\right)$$

For the circle, at the point of intersection with the ellipse,

$$v_{\text{circle}}^2 = \frac{\mu}{r} = \frac{\mu}{\dfrac{h^2}{\mu}\dfrac{1}{(1+e\cos\theta)}} = \frac{\mu^2}{h^2}\left(1 + e\cos\theta\right)$$

Equating the two previous expressions,

$$v_{\text{ellipse}}^2 = v_{\text{circle}}^2 \rightarrow \frac{\mu^2}{h^2}\left(e^2 + 2e\cos\theta + 1\right) = \frac{\mu^2}{h^2}\left(1 + e\cos\theta\right)$$

$$\therefore e^2 + 2e\cos\theta + 1 = 1 + e\cos\theta$$

$$\therefore e^2 + e\cos\theta = 0$$

$$\therefore e + \cos\theta = 0$$

$$\therefore \boxed{\theta = \cos^{-1}(-e)}$$

■ P2.19

The flight path angle γ is given by

$$\tan \gamma = \frac{e \sin \theta}{1 + e \cos \theta}$$

In the case at hand, $\theta = \cos^{-1}(-e)$, giving

$$\tan \gamma = \frac{e \times \sin \left[\arccos(-e) \right]}{1 + e \cos \left[\arccos(-e) \right]} = \frac{e \times \sin \left[\arccos(-e) \right]}{1 - e^2}$$

But $\sin[\arccos(-e)] = \sqrt{1 - e^2}$, so that

$$\tan \gamma = \frac{e \times \sqrt{1 - e^2}}{1 - e^2} = \frac{e}{\sqrt{1 - e^2}}$$

$$\boxed{\gamma = \tan^{-1} \left(\frac{e}{\sqrt{1 - e^2}} \right)}$$

■ **P2.20 (Precision: Minimum)**

Part (a): The eccentricity of the orbit is given by

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{100,000 - 10,000}{100,000 + 10,000} = \boxed{0.818}$$

Since this eccentricity value is between 0 and 1, the orbit is elliptical.

Part (b): The semimajor axis of the orbit is

$$a = \frac{r_{\text{apogee}} + r_{\text{perigee}}}{2} = \frac{100,000 + 10,000}{2} = \boxed{55,000 \, \text{km}}$$

Part (c): The period of the orbit is

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398,600}} \times 55,000^{3/2} = 128,400 \, \text{s}$$

$$\therefore \boxed{T = 35.7 \, \text{h}}$$

Part (d): The specific energy of the orbit is

$$\varepsilon = -\frac{\mu}{2a} = -\frac{398,600}{2 \times 55,000} = \boxed{-3.62 \, \text{km}^2/\text{s}^2}$$

Part (e): The true anomaly θ can be found from the orbit equation (2.72) on page 81:

$$r = a \frac{1 - e^2}{1 + e \cos \theta}$$

Substituting $r = 6378 + 10,000 = 16,378$ and other variables brings to

$$16,378 = 55,000 \times \frac{1 - 0.818^2}{1 + 0.818 \times \cos \theta}$$

$$\therefore 16,378 = \frac{18,200}{1 + 0.818 \times \cos \theta}$$

$$\therefore 1 + 0.818 \cos \theta = \frac{18,200}{16,378}$$

$$\therefore 1 + 0.818 \cos \theta = 1.11$$

$$\therefore 0.818 \cos \theta = 0.11$$

$$\therefore \cos \theta = 0.134$$

$$\therefore \theta = \arccos(0.134) = \boxed{82.3°}$$

Part (f): We first compute the specific angular momentum via equation (2.50):

$$r_p = \frac{h^2}{\mu} \frac{1}{1 + e} \rightarrow h = \sqrt{\mu(1 + e) r_{\text{perigee}}}$$

$$\therefore h = \sqrt{398,600 \times (1 + 0.818) \times 10,000} = 85,130 \, \text{km}^2/\text{s}$$

Then, the azimuthal component of velocity is calculated to be

$$v_\perp = \frac{h}{r} = \frac{85{,}130}{6378 + 10{,}000} = \boxed{5.20 \text{ km/s}}$$

while the radial component is given by equation (2.49):

$$v_r = \frac{\mu}{h} e \sin\theta = \frac{398{,}600}{85{,}130} \times 0.818 \times \sin(82.3°) = \boxed{3.80 \text{ km/s}}$$

Part (g): Finding the speeds at the perigee and apogee is straightforward:

$$v_{\text{perigee}} = \frac{h}{r_{\text{perigee}}} = \frac{85{,}130}{10{,}000} = \boxed{8.51 \text{ km/s}}$$

$$v_{\text{apogee}} = \frac{h}{r_{\text{apogee}}} = \frac{85{,}130}{100{,}000} = \boxed{0.851 \text{ km/s}}$$

■ **P2.21 (Precision: Minimum)**

The radius of the perigee is $r_{\text{perigee}} = 6378 + 400 = 6778$ km, while that of the apogee is $r_{\text{apogee}} = 6378 + 800 = 7178$ km. The semimajor axis of the orbit is determined as

$$a = \frac{r_{\text{perigee}} + r_{\text{apogee}}}{2} = \frac{6778 + 7178}{2} = 6978 \text{ km}$$

The orbital period is then

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398{,}600}} \times 6978^{3/2} = 5800 \text{ s}$$

$$\therefore T = 96.7 \text{ min}$$

The time required to coast from perigee to apogee should equal $T/2 = 48.35$ minutes.

■ **P2.22 (Precision: Minimum)**

Part (a): The radius of the apogee is $r_{\text{apogee}} = 6378 + 2000 = 8378$ km, while the radius of the perigee is $r_{\text{perigee}} = 6378 + 500 = 6878$ km. The eccentricity of the orbit is determined as

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{8378 - 6878}{8378 + 6878} = \boxed{0.0983}$$

Part (b): We first compute the specific angular momentum via equation (2.50):

$$h = \sqrt{\mu(1+e)r_{\text{perigee}}} = \sqrt{398{,}600 \times (1 + 0.0983) \times 6878}$$

$$\therefore h = 54{,}900 \text{ km}^2/\text{s}$$

so that

$$v_{\text{perigee}} = \frac{h}{r_{\text{perigee}}} = \frac{54{,}900}{6878} = \boxed{7.98 \text{ km/s}}$$

$$v_{\text{apogee}} = \frac{h}{r_{\text{apogee}}} = \frac{54{,}900}{8378} = \boxed{6.55 \text{ km/s}}$$

Part (c): The period of a typical elliptical orbit is given by equation (2.82):

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 = \frac{2\pi}{398{,}600^2}\left(\frac{54{,}900}{\sqrt{1-0.0983^2}}\right)^3 = 6640 \text{ s}$$

$$\therefore \boxed{T = 111 \text{ min}}$$

■ **P2.24 (Precision: Minimum)**

We first establish the radial and azimuthal components of velocity:

$$v_r = v \sin\gamma = 10 \times \sin 15° = 2.59 \text{ km/s}$$
$$v_\perp = v \cos\gamma = 10 \times \cos 15° = 9.66 \text{ km/s}$$

The specific angular momentum is then

$$h = r v_\perp = (6378 + 1000) \times 9.66 = 71{,}300 \text{ km}^2/\text{s}$$

Next, we insert the available data into (2.71) and (2.72) to obtain

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$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta} \to 7378 = \frac{71,300^2}{398,600} \times \frac{1}{1 + e\cos\theta}$$

$$\therefore e\cos\theta = 0.729 \quad (I)$$

In a similar manner, we substitute the available data into the equation for radial velocity (eq. (2.49)):

$$v_r = \frac{\mu}{h} e\sin\theta \to 2.59 = \frac{398,600}{71,300} e\sin\theta$$

$$\therefore e\sin\theta = 0.463 \quad (II)$$

Dividing (II) by (I) gives the true anomaly θ:

$$\frac{e\sin\theta}{e\cos\theta} = \frac{0.463}{0.729} \to \tan\theta = 0.635$$

$$\therefore \theta = \arctan(0.635) = \boxed{32.4^\circ}$$

Now, substituting $\theta = 32.4^\circ$ into (I) and solving for eccentricity,

$$e\cos\theta = 0.729 \to e = \frac{0.729}{\cos(32.4^\circ)} = 0.863$$

Finally, we determine the period using equation (2.82):

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 = \frac{2\pi}{398,600^2}\left(\frac{71,300}{\sqrt{1-0.863^2}}\right)^3 = 111,200\,\text{s}$$

$$\therefore \boxed{T = 30.9\,\text{h}}$$

■ **P2.29** (Precision: Minimum)

Parts (a) and (b): Inserting the available data into eqs. (2.71)/(2.72), we have, for each data point,

$$r_1 = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta_1} \to (6378 + 1000) = \frac{h^2}{398,600} \frac{1}{(1 + e\cos 40^\circ)} \quad (I)$$

$$r_2 = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta_2} \to (6378 + 2000) = \frac{h^2}{398,600} \frac{1}{(1 + e\cos 150^\circ)} \quad (II)$$

Dividing (II) by (I) and solving for e, we obtain

$$\frac{(6378 + 1000)}{(6378 + 2000)} = \frac{\cancel{h^2/398,600} \times \dfrac{1}{1 + e\cos 40^\circ}}{\cancel{h^2/398,600} \times \dfrac{1}{1 + e\cos 150^\circ}}$$

$$\therefore \frac{(6378 + 1000)}{(6378 + 2000)} = \frac{1 + e\cos 150^\circ}{1 + e\cos 40^\circ}$$

Entering this linear equation into Mathematica to save time, we obtain $e = 0.0775$, as shown.

```
In[167]:= Solve[ (6378 + 1000)/(6378 + 2000.) == (1 + e * Cos[150 °])/(1 + e * Cos[40 °]), e]
```

⋯ **Solve**: Solve was unable to solve the system with inexact coefficients.

The answer was obtained by solving a corresponding exact system and numericizing the result.

```
Out[167]=
        {{e → 0.0774747}}
```

Now, substituting e into either of equations (I) or (II) yields the specific angular momentum h:

$$(6378 + 1000) = \frac{h^2}{398,600} \frac{1}{(1 + 0.0775 \times \cos 40^\circ)}$$

$$\therefore h = 55,820 \text{ km}^2/\text{s}$$

The perigee radial distance is determined next:

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$$r_{\text{perigee}} = \frac{h^2}{\mu} \frac{1}{1+e} = \frac{55{,}820^2}{398{,}600} \times \frac{1}{1+0.0775} = 7250 \text{ km}$$

Subtracting the earthly radius:

$$z_{\text{perigee}} = r_{\text{perigee}} - 6378 = 7250 - 6378 = \boxed{872 \, \text{km}}$$

Part (c): The semimajor axis can be found via equation (2.73):

$$a = \frac{r_{\text{perigee}}}{1-e} = \frac{7250}{1-0.0775} = \boxed{7860 \text{ km}}$$

■ **P3.5**

On the minor axis, the eccentric anomaly is $\pi/2$, so that

$$M_B = E_B - e \sin E_B = \frac{\pi}{2} - e \sin\left(\frac{\pi}{2}\right)$$

$$\therefore M_B = \frac{\pi}{2} - e \ \ \text{(I)}$$

Now, the time required to reach a given angle is related to the mean anomaly M_B as

$$t_B = \frac{T}{2\pi} M_B \rightarrow M_B = \frac{2\pi}{T} t_B$$

so that, substituting in (I),

$$M_B = \frac{\pi}{2} - e = \frac{2\pi}{T} t_B$$

$$\therefore \frac{1}{4} - \frac{e}{2\pi} = \frac{1}{T} t_B$$

$$\therefore \boxed{t_B = \left(\frac{1}{4} - \frac{e}{2\pi}\right) T}$$

■ **P3.6** (Precision: Average)

We first determine the eccentric anomaly associated with point B:

$$\tan\frac{E_B}{2} = \sqrt{\frac{1-e}{1+e}} \tan\left(\frac{\theta_B}{2}\right) = \sqrt{\frac{1-0.3}{1+0.3}} \times \tan\left(\frac{\pi/2}{2}\right) = 0.734$$

$$\therefore E_B = 2 \arctan(0.734) = 1.266 \text{ rad}$$

Next, we compute the mean anomaly M_B:

$$M_B = E_B - e \sin E_B = 1.266 - 0.3 \times \sin(1.266) = 0.980 \text{ rad}$$

The time required to fly from P to B is then

$$t_B = \frac{M_B}{2\pi} T = \frac{0.980}{2\pi} \times T = \boxed{0.156T}$$

■ **P3.7** (Precision: Average)

We combine equations (see Section 2.7)

$$r = \frac{h^2}{\mu} \frac{1}{1+e\cos\theta}$$

and

$$r_{\text{perigee}} = \frac{h^2}{\mu} \frac{1}{1+e}$$

to obtain

$$r = \frac{r_{\text{perigee}}(1+e)}{1+e\cos\theta}$$

Substituting $r = 2r_{perigee}$ and solving for anomaly θ, we obtain

$$2\,\cancel{r_{\text{perigee}}} = \frac{\cancel{r_{\text{perigee}}}(1+0.5)}{1+0.5\cos\theta}$$

$$\therefore 2 = \frac{1.5}{1+0.5\cos\theta}$$

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Solving this equation with Mathematica to save time, we obtain $\cos \theta = -0.5$, so that $\theta = 120°$.

In[285]:= $\textbf{Solve}\left[2 == \dfrac{1.5}{1 + 0.5 * \textbf{Cos}\theta}, \textbf{Cos}\theta\right]$

Out[285]=
$\{\{\textbf{Cos}\theta \to -0.5\}\}$

We proceed to compute eccentric anomaly E_B,

$$\tan\left(\frac{E_B}{2}\right) = \sqrt{\frac{1-e}{1+e}} \tan\left(\frac{\theta_B}{2}\right) = \sqrt{\frac{1-0.5}{1+0.5}} \times \tan\left(\frac{120°}{2}\right) = 1.0$$

$$\therefore E_B = 2 \arctan(1.0) = 2 \times \frac{\pi}{4} = \frac{\pi}{2} \text{ rad}$$

and then the mean anomaly M_B:

$$M_B = E_B - e \sin E_B = \frac{\pi}{2} - 0.5 \times \sin\left(\frac{\pi}{2}\right) = 1.071 \text{ rad}$$

Finally, the time required to fly from P to B is

$$t_B = \frac{M_B}{2\pi} T = \frac{1.071}{2\pi} \times T = \boxed{0.170T}$$

■ **P3.9** (Precision: Average)

Part (a): The semimajor axis of the orbit is

$$a = \frac{1}{2}\left(r_{\text{apogee}} + r_{\text{perigee}}\right) = \frac{1}{2} \times (10,000 + 7000) = 8500 \text{ km}$$

The corresponding period is

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398,600}} \times 8500^{3/2} = 7799 \text{ s}$$

and the eccentricity is

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{10,000 - 7000}{10,000 + 7000} = 0.176$$

Consider first conditions at $t_1 = 0.5$ h = 1800 s. The mean anomaly M_1 is

$$M_1 = \frac{2\pi t_1}{T} = \frac{2\pi \times 1800}{7799} = 1.450 \text{ rad}$$

while the eccentric anomaly can be found as

$$E_1 - e \sin E_1 = M_1$$
$$\therefore E_1 - 0.176 \sin E_1 = 1.450 \text{ rad}$$

This transcendental equation can be solved for E_1 in MATLAB by creating an anonymous function and then applying *fsolve*:

```
>> orbit = @(E1) (E1 - 0.176*sin(E1) - 1.45)
x0 = 1.0
fsolve(orbit, x0)
```

This returns $E_1 = 1.626$ rad. The corresponding anomaly θ_1 follows as

$$\tan\left(\frac{\theta_1}{2}\right) = \sqrt{\frac{1+e}{1-e}} \tan\left(\frac{E_1}{2}\right) = \sqrt{\frac{1+0.176}{1-0.176}} \tan\left(\frac{1.626}{2}\right) = 1.262$$

$$\therefore \theta_1 = 2 \arctan(1.262) = 1.801 \text{ rad}$$
$$\therefore \theta_1 = 103.2°$$

Calculations for $t_2 = 1.5$ h = 5400 s are analogous:

$$M_2 = \frac{2\pi t_2}{T} = \frac{2\pi \times 5400}{7799} = 4.350 \text{ rad}$$

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$$E_2 - e\sin E_2 = M_2$$

$$\therefore E_2 - 0.176\sin E_2 = 4.350$$

$$\therefore E_2 = 4.197 \text{ rad}$$

$$\tan\left(\frac{\theta_2}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E_2}{2}\right) = \sqrt{\frac{1+0.176}{1-0.176}}\tan\left(\frac{4.197}{2}\right) = -2.049$$

$$\therefore \theta_2 = 2\arctan(-2.049) = -2.234 \text{ rad}$$

$$\therefore \theta_2 = -128.0^\circ = 232.0^\circ$$

It remains to compute the anomaly $\Delta\theta$ swept by the orbit:

$$\Delta\theta = \theta_2 - \theta_1 = 232.0^\circ - 103.2^\circ = \boxed{128.8^\circ}$$

Part (b): This part is started by computing the specific angular momentum h,

$$h = \sqrt{\mu r_{\text{perigee}}(1+e)} = \sqrt{398,600 \times 7000 \times (1+0.176)} = 57,280 \text{ km}^2/\text{s}$$

$$\Delta A = \frac{1}{2}h\Delta t = \frac{1}{2} \times 57,280 \times 3600 = 1.031 \times 10^8 \text{ km}^2$$

$$\therefore \boxed{\Delta A = 103.1 \times 10^6 \text{ km}^2}$$

■ **P3.10** (Precision: Average)

Part (a): The first step is to determine the apogee radius, which in turn requires the semimajor axis a:

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} \rightarrow 14 \times 3600 = \frac{2\pi}{\sqrt{398,600}}a^{3/2}$$

$$\therefore a = 29,490 \text{ km}$$

Therefore,

$$a = \frac{1}{2}\left(r_{\text{perigee}} + r_{\text{apogee}}\right) \rightarrow 29,490 = \frac{1}{2} \times \left(10,000 + r_{\text{apogee}}\right)$$

$$\therefore r_{\text{apogee}} = 48,980 \text{ km}$$

The eccentricity of the orbit easily follows:

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{48,980 - 10,000}{48,980 + 10,000} = 0.661$$

The mean anomaly is then

$$M = 2\pi\frac{t}{T} = 2\pi \times \frac{10}{14} = 4.488 \text{ rad}$$

while the eccentric anomaly is

$$E - E \times e\sin E = M \rightarrow E - 0.661\sin E = 4.488$$

Solving for E with MATLAB's *fsolve* gives $E = 3.992$ rad.

```
>> orbit = @(E1) (E1 - 0.661*sin(E1) - 4.488)
x0 = 1.0
fsolve(orbit, x0)
ans =

    3.9915
```

Next, we compute true anomaly θ:

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1+0.661}{1-0.661}} \times \tan\left(\frac{3.992}{2}\right) = -4.888$$

$$\therefore \theta = 2\arctan(-4.888) = -2.738 \text{ rad} = -156.9^\circ$$

$$\therefore \theta = 203.1^\circ$$

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Finally, the radial position is

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\theta} = \frac{29,490\times\left(1-0.661^2\right)}{1+0.661\times\cos\left(203.1^\circ\right)} = \boxed{42,360\ \text{km}}$$

Part (b): The velocity can be determined with the energy equation:

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \rightarrow \frac{v^2}{2} - \frac{398,600}{42,360} = -\frac{398,600}{2\times 29,490}$$

$$\therefore \frac{v^2}{2} - 9.410 = -6.758$$

$$\therefore v = \sqrt{2\left(-6.758+9.410\right)} = \boxed{2.303\ \text{km/s}}$$

Part (c): To find the radial component of velocity, we first require the specific angular momentum h:

$$h = \sqrt{\mu a\left(1-e^2\right)} = \sqrt{398,600\times 29,490\times\left(1-0.661^2\right)} = 81,360\ \text{km}^2/\text{s}$$

Then,

$$v_r = \frac{\mu}{h}e\sin\theta = \frac{398,600}{81,360}\times 0.661\times\sin\left(203.1^\circ\right) = \boxed{-1.271\ \text{km/s}}$$

■ **P3.14 (Precision: Average)**
Referring to Table A.2, we see that the radius of influence of the earth is 925,000 km. Substituting this and other pertaining data into the orbit equation, we can solve for the true anomaly θ:

$$r = \frac{r_{\text{perigee}}\left(1+e\right)}{1+e\cos\theta} \rightarrow 925,000 = \frac{\left(6378+200\right)\times\left(1+1\right)}{1+1\times\cos\theta}$$

$$\therefore \cos\theta = -0.986$$

$$\therefore \theta = 170.4^\circ$$

The mean anomaly is given by Barker's equation:

$$M_p = \frac{1}{2}\tan\left(\frac{\theta}{2}\right) + \frac{1}{6}\tan^3\left(\frac{\theta}{2}\right) = \frac{1}{2}\tan\left(\frac{170.4^\circ}{2}\right) + \frac{1}{6}\tan^3\left(\frac{170.4^\circ}{2}\right)$$

$$\therefore M_p = 287.4$$

We also need the specific angular momentum h:

$$r_{\text{perigee}} = \frac{h^2}{2\mu} \rightarrow h = \sqrt{2\mu r_{\text{perigee}}}$$

$$\therefore h = \sqrt{2\times 398,600\times 6578} = 72,420\ \text{km}^2/\text{s}$$

Finally, the time required for the spacecraft to leave earth's SOI is

$$t = \frac{h^3}{\mu^2}M_p = \frac{72,420^3}{398,600^2}\times 287.4 = 687,000\ \text{s} = \boxed{7.951\ \text{days}}$$

(The large difference relatively to the answer provided in the text, which is 7.77 d, is due to my somewhat reckless roundoff of true anomaly θ; entering $\theta = 170.33^\circ$ instead of $\theta = 170.4^\circ$ into Barker's formula would yield $M_p \approx 280.87$, which can be used to find $t \approx 7.7698$ d, a slightly better approximation.)

■ **P3.15 (Precision: Average)**
Part (a): We first determine the specific angular momentum h,

$$h = \sqrt{\mu r_{\text{perigee}}\left(1+e\right)} = \sqrt{398,600\times 6600\times\left(1+1\right)} = 72,540\ \text{km}^2/\text{s}$$

and then the mean anomaly for $\theta = 90^\circ$,

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$$M_{p,\theta=90°} = \frac{1}{2}\tan\left(\frac{\theta}{2}\right) + \frac{1}{6}\tan^3\left(\frac{\theta}{2}\right) = \frac{1}{2}\times\tan(45°) + \frac{1}{6}\times\tan^3(45°)$$

$$\therefore M_{p,\theta=90°} = 0.667$$

The corresponding time $t_{\theta=90°}$ is given by equation (3.31) in the text,

$$t_{\theta=90°} = \frac{h^3}{\mu^2}M_{p,\theta=90°} = \frac{72,540^3}{398,600^2}\times 0.667 = 1602 \text{ s}$$

The time required to coast from coast is twice this value, or $t_{\theta=-90° \text{to} 90°} = 2 \times$ 1602 = 3204 s = 53.4 min.

Part (b): The mean anomaly 36 hours after passing through the perigee is

$$M_p = \frac{\mu^2 t}{h^3} = \frac{398,600^2 \times (36 \times 3600)}{72,540^3} = 53.94$$

so that, solving Barker's equation for $tan(\theta/2)$, we get

$$\tan\left(\frac{\theta}{2}\right) = \left[3M_p + \sqrt{(3M_p)^2 + 1}\right]^{1/3} - \left[3M_p + \sqrt{(3M_p)^2 + 1}\right]^{-1/3}$$

$$\therefore \tan\left(\frac{\theta}{2}\right) = \left[3\times 53.94 + \sqrt{(3\times 53.94)^2 + 1}\right]^{1/3} - \left[3\times 53.94 + \sqrt{(3\times 53.94)^2 + 1}\right]^{-1/3}$$

$$\therefore \tan\left(\frac{\theta}{2}\right) = 6.720$$

or

$$\theta = 2\arctan(6.720) = 163.1°$$

The corresponding radial position r is

$$r = \frac{h^2}{\mu}\frac{1}{1+\cos\theta} = \frac{72,540^2}{398,600}\times\frac{1}{1+\cos(163.1°)} = \boxed{305,680 \text{ km}}$$

■ **P3.16 (Precision: Average)**

Part (a): The velocity at perigee is

$$v_{\text{perigee}} = 1.2\sqrt{\frac{2\mu}{r_{\text{perigee}}}} = 1.2\times\sqrt{\frac{2\times 398,600}{6600}} = 13.19 \text{ km/s}$$

The specific angular momentum is

$$h = r_{\text{perigee}}v_{\text{perigee}} = 6600\times 13.19 = 87,050 \text{ km}^2/\text{s}$$

The eccentricity of the trajectory can be found from the orbit equation:

$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e} \rightarrow 6600 = \frac{87,050^2}{398,600}\times\frac{1}{1+e}$$

$$\therefore e = 1.880$$

Now, the eccentric anomaly F is related to the true anomaly θ by equation (3.44a):

$$\tanh\left(\frac{F}{2}\right) = \sqrt{\frac{e-1}{e+1}}\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1.88-1}{1.88+1}}\times\tan\left(\frac{90°}{2}\right) = 0.553$$

$$\therefore F = 2\tanh^{-1}(0.553) = 1.245$$

The mean anomaly at $\theta = 90°$ then becomes

$$M_{h,90°} = e\sinh(F_{90°}) - F_{90°} = 1.88\times\sinh(1.245) = 1.245 = 1.749$$

so that, solving equation (3.34) for time,

$$M_{h,90°} = \frac{\mu^2}{h^3}(e^2-1)^{3/2}t \rightarrow 1.749 = \frac{398,600^2}{87,050^3}\times(1.88^2-1)^{3/2}t$$

$$\therefore 1.749 = 9.718\times 10^{-4}t$$

$$\therefore t_{90°} = \frac{1.749}{9.718\times 10^{-4}} = 1800 \text{ s}$$

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The time required for the spacecraft to coast from −90 degrees to +90 degrees is twice this value, namely

$$t_{-90° \text{ to } 90°} = 2 \times 1800 = 3600 \text{ s} = \boxed{1.0 \text{ h}}$$

Part (b): We begin by computing the mean anomaly 24 hours after the craft has passed through perigee:

$$M_h = \frac{\mu^2}{h^3}\left(e^2 - 1\right)^{3/2} t = \frac{398,600^2}{87,050^3} \times \left(1.88^2 - 1\right)^{3/2} \times (24 \times 3600) = 83.96$$

To find the eccentric anomaly F, we must solve Kepler's equation for a hyperbola (eq. (3.40)):

$$M_h = e\sinh F - F \;\rightarrow\; 83.96 = 1.88\sinh F - F$$

This transcendental equation can be solved for F using MATLAB's *fzero* command:

```
>> f = @(F) 83.96 - 1.88*sinh(F) + F;
x0 = 1;
fzero(f,x0)
ans =
    4.5450
```

That is, $F = 4.545$. We proceed to determine the true anomaly θ:

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{F}{2}\right) = \sqrt{\frac{1.88+1}{1.88-1}} \times \tanh\left(\frac{4.545}{2}\right) = 1.771$$

$$\therefore \theta = 2\arctan(1.771) = 121.1°$$

Finally, we establish the radial position r:

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} = \frac{87,050^2}{398,600} \times \frac{1}{1+1.88\times\cos(121.1°)} = \boxed{657,400 \text{ km}}$$

■ **P3.19** (Precision: Average)

MATLAB function *kepler_U* can be used to implement Algorithm 3.3:

```
clear
global mu
mu = 398600;

ro = 7200;
vro = 1;
a = 10000;
dt = 3600;

x = kepler_U(dt, ro, vro, 1/a);
fprintf('\n Initial radial coordinate = %g', ro)
fprintf('\n Initial radial velocity = %g', vro)
fprintf('\n Elapsed time = %g', dt)
fprintf('\n Semimajor axis = %g\n', a)
fprintf('\n Universal anomaly = %g\n', x)
```

The output is:

```
Initial radial coordinate = 7200
Initial radial velocity = 1
Elapsed time = 3600
Semimajor axis = 10000

Universal anomaly = 229.341
```

As shown, the universal anomaly is $\chi = 229.341$ km$^{1/2}$. Let us confirm this result using equation 3.58. The first step is to solve the energy equation for v:

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \;\rightarrow\; v = \sqrt{2\mu\left(\frac{1}{r} - \frac{1}{2a}\right)}$$

$$\therefore v = \sqrt{2 \times 398,600 \times \left(\frac{1}{7200} - \frac{1}{2 \times 10,000}\right)} = 8.418 \text{ m/s}$$

Then, the azimuthal component of velocity follows from the Pythagorean theorem:

$$v_\perp = \sqrt{v^2 - v_r^2} = \sqrt{8.418^2 - 1.0^2} = 8.358 \text{ km/s}$$

The specific angular momentum is

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$$h = rv_\perp = 7200 \times 8.358 = 60{,}180 \text{ km}^2/\text{s}$$

To compute the true anomaly, we make use of two equations. The first is the equation for radial velocity v_r:

$$v_r = 1.0 \;\rightarrow\; \frac{\mu}{h} e \sin\theta = 1.0$$

$$\therefore \frac{398{,}600}{60{,}180} e \sin\theta = 1.0 \;\; \text{(I)}$$

The second is the orbit equation:

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta} \;\rightarrow\; 7200 = \frac{60{,}180^2}{398{,}600} \times \frac{1}{1 + e\cos\theta} \;\; \text{(II)}$$

Equations (I) and (II) can be solved simultaneously using MATLAB's *fsolve* function; in the following code snippet, *x(1)* is eccentricity and *x(2)* is true anomaly.

```
function f = orbitProb(x)
f(1) = 398600/60180*x(1)*sin(x(2)) - 1;
f(2) = 7200 - 60180^2/398600*1/(1 + x(1)*cos(x(2)));
>> fun = @orbitProb;

>> fun = @orbitProb;
x0 = [0.1, 0]
x = fsolve(fun,x0)
x =
    0.3023    0.5229
```

Hence, $e = 0.302$ and $\theta = 0.523$ rad = 29.97°. Then, we compute the time at the initial true anomaly as follows:

$$\tan\left(\frac{E_1}{2}\right) = \sqrt{\frac{1-e}{1+e}} \tan\left(\frac{\theta_1}{2}\right) = \sqrt{\frac{1-0.302}{1+0.302}} \times \tan\left(\frac{0.523}{2}\right) = 0.196$$

$$\therefore E_1 = 2\arctan(0.196) = 0.387 \text{ rad}$$

$$M_1 = E_1 - e\sin E_1 = 0.387 - 0.302 \times \sin(0.387) = 0.273 \text{ rad}$$

$$t_1 = \frac{M_1}{2\pi}T = \frac{0.273}{2\pi} \times \underbrace{\left(\frac{2\pi}{\sqrt{398{,}600}} \times 10{,}000^{3/2}\right)}_{=9952}$$

$$t_1 = 432.4 \text{ s}$$

Then, we proceed to determine *E* one hour later:

$$t_2 = t_1 + 3600 = 4032 \text{ s}$$

$$M_2 = \frac{2\pi}{T}t_2 = \frac{2\pi}{9952} \times 4032 = 2.546 \text{ rad}$$

$$E_2 - e\sin(E_2) = M_2 \;\rightarrow\; E_2 - 0.302\sin(E_2) = 2.546$$

$$\therefore E_2 = 2.680 \text{ rad}$$

At last, the universal anomaly is given by equation (3.58):

$$\chi = \sqrt{a}\,(E_2 - E_1) = \sqrt{10{,}000} \times (2.680 - 0.387) = \boxed{229.3 \text{ km}^{1/2}}$$

■ **P3.20** (Precision: Average)

Algorithm 3.4 can be executed with the code *rv_from_r0v0*.m; all we have to do is insert as inputs the initial position vector **r₀**, the initial velocity vector **v₀**, and the time $t = 2 \text{ h} = 2 \times 3600 = 7200 \text{ s}$.

```
global mu
mu = 398600;

R0 = [20000, -105000, -19000];
V0 = [0.9, -3.4, -1.5];
t = 2*3600;

[R, V] = rv_from_r0v0(R0, V0, t);

fprintf('\n Initial position vector (km):')
fprintf('\n r0 = (%g, %g, %g)\n', R0(1), R0(2), R0(3))
fprintf('\n Initial velocity vector (km/s):')
```

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```
fprintf('\n v0 = (%g, %g, %g)\n', V0(1), V0(2), V0(3))
fprintf('\n\n Elapsed time = %g s\n', t)
fprintf('\n Final position vector (km):')
fprintf('\n r = (%g, %g, %g)\n', R(1), R(2), R(3))
fprintf('\n Final velocity vector (km/s):')
fprintf('\n v = (%g, %g, %g)', V(1), V(2), V(3))
```

The output is:

```
Elapsed time = 7200 s

Final position vector (km):
r = (26337.8, -128752, -29655.9)

Final velocity vector (km/s):
v = (0.862796, -3.2116, -1.46129)
```

Hence, after two hours, the final position and velocity vectors are, respectively:

$$\boxed{\mathbf{r} = 26,337.8\mathbf{i} - 128,752\mathbf{j} - 26,655.9\mathbf{k}} \ \text{km}$$

$$\boxed{\mathbf{v} = 0.862796\mathbf{i} - 3.2116\mathbf{j} - 1.46129\mathbf{k}} \ \text{km/s}$$

■ P4.3 (Precision: Average)

The norms of the inertial position and velocity vectors are, respectively,

$$r = \|\mathbf{r}\| = \sqrt{2500^2 + 16,000^2 + 4000^2} = 16,681 \ \text{km}$$

$$v = \|\mathbf{v}\| = \sqrt{(-3)^2 + (-1)^2 + 5^2} = 5.916 \ \text{km/s}$$

The radial component of velocity is such that

$$v_r = \frac{\mathbf{r} \cdot \mathbf{v}}{r} = \frac{2500 \times (-3) + 16,000 \times (-1) + 4000 \times 5}{16,681} = -0.210 \ \text{km/s}$$

The angular momentum vector is given by the cross product

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 2500 & 16,000 & 4000 \\ -3 & -1 & 5 \end{vmatrix} = 84,000\mathbf{I} - 24,500\mathbf{J} + 45,500\mathbf{K} \ (\text{km}^2/\text{s})$$

which has norm $h = \|\mathbf{h}\| = 98,620$ km²/s. The second orbital element we require is the inclination i:

$$i = \cos^{-1}\left(\frac{h_z}{h}\right) = \cos^{-1}\left(\frac{45,500}{98,620}\right) = \boxed{62.52^\circ}$$

The node line vector **N** is, in turn,

$$\mathbf{N} = \mathbf{K} \times \mathbf{h} = 24,500\mathbf{I} + 84,000\mathbf{J} \ (\text{km}^2/\text{s})$$

and the corresponding magnitude is $N = \|\mathbf{N}\| = 87,500$ km². Then, the right ascension of the ascending node Ω becomes

$$\Omega = \cos^{-1}\left(\frac{N_x}{N}\right) = \cos^{-1}\left(\frac{24,500}{87,500}\right) = \boxed{73.74^\circ}$$

The eccentricity vector **e** is given by equation (4.10):

$$\mathbf{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - rv_r\mathbf{v}\right]$$

$$\therefore \mathbf{e} = \frac{1}{398,600} \times \left[\left(5.916^2 - \frac{396,800}{16,681}\right)\mathbf{r} - 16,681 \times (-0.210)\mathbf{v}\right]$$

$$\therefore \mathbf{e} = \frac{1}{398,600} \times [11.21\mathbf{r} + 3503\mathbf{v}]$$

$$\therefore \mathbf{e} = \frac{1}{398,600} \times [11.21(2500\mathbf{I} + 16,000\mathbf{J} + 4000\mathbf{K}) + 3503(-3\mathbf{I} - \mathbf{J} + 5\mathbf{K})]$$

$$\therefore \mathbf{e} = 0.0439\mathbf{I} + 0.441\mathbf{J} + 0.156\mathbf{K}$$

The corresponding magnitude is $e = \|\mathbf{e}\| = 0.468$. The penultimate orbital element we need is the argument of perigee, namely

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$$\omega = \cos^{-1}\left(\frac{\mathbf{N}}{N} \cdot \frac{\mathbf{e}}{e}\right) = \cos^{-1}\left(\frac{24,500\mathbf{I} + 84,000\mathbf{J}}{87,500} \cdot \frac{0.0439\mathbf{I} + 0.441\mathbf{J} + 0.156\mathbf{K}}{0.468}\right)$$

$$\therefore \omega = \cos^{-1}(0.931) = \boxed{21.43^\circ}$$

Finally, the true anomaly is given by the equation listed in topic 13 of algorithm 4.2:

$$\theta = \cos^{-1}\left(\frac{\mathbf{e}}{e} \cdot \frac{\mathbf{r}}{r}\right) = \cos^{-1}\left(\frac{0.0439\mathbf{I} + 0.441\mathbf{J} + 0.156\mathbf{K}}{0.468} \cdot \frac{2500\mathbf{I} + 16,000\mathbf{J} + 4000\mathbf{K}}{16,681}\right)$$

$$\therefore \theta = \cos^{-1}(0.997)$$

Now, note that $\mathbf{e} \cdot \mathbf{r} = -3500$, which means that $\theta \in (180^\circ, 360^\circ)$. Thus, the true anomaly can only be (see eq. (4.31a))

$$\theta = 360^\circ - \cos^{-1}(0.997) = 360^\circ - 4.439^\circ = \boxed{355.6^\circ}$$

The six orbital elements are summarized below.

Orbital Element	Value
Specific angular momentum, h	98,620 km²/s
Inclination, i	62.52°
Right ascension of ascending node, Ω	73.74°
Eccentricity, e	0.468
Argument of perigee, ω	21.43°
True anomaly, θ	355.6°

■ **P4.5** (Precision: Average)

The calculations are identical to those of Problem 4.3. We can speed things up by using MATLAB code *coe_from_sv* as provided in Appendix D.18. The inputs are vectors **r** and **v** and the gravitational parameter μ.

```
r = [6500, -7500, -2500];
v = [4, 3, -3];
mu = 398600;
elements = coe_from_sv(r,v,mu);
fprintf('The specific angular momentum is %g\n', elements(1))
fprintf('The eccentricity is %g\n', elements(2))
fprintf('The R.A. of the ascending node is %g\n', elements(3))
fprintf('The inclination is %g\n', elements(4))
fprintf('The argument of perigee is %g\n', elements(5))
fprintf('The true anomaly is %g\n', elements(6))
```

Here's the output:

```
The specific angular momentum is 58655.8
The eccentricity is 0.222606
The R.A. of the ascending node is 1.87747
The inclination is 0.566272
The argument of perigee is 1.2629
The true anomaly is 2.35141
```

The orbital parameters are summarized below.

Orbital Element	Value
Specific angular momentum, h	58,656 km²/s
Inclination, i	0.566 rad (32.43°)
Right ascension of ascending node, Ω	1.877 rad (107.5°)
Eccentricity, e	0.223
Argument of perigee, ω	1.263 rad (72.36°)
True anomaly, θ	2.351 rad (134.7°)

■ **P4.6** (Precision: Average)

The magnitudes of the position and eccentricity vectors we were given are, respectively,

$$r = \|\mathbf{r}\| = \sqrt{(-6000)^2 + (-1000)^2 + 5000^2} = 7874 \text{ km}$$

$$e = \|\mathbf{e}\| = \sqrt{0.4^2 + 0.5^2 + 0.6^2} = 0.877$$

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As stated, the satellite is approaching perigee; thus, the true anomaly may be computed from the second of eqs. (4.13a):

$$\theta = 360^\circ - \cos^{-1}\left(\frac{\mathbf{e}}{e} \cdot \frac{\mathbf{r}}{r}\right) = 360^\circ - \cos^{-1}\left(\frac{-0.4\mathbf{I} - 0.5\mathbf{J} - 0.6\mathbf{K}}{0.877} \cdot \frac{-6000\mathbf{I} - 1000\mathbf{J} - 5000\mathbf{K}}{7874}\right)$$

$$\therefore \theta = 360^\circ - \cos^{-1}(0.854) = 360^\circ - 31.35^\circ = \boxed{328.65^\circ}$$

■ P4.7 (Precision: Average)

We first compute the unit normal **w** to the orbital plane:

$$\mathbf{w} = \frac{\mathbf{r} \times \mathbf{e}}{\|\mathbf{r} \times \mathbf{e}\|}$$

where

$$\mathbf{r} \times \mathbf{e} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -6600 & -1300 & -5200 \\ -0.4 & -0.5 & -0.6 \end{vmatrix} = -1820\mathbf{I} - 1880\mathbf{J} + 2780\mathbf{K} \text{ km}$$

and the corresponding norm is

$$\|\mathbf{r} \times \mathbf{e}\| = \sqrt{1820^2 + 1880^2 + 2780^2} = 3818 \text{ km}$$

so that

$$\mathbf{w} = \frac{-1820\mathbf{I} - 1880\mathbf{J} + 2780\mathbf{K}}{3818} = -0.477\mathbf{I} - 0.492\mathbf{J} + 0.728\mathbf{K}$$

It remains to compute the inclination i:

$$i = \cos^{-1} w_z = \cos^{-1}(0.728) = \boxed{43.28^\circ}$$

■ P4.14 (Precision: Average)

The solution procedure is analogous to the one adopted in Example 3.7, page 175. We begin by computing the magnitudes

$$r_0 = \sqrt{5000^2 + 8000^2 + 2100^2} = 9665 \text{ km}$$

$$v_0 = \sqrt{4^2 + 3.5^2 + 3^2} = 6.103 \text{ km/s}$$

We also need the radial component of velocity,

$$v_{r,0} = \frac{\mathbf{v}_0 \cdot \mathbf{r}_0}{r_0} = \frac{-5000 \times (-4) - 8000 \times (3.5) - 2100 \times (-3)}{9665} = -0.176 \text{ km/s}$$

and the reciprocal α of the semimajor axis,

$$\alpha = \frac{2}{r_0} - \frac{v_0^2}{\mu} = \frac{2}{9665} - \frac{6.103^2}{398,600} = 1.135 \times 10^{-4} \text{ km}^{-1}$$

We can speed things up by calculating the universal anomaly with *kepler_U*:

```
global mu
mu = 398600;
dt = 50*60;
ro = 9665;
vro = -0.176;
a = 1.135e-4;
x = kepler_U(dt, ro, vro, a)

x =

   212.3778
```

As shown, *kepler_U* returns $\chi \approx 212.4$ km$^{1/2}$. Next, Lagrange coefficients f and g can be determined with code *f_and_g.m* (Appendix D.15 in the text):

```
x = 212.4;
t = 50*60;
ro = 9665;
a = 1.135e-4;
[fLag, gLag] = f_and_g(x, t, ro, a)

 fLag =

   -0.4933

gLag =
```

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```
    1.0445e+03
```

As shown, *f_and_g* returns $f \approx -0.493$ and $g \approx 1045$ s^{-1}. Substituting into equation (3.67) brings to

$$\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0 = -0.493 \times (-5000\mathbf{I} - 8000\mathbf{J} - 2100\mathbf{K})$$

$$+1045 \times (-4\mathbf{I} + 3.5\mathbf{J} - 3\mathbf{K}) = \boxed{-1715\mathbf{I} + 7601\mathbf{J} - 2100\mathbf{K}} \text{ km}$$

Finally, to find the updated velocity vector **v**, we need derivatives $\dot{f}$ and $\dot{g}$, which are given by equations (3.69c) and (3.69d), respectively. The calculation of these coefficients can be automated with the accompanying function *fDot_and_gDot*:

```
x = 212.4;
t = 50*60;
ro = 9665;
r = norm([-1715,7601,-2100]);
a = 1.135e-4;
[fDotLag, gDotLag] = fDot_and_gDot(x, r, ro, a)

fDotLag =

  -5.8499e-04

gDotLag =

  -0.7884
```

As shown, the results are $\dot{f} \approx -5.850 \times 10^{-4}$ s^{-1} and $\dot{g} = -0.788$. Substituting into equation (3.68) yields

$$\mathbf{r} = \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0 = -(5.850 \times 10^{-4}) \times (-5000\mathbf{I} - 8000\mathbf{J} - 2100\mathbf{K})$$

$$-0.788 \times (-4\mathbf{I} + 3.5\mathbf{J} - 3\mathbf{K}) = \boxed{6.077\mathbf{I} + 1.922\mathbf{J} + 3.593\mathbf{K}} \text{ km/s}$$

In the second method, we first compute the classical orbital elements using the *coe_from_sv* code:

```
R = [-5000, -8000, -2100];
V = [-4, 3.5, -3];
mu = 398600;
elmnts = coe_from_sv(R,V,mu);
fprintf('The spec. ang. momentum is %g\n', elmnts(1))
fprintf('The eccentricity is %g\n', elmnts(2))
fprintf('The right ascension of ascending node is %g\n', elmnts(3))
fprintf('The inclination of the orbit is %g\n', elmnts(4))
fprintf('The argument of the perigee is %g\n', elmnts(5))
fprintf('The true anomaly is %g\n', elmnts(6))

The spec. ang. momentum is 58963
The eccentricity is 0.100957
The right ascension of ascending node is 1.3633
The inclination of the orbit is 2.56718
The argument of the perigee is 0.150736
The true anomaly is 3.40226
```

The true anomaly $\theta = 3.402$ rad $= 194.9°$, and the period T is calculated as

$$T = \frac{2\pi}{\mu^2} \left(\frac{h}{\sqrt{1-e^2}}\right)^3 = \frac{2\pi}{398,600^2} \times \left(\frac{58,963}{\sqrt{1-0.101^2}}\right)^3 = 8232 \text{ s}$$

We proceed to determine the time since perigee passage at true anomaly $\theta = 194.9°$:

$$\tan\left(\frac{E_0}{2}\right) = \sqrt{\frac{1-e}{1+e}} \tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-0.101}{1+0.101}} \times \tan\left(\frac{194.9°}{2}\right) = -6.901$$

$$\therefore E_0 = 2\arctan(-6.901) = -2.854 \text{ rad}$$

$$M_0 = E_0 - e\sin E_0 = -2.854 - 0.101 \times \sin(-2.854) = -2.825 \text{ rad}$$

$$t_0 = \frac{M_0}{2\pi} T = \frac{-2.825}{2\pi} \times 8232 = -3701 \text{ s}$$

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The negative sign means time *until* perigee passage. We press on to update the true anomaly of the spacecraft at $t = t_0 + 50 \times 60 = -701$ s:

$$M = 2\pi \frac{t}{T} = 2\pi \times \frac{(-701)}{8232} = -0.535 \text{ rad}$$

$$M = E - e\sin E \rightarrow -0.535 = E - 0.101\sin E$$

$$\therefore E = -0.591 \text{ rad (Numerical solution)}$$

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}} \tan\left(\frac{E}{2}\right) = \sqrt{\frac{1+0.101}{1-0.101}} \times \tan\left(\frac{-0.591}{2}\right) = -0.337$$

$$\therefore \theta = 2\arctan(-0.337) = -37.25°$$

Then, position vector $\{r\}_{\bar{x}}$ and velocity vector $\{v\}_{\bar{X}}$ in perifocal coordinates are given by equations (4.45) and (4.46), respectively:

$$\{r\}_{\bar{X}} = \frac{h^2}{\mu} \frac{1}{1+e\cos\theta} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \frac{58,963^2}{398,600} \times \frac{1}{1+0.101\times\cos(-37.25°)} \begin{bmatrix} \cos(-37.25°) \\ \sin(-37.25°) \\ 0 \end{bmatrix}$$

$$\therefore \{r\}_{\bar{X}} = 8073 \begin{bmatrix} \cos(-37.25°) \\ \sin(-37.25°) \\ 0 \end{bmatrix}$$

$$\therefore \{r\}_{\bar{X}} = \begin{bmatrix} 6426 \\ -4887 \\ 0 \end{bmatrix} \text{ km}$$

$$\{v\}_{\bar{X}} = \frac{\mu}{h} \begin{bmatrix} -\sin\theta \\ e+\cos\theta \\ 0 \end{bmatrix} = \frac{398,600}{58,963} \times \begin{bmatrix} -\sin(-37.25°) \\ 0.101+\cos(-37.25°) \\ 0 \end{bmatrix}$$

$$\therefore \{v\}_{\bar{X}} = \begin{bmatrix} 4.092 \\ 6.064 \\ 0 \end{bmatrix} \text{ km/s}$$

Next, we calculate the matrix $[Q]_{\bar{x}X}$ of the transformation from perifocal to geocentric equatorial coordinates:

$$[Q]_{\bar{X}x} = \begin{bmatrix} -\sin\Omega\cos i\sin\omega + \cos\Omega\cos\omega & \cos\Omega\cos i\sin\omega + \sin\Omega\cos\omega & \sin i\sin\omega \\ -\sin\Omega\cos i\cos\omega - \cos\Omega\sin\omega & \cos\Omega\cos i\cos\omega - \sin\Omega\sin\omega & \sin i\cos\omega \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix}$$

$$\therefore [Q]_{\bar{X}x} = \begin{bmatrix} 0.327 & 0.781 & 0.532 \\ 0.941 & -0.318 & -0.112 \\ 0.0816 & 0.537 & -0.840 \end{bmatrix}$$

We now have enough information to carry out the transformation from perifocal to geocentric equatorial components:

$$\{r\}_{\bar{X}} = [Q]_{\bar{x}\bar{X}} \{r\}_{\bar{x}} = \begin{bmatrix} 0.327 & 0.781 & 0.532 \\ 0.941 & -0.318 & -0.112 \\ 0.0816 & 0.537 & -0.840 \end{bmatrix} \begin{bmatrix} 6426 \\ -4887 \\ 0 \end{bmatrix}$$

```
>> QxX = [0.327, 0.781, 0.532; 0.941, -0.318, -0.112; 0.0816, 0.537,
-0.840];
>> rx = [6426, -4887, 0]';
>> mtimes(QxX,rx)

ans =

   1.0e+03 *

   -1.7154
    7.6009
```

```
    -2.1000
```

That is,

$$\boxed{\mathbf{r} = -1715\mathbf{I} + 7601\mathbf{J} - 2100\mathbf{K}} \text{ km}$$

Likewise for the velocity vector:

$$\{\mathbf{v}\}_{\bar{X}} = [\mathbf{Q}]_{\bar{x}\bar{X}}\{\mathbf{v}\}_{\bar{x}} = \begin{bmatrix} 0.327 & 0.781 & 0.532 \\ 0.941 & -0.318 & -0.112 \\ 0.0816 & 0.537 & -0.840 \end{bmatrix}\begin{bmatrix} 4.092 \\ 6.064 \\ 0 \end{bmatrix}$$

```
>> vx = [4.092, 6.064, 0]';
>> mtimes(QxX,vx)

ans =

    6.0741
    1.9222
    3.5903
```

That is,

$$\boxed{\mathbf{v} = 6.074\mathbf{I} + 1.922\mathbf{J} + 3.590\mathbf{K}} \text{ km/s}$$

■ **P4.15** (Precision: Average)

Part (a): The specific angular momentum for this orbit is

$$h = \sqrt{\mu(1+e)r_{\text{perigee}}} = \sqrt{398,600 \times (1+1.5) \times (6378+300)} = 81,576 \text{ km}^2/\text{s}$$

The perifocal radial position is given by equation (4.45):

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \frac{81,576^2}{398,600} \times \frac{1}{1+1.5\cos(0)}\begin{bmatrix} \cos(0) \\ \sin(0) \\ 0 \end{bmatrix}$$

$$\therefore \{\mathbf{r}\}_{\bar{x}} = 6678\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{\{\mathbf{r}\}_{\bar{x}} = 6678\mathbf{p}} \text{ km}$$

The perifocal velocity is given by equation (4.46):

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{bmatrix} -\sin\theta \\ e+\cos\theta \\ 0 \end{bmatrix} = \frac{398,600}{81,576} \times \begin{bmatrix} -\sin(0) \\ 1.5+\cos(0) \\ 0 \end{bmatrix}$$

$$\therefore \{\mathbf{v}\}_{\bar{x}} = 4.886 \times \begin{bmatrix} 0 \\ 2.5 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{\{\mathbf{v}\}_{\bar{x}} = 12.22\mathbf{q}} \text{ km/s}$$

Part (b): To compute the geocentric equatorial components, we first write down the transformation matrix

$$[\mathbf{Q}]_{\bar{X}\bar{x}} = \begin{bmatrix} -\sin\Omega\cos i\sin\omega + \cos\Omega\cos\omega & \cos\Omega\cos i\sin\omega + \sin\Omega\cos\omega & \sin i\sin\omega \\ -\sin\Omega\cos i\cos\omega - \cos\Omega\sin\omega & \cos\Omega\cos i\cos\omega - \sin\Omega\sin\omega & \sin i\cos\omega \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix}$$

$$\therefore [\mathbf{Q}]_{\bar{X}\bar{x}} = \begin{bmatrix} -0.297 & 0.848 & 0.439 \\ -0.801 & -0.472 & 0.369 \\ 0.520 & -0.242 & 0.819 \end{bmatrix}$$

Then, we perform the pertaining matrix multiplication using MATLAB:

```
>> QxX = [-0.297, 0.848, 0.439; -0.801, -0.472, 0.369; 0.520, -0.242,
0.819];
>> rx = [6678, 0, 0]';
>> mtimes(QxX, rx)
ans =
```

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```
1.0e+03 *

 -1.9834
 -5.3491
  3.4726
```

Thus,

$$\{\mathbf{r}\}_{\bar{X}} = [\mathbf{Q}]_{\overline{x}\overline{x}} \{\mathbf{r}\}_{\overline{x}} = \boxed{-1983\mathbf{I} - 5349\mathbf{J} + 3473\mathbf{K}} \text{ km}$$

Proceeding similarly with the velocity vector:

```
>> vx = [0, 12.22, 0]';
mtimes(QxX, vx)

ans =

   10.3626
   -5.7678
   -2.9572
```

Thus,

$$\{\mathbf{v}\}_{\bar{X}} = [\mathbf{Q}]_{\overline{x}\overline{x}} \{\mathbf{v}\}_{\overline{x}} = \boxed{10.36\mathbf{I} - 5.768\mathbf{J} - 2.957\mathbf{K}} \text{ km/s}$$

■ P4.16 (Precision: Average)

Part (a): The true anomaly at time $t = 2 \times 3600 = 7200$ s is calculated as

$$M_h = \frac{\mu^2}{h^3}\left(e^2 - 1\right)^{3/2} t = \frac{398,600^2}{81,576^3} \times \left(1.5^2 - 1\right)^{3/2} \times 7200 = 2.945$$

$$e\sinh F - F = M_h \;\rightarrow\; 1.5 \times \sinh F - F = 2.945$$

$$\therefore F = 1.886 \;\left(\text{Numerical solution}\right)$$

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{e+1}{e-1}}\,\tanh\left(\frac{F}{2}\right) = \sqrt{\frac{1.5+1}{1.5-1}} \times \tanh\left(\frac{1.886}{2}\right) = 1.647$$

$$\therefore \theta = 2\arctan\left(1.647\right) = 117.5^{\text{o}}$$

The distance vector in perifocal coordinates is given by equation (4.45):

$$\{\mathbf{r}\}_{\overline{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{bmatrix}\cos\theta \\ \sin\theta \\ 0\end{bmatrix} = \frac{81,576^2}{398,600} \times \frac{1}{1+1.5\times\cos\left(117.5^{\text{o}}\right)}\begin{bmatrix}\cos\left(117.5^{\text{o}}\right) \\ \sin\left(117.5^{\text{o}}\right) \\ 0\end{bmatrix}$$

$$\therefore \{\mathbf{r}\}_{\overline{x}} = 54,315\begin{bmatrix}-0.462 \\ 0.887 \\ 0\end{bmatrix} = \boxed{-25,093\mathbf{p} + 48,177\mathbf{q}} \text{ km}$$

The velocity vector in perifocal coordinates is given by equation (4.46):

$$\{\mathbf{v}\}_{\overline{x}} = \frac{\mu}{h}\begin{bmatrix}-\sin\theta \\ e+\cos\theta \\ 0\end{bmatrix} = \frac{398,600}{81,576} \times \begin{bmatrix}-\sin\left(117.5^{\text{o}}\right) \\ 1.5+\cos\left(117.5^{\text{o}}\right) \\ 0\end{bmatrix}$$

$$\therefore \{\mathbf{v}\}_{\overline{x}} = 4.886 \times \begin{bmatrix}-0.887 \\ 1.038 \\ 0\end{bmatrix} = \boxed{-4.334\mathbf{p} + 5.072\mathbf{q}} \text{ km/s}$$

Part (b): The same transformation matrix used in Problem 4.15 also applies here:

$$[\mathbf{Q}]_{\overline{X}x} = \begin{bmatrix} -0.297 & 0.848 & 0.439 \\ -0.801 & -0.472 & 0.369 \\ 0.520 & -0.242 & 0.819 \end{bmatrix}$$

The matrix multiplications we require can be automated with MATLAB; firstly, for the position vector in geocentric coordinates:

```
>> QxX = [-0.297, 0.848, 0.439; -0.801, -0.472, 0.369; 0.520, -0.242,
0.819];
rx = [-25093, 48177, 0]';

mtimes(QxX,rx)

ans =
   1.0e+04 *
    4.8307
   -0.2640
   -2.4707
```

Thus,

$$\{\mathbf{r}\}_{\bar{X}} = [\mathbf{Q}]_{\bar{x}\bar{X}} \{\mathbf{r}\}_{\bar{x}} = \boxed{48,307\mathbf{I} - 2640\mathbf{J} - 24,707\mathbf{K}} \ \text{km}$$

Similarly, the velocity in geocentric coordinates is such that

```
>> vx = [-4.334, 5.072, 0]';

mtimes(QxX,vx)

ans =
    5.5883
    1.0776
   -3.4811
```

That is,

$$\{\mathbf{v}\}_{\bar{X}} = [\mathbf{Q}]_{\bar{x}\bar{X}} \{\mathbf{v}\}_{\bar{x}} = \boxed{5.588\mathbf{I} + 1.078\mathbf{J} - 3.481\mathbf{K}} \ \text{km/s}$$

■ **P4.18** (Precision: Average)

Part (a): The calculations are identical to the ones in Problem 4.15. Note that the specific angular momentum is missing; calculating it is effortless:

$$h = \sqrt{\mu(1+e)r_{\text{perigee}}} = \sqrt{398,600\times(1+1.2)\times6578} = 75,950 \ \text{km}^2/\text{s}$$

Then, the perifocal state vector can be found from equations (4.45) and (4.46):

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{bmatrix}\cos\theta\\\sin\theta\\0\end{bmatrix} = \frac{75,950^2}{398,600}\times\frac{1}{1+1.2\times\cos0}\times\begin{bmatrix}\cos0\\\sin0\\0\end{bmatrix}$$

$$\therefore \{\mathbf{r}\}_{\bar{x}} = 6578\times\begin{bmatrix}1\\0\\0\end{bmatrix} = \boxed{6578\mathbf{p} \ \text{km}}$$

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{bmatrix}-\sin\theta\\e+\cos\theta\\0\end{bmatrix} = \frac{398,600}{75,950}\times\begin{bmatrix}-\sin0\\1.2+\cos0\\0\end{bmatrix}$$

$$\therefore \{\mathbf{v}\}_{\bar{x}} = 5.25\times\begin{bmatrix}0\\1.2\\0\end{bmatrix} = \boxed{6.3\mathbf{q} \ \text{km/s}}$$

Part (b): Again, the calculations are identical to part (*b*) of Problem 4.15. We can speed things up by using the MATLAB code *sv_from_coe.m* in Appendix D.22. This function takes two inputs: The first is a vector containing six orbital elements; the second is the gravitational parameter μ.

```
mu = 398600; %Gravitational parameter
ecc = 1.2; %Eccentricity
rperigee = 6578; %Radial distance of perigee
inc = 50*pi/180; %Inclination
RA = 75*pi/180; %Right ascension of ascending node
w = 80*pi/180; %Argument of perigee
TA = 0; %True anomaly
h = sqrt(mu*(1+ecc)*rperigee); %Specific angular momentum
elements = [h, ecc, RA, inc, w, 0];
[r,v] = sv_from_coe(elements, mu)
```

Here's the output:

```
r =
```

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```
  1.0e+03 *

      -3.7265    2.1811    4.9625
v =

      -4.1878  -10.6496    1.5359
```

That is,

$$\boxed{\{\mathbf{r}\}_{\bar{X}} = -3727\mathbf{I} + 2181\mathbf{J} + 4963\mathbf{K}}\ \text{km}$$

and

$$\boxed{\{\mathbf{v}\}_{\bar{X}} = -4.188\mathbf{I} + 10.65\mathbf{J} + 1.536\mathbf{K}}\ \text{km/s}$$

■ **P4.19 (Precision: Average)**

Part (a): The first step is to update the true anomaly θ for 2 hours past perigee:

$$M_h = \frac{\mu^2}{h^3}\left(e^2 - 1\right)^{3/2} t = \frac{398,600^2}{75,950^3} \times \left(1.2^2 - 1\right)^{3/2} \times 7200 = 0.762$$

$$e\sinh(F) - F = M_h \ \rightarrow\ 1.2\sinh(F) - F = 0.762$$

$$\therefore F = 1.317\ \text{(Numerical solution)}$$

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{F}{2}\right) \ \rightarrow\ \tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1.2+1}{1.2-1}}\tanh\left(\frac{1.317}{2}\right) = 1.915$$

$$\therefore \theta = 2\arctan(1.915) = 124.9^{\circ}$$

As before, the perifocal state vectors can be found from equations (4.45) and (4.46):

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{bmatrix}\cos\theta\\\sin\theta\\0\end{bmatrix} = \frac{75,950^2}{398,600}\times\frac{1}{1+1.2\times\cos(124.9^{\circ})}\times\begin{bmatrix}\cos(124.9^{\circ})\\\sin(124.9^{\circ})\\0\end{bmatrix}$$

$$\therefore \{\mathbf{r}\}_{\bar{x}} = 46,173\times\begin{bmatrix}-0.572\\0.820\\0\end{bmatrix} = \boxed{-26,411\mathbf{p} + 37,862\mathbf{q}}\ \text{km}$$

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{bmatrix}-\sin\theta\\e+\cos\theta\\0\end{bmatrix} = \frac{398,600}{75,950}\times\begin{bmatrix}-\sin(124.9^{\circ})\\1.2+\cos(124.9^{\circ})\\0\end{bmatrix}$$

$$\therefore \{\mathbf{v}\}_{\bar{x}} = 5.248\times\begin{bmatrix}-0.820\\0.628\\0\end{bmatrix} = \boxed{-4.303\mathbf{p} + 3.260\mathbf{q}}\ \text{km/s}$$

Part (b): We can employ the same MATLAB routine as in Problem 4.18, remembering to update the true anomaly to $\theta = 124.9^{\circ}$:

```
mu = 398600; %Gravitational parameter
ecc = 1.2; %Eccentricity
rperigee = 6578; %Radial distance of perigee
inc = 50*pi/180; %Inclination
RA = 75*pi/180; %Right ascension of ascending node
w = 80*pi/180; %Argument of perigee
TA = 124.8*pi/180; %% True anomaly after 2 h = 124.9 deg
h = sqrt(mu*(1+ecc)*rperigee); %Specific angular momentum
elements = [h, ecc, RA, inc, w, TA];
[r,v] = sv_from_coe(elements, mu)
```

Here's the output:

```
r =

   1.0e+04 *

   0.1170   -4.3470   -1.4755

v =

   1.2435   -4.4751   -2.8118
```

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That is,

$$\{\mathbf{r}\}_{\bar{X}} = 1170\mathbf{I} - 43,470\mathbf{J} - 14,755\mathbf{K} \ \text{km}$$

and

$$\{\mathbf{v}\}_{\bar{X}} = 1.244\mathbf{I} - 4.475\mathbf{J} - 2.812\mathbf{K} \ \text{km/s}$$

■ **P5.4** **(Precision: Average)**

The magnitudes of the given position vectors are

$$\|\mathbf{r}_1\| = \sqrt{3600^2 + 4600^2 + 3600^2} = 6861 \ \text{km}$$

$$\|\mathbf{r}_2\| = \sqrt{5500^2 + 6240^2 + 5200^2} = 9810 \ \text{km}$$

Assume a prograde trajectory. The *z*-component of vector $\mathbf{r}_1 \times \mathbf{r}_2 = 4.776\times10^8$ km, which is positive and hence indicates that $\Delta\theta$ can be determined with the first of eqs. (5.26):

$$\Delta\theta = \cos^{-1}\left(\frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}\right) = \cos^{-1}\left[\frac{3600\times(-5500) + 4600\times6240 + 3600\times(-5200)}{6861\times9810}\right]$$

$$\therefore \Delta\theta = \cos^{-1}(-0.156) = 98.97º$$

We also need parameter *A*:

$$A = \sin(\Delta\theta)\sqrt{\frac{r_1 r_2}{1 - \cos(\Delta\theta)}} = \sin(98.97º)\times\sqrt{\frac{6861\times9810}{1 - \cos(98.97º)}} = 7537 \ \text{km}$$

To compute function *F(z)*, we refer to equation (5.40):

$$F(z) = \left[\frac{1}{C(z)}\left(r_1 + r_2 + A\frac{zS(z)-1}{\sqrt{C(z)}}\right)\right]^{3/2} S(z) + A\sqrt{r_1 + r_2 + A\frac{zS(z)-1}{\sqrt{C(z)}}} - \sqrt{\mu}\Delta t$$

The following MATLAB script can be used to solve the equation *F(z) = 0*:

```
mu = 398600;
deltat = 1800;
r1 = 6861;
r2 = 9810;
A = 7537;
y = @(z) ((r1 + r2 + A*(z*stumpS(z)-
1)/sqrt(stumpC(z)))/stumpC(z))^1.5*stumpS(z) ...
    + A*sqrt(r1 + r2 + A*(z*stumpS(z)-1)/sqrt(stumpC(z))) -
sqrt(mu)*deltat;
z0 = 1;
fzero(y,z0)

ans =

    2.0151
```

The solution is *z* = 2.015; since *z* > 0, the orbit is an ellipse. We proceed to evaluate the Stumpf functions at *z*:

```
>> [stumpS(2.015), stumpC(2.015)]

ans =

    0.1507    0.4215
```

Therefore, *S(2.015)* = 0.151 and *C(2.015)* = 0.422. We proceed to compute Lagrange parameters f and g, which first requires

$$y = r_1 + r_2 + A\frac{zS(z)-1}{\sqrt{C(z)}} = 6861 + 9810 + 7537\times\frac{2.015\times0.151-1}{\sqrt{0.422}} = 8599$$

so that

$$f = 1 - \frac{y}{r_1} = 1 - \frac{8599}{6861} = -0.253$$

$$g = A\sqrt{\frac{y}{\mu}} = 7537\times\sqrt{\frac{8599}{398,600}} = 1107 \ \text{s}$$

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and, using equation (5.28):

$$\mathbf{v}_1 = \frac{1}{g}(\mathbf{r}_2 - f\mathbf{r}_1) = \frac{1}{1107} \times \left[(-5500\mathbf{I} + 6240\mathbf{J} - 5200\mathbf{K}) - (-0.253) \times (3600\mathbf{I} + 4600\mathbf{J} + 3600\mathbf{K})\right]$$

$$\therefore \mathbf{v}_1 = -4.146\mathbf{I} + 6.688\mathbf{J} - 3.875\mathbf{K} \ \text{km/s}$$

The magnitude of $\mathbf{v}_1$ is 8.771 km/s. It remains to compute the specific energy ε:

$$\varepsilon = \frac{v_1^2}{2} - \frac{\mu}{r_1} = \frac{8.771^2}{2} - \frac{398,600}{6861} = \boxed{-19.63 \ (\text{km/s})^2}$$

■ P5.5 (Precision: Average)

Since we have velocity and position vectors, we may compute the specific angular momentum vectorially:

$$\mathbf{h} = \mathbf{r}_1 \times \mathbf{v}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3600 & 4600 & 3600 \\ -4.146 & 6.688 & -3.875 \end{vmatrix} = -41,902\mathbf{i} - 976\mathbf{j} + 43,148\mathbf{k} \ (\text{km}^2/\text{s})$$

$$\therefore \|\mathbf{h}\| = \sqrt{41,902^2 + 976^2 + 43,148^2} = 60,154 \ (\text{km}^2/\text{s})$$

We also need the orbital eccentricity,

$$e = \sqrt{1 + \frac{h^2}{\mu^2}\left(v_1^2 - \frac{2\mu}{r_1}\right)} = \sqrt{1 + \frac{60,154^2}{398,600^2} \times \left(8.771^2 - \frac{2 \times 398,600}{6861}\right)} = 0.325$$

so that

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{60,154^2}{398,600} \times \frac{1}{1+0.325} = 6851 \ \text{km}$$

Deducting the earthly radius,

$$z_p = r_p - 6378 = \boxed{473 \ \text{km}}$$

In turn, the inclination i is

$$i = \cos^{-1}\left(\frac{h_z}{h}\right) = \cos^{-1}\left(\frac{43,148}{60,154}\right) = \boxed{44.17^\circ}$$

■ P5.6 (Precision: Average)

We can easily solve this Lambert's problem with code *lambert.m*, which is provided in Appendix D.25. This code takes four inputs. The first two inputs are position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$; the third is the time of flight from $\mathbf{r}_1$ to $\mathbf{r}_2$; the fourth is a string to which we assign value '*pro*' if the orbit is prograde or '*retro*' if the orbit is retrograde. The outputs are velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Implementing the code in the problem at hand is straightforward:

```
global mu
mu = 398600;
r1 = [5644, -2830, -4170];
r2 = [-2240, 7320, -4980];
dt = 20*60;
string = 'pro';

[v1, v2] = lambert(r1, r2, dt, string);

fprintf('Solution:\n')
fprintf('\n v1 (km/s) = [%g, %g, %g]', v1(1), v1(2), v1(3))
fprintf('\n v2 (km/s) = [%g, %g, %g]\n', v2(1), v2(2), v2(3))
```

Here's the output:

```
v1 (km/s) = [-4.13223, 9.01237, -4.3781]

v2 (km/s) = [-7.28524, 6.31978, 2.5272]
```

That is,

$$\boxed{\mathbf{v}_2 = -7.285\mathbf{I} + 6.320\mathbf{J} + 2.527\mathbf{K}} \ \text{km/s}$$

$$\boxed{\mathbf{v}_1 = -4.132\mathbf{I} + 9.012\mathbf{J} - 4.378\mathbf{K}} \ \text{km/s}$$

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■ **P5.7** (Precision: Average)

The orbital elements can be determined by using *coe_from_sv.m* with **r₁** and **v₁** as inputs. That is:

```
global mu
mu = 398600;
r1 = [5644, -2830, -4170];
v1 = [-4.132, 9.012, -4.378];
elements = coe_from_sv(r1, v1, mu);

fprintf('Spec. Ang. Momentum (km^2/s): %g\n', elements(1))
fprintf('Eccentricity: %g\n', elements(2))
fprintf('R.A. of Ascending Node (deg): %g\n', elements(3)*180/pi)
fprintf('inclination (deg): %g\n', elements(4)*180/pi)
fprintf('Argument of Perigee(deg): %g\n', elements(5)*180/pi)
fprintf('True Anomaly (deg): %g\n', elements(6)*180/pi)
fprintf('Semimajor axis (km): %g\n', elements(7))
```

Here's the output:

```
Spec. Ang. Momentum (km^2/s): 76093.6
Eccentricity: 1.20037
R.A. of Ascending Node (deg): 130.007
inclination (deg): 59.0184
Argument of Perigee(deg): 259.981
True Anomaly (deg): 320.021
Semimajor axis (km): -32948
```

The six orbital elements are summarized below.

Orbital Element	Value
Specific angular momentum, h	76,094 km²/s
Inclination, i	59.02°
Right ascension of ascending node, Ω	130.0°
Eccentricity, e	1.200
Argument of perigee, ω	260.0°
True anomaly, θ	320.0°

Finally, we were also asked to compute the perigee altitude:

$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{76,094^2}{398,600}\times\frac{1}{1+1.20} = 6603 \text{ km}$$

$$\therefore z_{\text{perigee}} = r_{\text{perigee}} - 6378 = \boxed{225 \text{ km}}$$

■ **P5.12** (Precision: Average)

The MATLAB code *rv_from_observe.m* can be used to compute the state vector of a space object from the given data. The pertaining code follows:

```
% Variables
% deg - Conversion factor used to convert from degrees to radians
% mu - Gravitational parameter
%
% Re - Equatorial radius of the earth (km)
% f - Earth's flattening factor
% wE - Angular velocity of the Earth (rad/s)
% omega - earth's angular velocity vector (rad/s) in the geocentric
%         equatorial frame
%
% rho - slant range of object (km)
% rhodot - range rate (km/s)
% A - azimuth (deg) of object relative to observation site
% Adot - Time rate of change of azimuth (deg/s)
% a - Elevation angle (deg) of object relative to observation site
% adot - Time rate of change of elevation angle (deg/s)
%
% theta - Local sidereal time (deg) of tracking site
% phi - Geodetic latitude (deg) of site
% H - Elevation of site (km)
%
% r - Geocentric equatorial position vector of object (km)
% v - Geocentric equatorial velocity vector of object (km/s)
%
% rp - Perigee radius (km)
% coe - Orbital elements [h e RA inc w TA a]
%                 h - Spec. angular momentum (km^2/2)
%                 e - Eccentricity
```

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```
%                  RA - Right ascension of the ascending node
(rad)
%                  inc - Inclination of the orbit (rad)
%                  w - Argument of perigee (rad)
%                  TA - True anomaly (rad)
%                  a - Semimajor axis (km)
global f Re wE mu
deg = pi/180;
f = 0.0033528;
Re = 6378;
wE = 7.2921e-5;
mu = 398600;

rho = 988;
rhodot = 4.86;
A = 36;
Adot = 0.59;
a = 36.6;
adot = -0.263;
theta = 40;
phi = 35;
H = 0;

[r,v] = rv_from_observe(rho, rhodot, A, Adot, a, adot, theta, phi,
H);
fprintf('\nState vector:\n')
fprintf('\n r (km) = [%g, %g, %g]', r(1), r(2), r(3))
fprintf('\n v (km/s) = [%g, %g, %g]\n', v(1), v(2), v(3))
```

Here's the output:

```
State vector:

 r (km) = [3794.66, 3792.71, 4501.31]
 v (km/s) = [-7.72483, 7.72134, 0.0186586]
```

That is:

$$\boxed{\mathbf{r} = 3795\mathbf{I} + 3793\mathbf{J} + 4501\mathbf{K}} \ \text{km}$$

$$\boxed{\mathbf{v} = -7.725\mathbf{I} + 7.721\mathbf{J} + 0.0187\mathbf{K}} \ \text{km/s}$$

■ **P5.13** **(Precision: Average)**

Equipped with position vector **r** and velocity vector **v**, we can easily obtain the orbital elements using code *coe_from_sv.m* in Appendix D.18:

```
deg = pi/180;
mu = 398600;

r = [3795, 3793, 4501];
v = [-7.725, 7.721, 0.0187];

elements = coe_from_sv(r,v,mu);

fprintf('\n The spec. angular momentum is (km^2/s): %g', elements(1))
fprintf('\n The eccentricity is: %g', elements(2))
fprintf('\n The right ascension is (deg): %g', elements(3)/deg)
fprintf('\n The inclination is (deg): %g', elements(4)/deg)
fprintf('\n The argument of perigee is (deg): %g', elements(5)/deg)
fprintf('\n The true anomaly is (deg): %g', elements(6)/deg)
fprintf('\n The semimajor axis is (km): %g\n', elements(7))
```

Here's the output:

```
 The spec. angular momentum is (km^2/s): 76491.2
 The eccentricity is: 1.09593
 The right ascension is (deg): 315.132
 The inclination is (deg): 39.9925
 The argument of perigee is (deg): 89.8108
 The true anomaly is (deg): 0.0767075
 The semimajor axis is (km): -73007.7
```

The orbital elements are summarized below.

Orbital Element	Value
Specific angular momentum, h	76,491 km²/s
Inclination, i	39.99°
Right ascension of ascending node, Ω	315.1°
Eccentricity, e	1.096
Argument of perigee, ω	89.81°
True anomaly, θ	0.0767°

■ **P5.14** **(Precision: Maximum)**

The local sidereal time θ, azimuth A, angular elevation a and slant range ρ are provided at three observation times. The rates are not provided, but we can use function *rv_from_observe.m*, provided in Appendix D.28, to find the position vectors at each of the times. The code is listed below; note that zeros are passed to the rate variables.

```
% Variables
% deg - Conversion factor used to convert from degrees to radians
%
% Re - Equatorial radius of the earth (km)
% f - Earth's flattening factor
%
% t – Vector of three observation times (min)
% rho – Vector of slant ranges (km) of the object at the three
observation
% times
% az – Vector of azimuths (deg) of the object relative to the
observation
% site at the 3 observation times
% el – Vector of elevation angles (deg) of the object relative to the
% observation site at the 3 observation times
%
% theta – Vector of local sidereal times (deg) of the tracking site
at
% the 3 observation times
% phi - Geodetic latitude (deg) of site
% H - Elevation of site (km)
%
% r - Geocentric equatorial position vector of object (km)

global f Re wE
deg = pi/180;
Re = 6378;
f = 0.0033528;

phi = -20;
H = 0.5;
t = [0, 2, 4];
theta = [60, 60.5014, 61.0027];
az = [165.931, 145.967, 2.40962];
el = [9.53549, 45.7711, 21.8825];
rho = [1214.89, 421.441, 732.079];

fprintf('\n\n Solution')
fprintf('\n\n   Time')
fprintf('\n  (min)   Geocentric position vector (km)\n')
for i = 1:3
    [r,v] = rv_from_observe(rho(i), 0, az(i), 0, el(i), ...
        0, theta(i), phi, H);
    fprintf('\n %5.1f      [%g, %g, %g]\n', t(i), r(1), r(2), r(3))
end
```

Here's the output:

```
  Time
 (min)   Geocentric position vector (km)

  0.0      [2641.68, 5158.02, -3328.73]

  2.0      [2908.04, 5474.36, -2500.03]

  4.0      [3118.6, 5685.65, -1623.34]
```

Accordingly, the three position vectors are

$$\mathbf{r_1} = 2641.7\mathbf{I} + 5158.0\mathbf{J} - 3328.7\mathbf{K} \;\; \text{km}$$
$$\mathbf{r_2} = 2908.0\mathbf{I} + 5474.4\mathbf{J} - 2500.0\mathbf{K} \;\; \text{km}$$

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$$\mathbf{r}_3 = 3118.6\mathbf{I} + 5685.7\mathbf{J} - 1623.3\mathbf{K} \;\; \text{km}$$

Now, we can apply Gibbs' method, implemented in file *gibbs.m* and listed in Appendix D.24, to establish the velocity vector $\mathbf{v_2}$ that corresponds to position vector $\mathbf{r_2}$. The pertaining code follows:

```
global mu
mu = 398600;

%Position vectors
r1 = [2642, 5158, -3329];
r2 = [2908, 5474, -2500];
r3 = [3119, 5686, -1623];

%Applying algorithm 5.1
[v2, ierr] = gibbs(r1,r2,r3);

%Abort if the 3 pos, vectors are not coplanar:
if ierr == true
    fprinf('\n These vectors are not coplanar. \n\n')
    return
end

%Output
fprintf(' Solution:')
fprintf('\n')
fprintf('\n v2 (km/s) = [%g, %g, %g]', v2(1), v2(2), v2(3))
fprintf('\n')
```

Here's the output:

Solution:

```
 v2 (km/s) = [1.994, 2.2064, 7.1306]
```

That is,

$$\mathbf{v}_2 = 1.994\mathbf{I} + 2.2064\mathbf{J} + 7.1306\mathbf{K} \;\; \text{km/s}$$

Finally, the state vector that describes the satellite at observation 2 is

$$\boxed{\mathbf{r}_2 = 2908.0\mathbf{I} + 5474.4\mathbf{J} - 2500.0\mathbf{K}} \;\; \text{km}$$

$$\boxed{\mathbf{v}_2 = 1.994\mathbf{I} + 2.2064\mathbf{J} + 7.1306\mathbf{K}} \;\; \text{km/s}$$

■ **P5.15** (Precision: Maximum)

We've already obtained the pertaining position and velocity vectors in Problem 5.14; all that's left to do is insert these into function *coe_from_sv.m* in Appendix D.18:

```
 r = [2908.0, 5474.4, -2500.0];

 v = [1.994, 2.2064, 7.1306];

 elements = coe_from_sv(r, v, 398600);

 fprintf('\n Spec. angular momentum (km^2/m): %g', elements(1))
 fprintf('\n Eccentricity: %g', elements(2))
 fprintf('\n Right ascension (deg): %g', elements(3)*180/pi)
 fprintf('\n Inclination (deg): %g', elements(4)*180/pi)
 fprintf('\n Argument of the perigee (deg): %g', elements(5)*180/pi)
 fprintf('\n True anomaly (deg): %g', elements(6)*180/pi)
```

Here's the output:

```
Spec. angular momentum (km^2/2): 51639.8
 Eccentricity: 0.0013425
 Right ascension (deg): 60.0011
 Inclination (deg): 94.9989
 Argument of the perigee (deg): 290.811
 True anomaly (deg): 47.1362
```

The orbital elements are summarized below.

Orbital Element	Value
Specific angular momentum, h	51,639.8 km^2/s
Inclination, i	95.00°
Right ascension of ascending node, Ω	60.00°
Eccentricity, e	0.001343
Argument of perigee, ω	290.81°
True anomaly, θ	47.136°

■ **P5.16/17** (Precision: Maximum)

We first use equations (5.56) and (5.57) to convert the given data into three tracking position vectors (**R₁**, **R₂**, **R₃**) and three direction cosine vectors ($\boldsymbol{\rho_1}$, $\boldsymbol{\rho_2}$, $\boldsymbol{\rho_3}$). Afterwards, these vectors and the three observation times are inserted in function *gauss.m*, which implements algorithms 5.5 (to compute an approximation of the state vector (**r**,**v**)) and 5.6 (to iteratively improve the estimated state vector).

```
global mu
mu = 398600;
Re = 6378; %Earthly radius
f = 1/298.26; %Earth's flattening
deg = pi/180; %Angle conversion factor

H = 0; %Elevation of observation site
phi = 29*pi/180; %Latitude of site
t = [0, 60, 120]; %Vector containing times
ra = [0, 65.9279, 79.85]*deg; %Vector of topocentric equatorial right
ascensions
dec = [51.511, 27.9911, 14.6609]*deg; %Vector of topocentric
declination values
theta = [0, 0.250684, 0.501369]*deg; %Vector of local sidereal times

%Equations 5.56 and 5.57
fact1 = Re/sqrt(1-(2*f - f*f)*sin(phi)^2);
fact2 = (Re*(1-f)^2/sqrt(1 - (2*f - f*f)*sin(phi)^2) + H)*sin(phi);
for i = 1:3
    R(i,1) = (fact1 + H)*cos(phi)*cos(theta(i));
    R(i,2) = (fact1 + H)*cos(phi)*sin(theta(i));
    R(i,3) = fact2;
    rho(i,1) = cos(dec(i))*cos(ra(i));
    rho(i,2) = cos(dec(i))*sin(ra(i));
    rho(i,3) = sin(dec(i));
end

%Algorithms 5.5 and 5.6:
 [r, v, r_old, v_old] = gauss(rho(1,:), rho(2,:), rho(3,:), ...
                        R(1,:), R(2,:), R(3,:), ...
                        t(1), t(2), t(3));

fprintf('\n\n Solution: \n')
fprintf('\n Without iterative improvement (Problem 5.16) \n')
fprintf('\n r (km) = [%g, %g, %g]', r_old(1), r_old(2), r_old(3))
fprintf('\n v (km/s) = [%g, %g, %g]', v_old(1), v_old(2), v_old(3))
fprintf('\n\n With iterative improvement (Problem 5.17) \n')
fprintf('\n r (km) = [%g, %g, %g]', r(1), r(2), r(3))
fprintf('\n v (km/s) = [%g, %g, %g]', v(1), v(2), v(3))
```

Here's the output:

```
Solution:

 Without iterative improvement (Problem 5.16)

 r (km) = [5788.09, 484.257, 3341.52]
 v (km/s) = [-0.460072, 8.05816, -0.265618]

 With iterative improvement (Problem 5.17)

 r (km) = [5788.42, 485.007, 3341.96]
 v (km/s) = [-0.460926, 8.0706, -0.266112]
```

As shown, the state vector without iterative improvement is constituted of

$$\mathbf{r} = 5788.09\mathbf{I} + 484.257\mathbf{J} + 3341.52\mathbf{K} \text{ km}$$
$$\mathbf{v} = -0.46007\mathbf{I} + 8.05816\mathbf{J} - 0.265618\mathbf{K} \text{ km/s}$$

In turn, the results *with* iterative improvement are

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$$\mathbf{r} = 5788.42\mathbf{I} + 485.007\mathbf{J} + 3341.96\mathbf{K} \text{ km}$$
$$\mathbf{v} = -0.460926\mathbf{I} + 8.0706\mathbf{J} - 0.266112\mathbf{K} \text{ km/s}$$

■ **P5.18** (Precision: Maximum)

At this point in the course, the student should know that the orbital elements can be easily established by entering the position and velocity vectors into *coe_from_sv.m*, which is provided in Appendix D.18. Here's the code:

```
r = [5788.42, 485.007, 3341.96];
v = [-0.460926, 8.0706, -0.266112];
elements = coe_from_sv(r, v, 398600);

fprintf('\n Spec. angular momentum (km^2/m): %g', elements(1))
fprintf('\n Eccentricity: %g', elements(2))
fprintf('\n Right ascension (deg): %g', elements(3)*180/pi)
fprintf('\n Inclination (deg): %g', elements(4)*180/pi)
fprintf('\n Argument of the perigee (deg): %g', elements(5)*180/pi)
fprintf('\n True anomaly (deg): %g', elements(6)*180/pi)
fprintf('\n Semimajor axis (km): %g\n', elements(7))
```

Here's the output:

```
 Spec. angular momentum (km^2/m): 54201.2
 Eccentricity: 0.100054
 Right ascension (deg): 270
 Inclination (deg): 30.0001
 Argument of the perigee (deg): 89.9993
 True anomaly (deg): 4.15098
 Semimajor axis (km): 7444.75
```

The orbital elements are summarized below.

Orbital Element	Value
Specific angular momentum, h	54,201.2 km^2/s
Inclination, i	30.00°
Right ascension of ascending node, Ω	270.00°
Eccentricity, e	0.100054
Argument of perigee, ω	89.9993°
True anomaly, θ	4.15098°

■ **P5.22/23** (Precision: Maximum)

We have the position vector set ($\mathbf{R_1}$, $\mathbf{R_2}$, $\mathbf{R_3}$), the direction cosine vectors ($\boldsymbol{\rho_1}, \boldsymbol{\rho_2}, \boldsymbol{\rho_3}$), and the corresponding times; we can use *gauss.m* (Appendix D.29) to obtain the state vector of the satellite. Here's the code:

```
t = [0, 60, 120];
R = [-1825.96, 3583.66, 4933.54; ...
    -1841.63, 3575.63, 4933.56; ...
    -1857.25, 3567.54, 4933.54];
rho = [-0.301687, 0.200673, 0.932049; ...
    -0.793090, -0.210324, 0.571640; ...
    -0.873085, -0.362969, 0.325539];

[r, v, r_old, v_old] = gauss(rho(1,:), rho(2,:), rho(3,:), ...
                        R(1,:), R(2,:), R(3,:), ...
                        t(1),   t(2),   t(3));

fprintf('\n\n Solution: \n')
fprintf('\n Without iterative improvement (Problem 5.22) \n')
fprintf('\n r (km) = [%g, %g, %g]', r_old(1), r_old(2), r_old(3))
fprintf('\n v (km/s) = [%g, %g, %g]', v_old(1), v_old(2), v_old(3))
fprintf('\n\n With iterative improvement (Problem 5.23) \n')
fprintf('\n r (km) = [%g, %g, %g]', r(1), r(2), r(3))
fprintf('\n v (km/s) = [%g, %g, %g]', v(1), v(2), v(3))
```

Here's the output:

```
Without iterative improvement (Problem 5.22)

 r (km) = [-2351.29, 3440.47, 5300.91]
 v (km/s) = [-6.62007, -3.88644, -0.414434]

 With iterative improvement (Problem 5.23)

 r (km) = [-2352.14, 3440.24, 5301.52]
```

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```
v (km/s) = [-6.63063, -3.89267, -0.415125]
```

As shown in the code, the state vector without iterative improvement is constituted of

$$\boxed{\begin{array}{l}\mathbf{r} = -2351.29\mathbf{I} + 3440.47\mathbf{J} + 5300.91\mathbf{K}\ \text{km} \\ \mathbf{v} = -6.62007\mathbf{I} - 3.88644\mathbf{J} - 0.414434\mathbf{K}\ \text{km/s}\end{array}}$$

In turn, the results *with* iterative improvement are

$$\boxed{\begin{array}{l}\mathbf{r} = -2352.14\mathbf{I} + 3440.24\mathbf{J} + 5301.52\mathbf{K}\ \text{km} \\ \mathbf{v} = -6.63063\mathbf{I} - 3.89267\mathbf{J} - 0.415125\mathbf{K}\ \text{km/s}\end{array}}$$

■ P5.24 (Precision: Maximum)

As usual, the orbital elements can be obtained with code *coe_from_sv.m* in Appendix D.18.

```
r = [-2352.14, 3440.24, 5301.52];
v = [-6.63063, -3.89267, -0.415125];
elements = coe_from_sv(r, v, 398600);

fprintf('\n Spec. angular momentum (km^2/m): %g', elements(1))
fprintf('\n Eccentricity: %g', elements(2))
fprintf('\n Right ascension (deg): %g', elements(3)*180/pi)
fprintf('\n Inclination (deg): %g', elements(4)*180/pi)
fprintf('\n Argument of the perigee (deg): %g', elements(5)*180/pi)
fprintf('\n True anomaly (deg): %g', elements(6)*180/pi)
fprintf('\n Semimajor axis (km): %g\n', elements(7))
```

Here's the output:

```
 Spec. angular momentum (km^2/m): 51924.7
 Eccentricity: 0.00306513
 Right ascension (deg): 27.9986
 Inclination (deg): 52.0013
 Argument of the perigee (deg): 92.6061
 True anomaly (deg): 1.32085
 Semimajor axis (km): 6764.16
```

The orbital elements are summarized below.

Orbital Element	Value
Specific angular momentum, h	51,924.7 km²/s
Inclination, i	52.0013°
Right ascension of ascending node, Ω	27.9986°
Eccentricity, e	0.00306513
Argument of perigee, ω	92.6061°
True anomaly, θ	1.32085°

■ P6.1 (Precision: Minimum)

Part (a): Considering first the circular orbit, we have a radius r_1 = 6378 + 400 = 6778 km and a speed

$$v_1 = \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{398,600}{6778}} = 7.668\ \text{km/s}$$

For the elliptical orbit, in turn, we have r_{a_2} = 6778 km and r_{p_2} = 6378 + 300 = 6678 km. The specific angular momentum then becomes

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_{a_2}r_{p_2}}{r_{a_2} + r_{p_2}}} = \sqrt{2 \times 398,600} \times \sqrt{\frac{6778 \times 6678}{6778 + 6678}} = 51,784\ \text{km}^2/\text{s}$$

whence we find the speed

$$v_{a_2} = \frac{h_2}{r_{a_2}} = \frac{51,784}{6778} = 7.640\ \text{km/s}$$

The value of Δv follows as

$$\Delta v = \left|v_{a_2} - v_1\right| = \left|7.640 - 7.668\right| = 0.028\ \text{km/s}$$
$$\therefore \Delta v = 28\ \text{m/s}$$

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Now, from the impulse theorem, denoting T as thrust,

$$T \times \Delta t = m \times \Delta v \rightarrow \Delta t = \frac{m \times \Delta v}{T}$$

$$\therefore \Delta t = \frac{125,000 \times 28}{50,000} = \boxed{70 \text{ s}}$$

Part (b): Since speed is not constant, the easiest way to proceed is to take an average value v_{avg} such that

$$v_{avg} = \frac{v_1 + (v_1 + \Delta v)}{2} = v_1 + \frac{\Delta v}{2} = 7.668 + \frac{0.028}{2} = 7.682 \text{ km/s}$$

and then multiply it by the burn time Δt:

$$\Delta s = v_{avg} \Delta t = 7.682 \times 70 = \boxed{538 \text{ km}}$$

Part (c): The ratio we aim for is

$$\frac{\Delta s}{\text{Orbit circumference}} = \frac{538}{2\pi \times 6778} = 0.0126 = \boxed{1.26\%}$$

Part (d): Applying the rocket equation with a specific impulse I_{sp} = 290 sec read from Table 6.1:

$$\frac{\Delta m}{m} = 1 - e^{-\Delta v/I_{sp}g_0} = 1 - \exp\left(-\frac{0.028}{290 \times 0.00981}\right) = \boxed{0.979\%}$$

■ **P6.8 (Precision: Average)**

Part (a): Orbit 1 is a circle associated with speed v_1, namely

$$v_1 = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398,600}{6378 + 300}} = 7.726 \text{ km/s}$$

Orbit 2 is a transfer ellipse with eccentricity

$$e_2 = \frac{r_{apogee,2} - r_{perigee,2}}{r_{apogee,2} + r_{perigee,2}} = \frac{(6378 + 3000) - (6378 + 300)}{(6378 + 3000) + (6378 + 300)} = 0.1682$$

and specific angular momentum

$$r_{perigee,2} = \frac{h_2^2}{\mu} \frac{1}{1 + e_2} \rightarrow 6678 = \frac{h_2^2}{398,600} \times \frac{1}{1 + 0.1682}$$

$$\therefore h_2 = 55,764 \text{ km}^2/\text{s}$$

so that

$$v_{perigee,2} = \frac{55,764}{6678} = 8.350 \text{ km/s}$$

$$v_{apogee,2} = \frac{55,764}{9378} = 5.946 \text{ km/s}$$

Orbit 3 is a circle with speed v_3 computed as

$$v_3 = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398,600}{6378 + 3000}} = 6.520 \text{ km/s}$$

We can proceed to determine the Δv's,

$$\Delta v_1 = v_{perigee,2} - v_1 = 8.350 - 7.726 = 0.624 \text{ km/s}$$
$$\Delta v_2 = v_3 - v_{apogee,2} = 6.520 - 5.946 = 0.574 \text{ km/s}$$

Lastly,

$$\Delta v_{total} = \Delta v_1 + \Delta v_2 = 0.624 + 0.574 = \boxed{1.198 \text{ km/s}}$$

Part (b): We first compute the period of the transfer ellipse:

$$a_2 = \frac{1}{2}\left(r_{perigee,2} + r_{apogee,2}\right) = \frac{1}{2} \times (6678 + 9378) = 8028 \text{ km/s}$$

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$$T_2 = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398,600}} \times 8028^{3/2} = 7159\,\text{s}$$

The orbit transfer time is half of this value, or $t_{\text{perigee to apogee}} = 3580\,\text{s} = 59.67$ min.

■ **P6.11** (Precision: Average)

Let r be the radius of the smaller orbit 1 and $3r$ be the radius of the bigger orbit 3. The speed of the spacecraft in smaller orbit 1 is

$$v_{A,1} = v_1 = \sqrt{\frac{\mu}{r_A}} = \sqrt{\frac{\mu}{r}}$$

For the transfer orbit 2, the angular momentum of the spacecraft is given by

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} = \sqrt{2\mu} \times \sqrt{\frac{r \times 3r}{r + 3r}} = 1.225\sqrt{\mu r}$$

The speed of the spacecraft for orbit 2 at point A is given by

$$v_{A,2} = \frac{h_2}{r_A} = \frac{1.225\sqrt{\mu r}}{r} = 1.225\sqrt{\frac{\mu}{r}}$$

while at point B,

$$v_{B,2} = \frac{h_2}{r_B} = \frac{1.225\sqrt{\mu r}}{3r} = 0.408\sqrt{\frac{\mu}{r}}$$

The speed of the spacecraft in bigger orbit 3 at point B is given by

$$v_{B,3} = \sqrt{\frac{\mu}{r_B}} = \sqrt{\frac{\mu}{3r}} = 0.577\sqrt{\frac{\mu}{r}}$$

The total change in speed required for the orbit transfer is

$$\Delta v = \left|v_{A,2} - v_{A,1}\right| + \left|v_{B,3} - v_{B,2}\right| = \left[\left|1.225 - 1\right| + \left|0.577 - 0.408\right|\right]\sqrt{\frac{\mu}{r}}$$

$$\therefore \boxed{\Delta v = 0.394\sqrt{\frac{\mu}{r}}}$$

■ **P6.13** (Precision: Minimum)

We first calculate the apogee radius of orbit 1, here labeled as r_B:

$$e_1 = \frac{r_B - r_A}{r_B + r_A} \rightarrow 0.3 = \frac{r_B - 7000}{r_B + 7000}$$

$$\therefore r_B = 13,000\ \text{km}$$

Similarly for orbit 2:

$$e_2 = \frac{r_D - r_C}{r_D + r_C} \rightarrow 0.5 = \frac{r_D - 32,000}{r_D + 32,000}$$

$$\therefore r_D = 96,000\ \text{km}$$

We proceed to compute the angular momenta:

$$h_1 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} = \sqrt{2 \times 398,600 \times \frac{7000 \times 13,000}{7000 + 13,000}} = 60,230\ \text{km}^2/\text{s}$$

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_C r_D}{r_C + r_D}} = \sqrt{2 \times 398,600 \times \frac{32,000 \times 96,000}{32,000 + 96,000}} = 138,320\ \text{km}^2/\text{s}$$

$$h_3 = \sqrt{2\mu}\sqrt{\frac{r_A r_D}{r_A + r_D}} = \sqrt{2 \times 398,600 \times \frac{7000 \times 96,000}{7000 + 96,000}} = 72,120\ \text{km}^2/\text{s}$$

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$$h_4 = \sqrt{2\mu}\sqrt{\frac{r_B r_C}{r_B + r_C}} = \sqrt{2 \times 398,600 \times \frac{13,000 \times 32,000}{13,000 + 32,000}} = 85,850 \text{ km}^2/\text{s}$$

and the pertaining velocities:

$$v_{A,1} = \frac{h_1}{r_A} = \frac{60,230}{7000} = 8.604 \text{ km/s}$$

$$v_{A,3} = \frac{h_3}{r_A} = \frac{72,120}{7000} = 10.303 \text{ km/s}$$

$$v_{B,1} = \frac{h_1}{r_B} = \frac{60,230}{13,000} = 4.633 \text{ km/s}$$

$$v_{B,4} = \frac{h_4}{r_B} = \frac{85,850}{13,000} = 6.604 \text{ km/s}$$

$$v_{C,2} = \frac{h_2}{r_C} = \frac{138,320}{32,000} = 4.323 \text{ km/s}$$

$$v_{C,4} = \frac{h_4}{r_C} = \frac{85,850}{32,000} = 2.683 \text{ km/s}$$

$$v_{D,2} = \frac{h_2}{r_D} = \frac{138,320}{96,000} = 1.441 \text{ km/s}$$

$$v_{D,3} = \frac{h_3}{r_D} = \frac{72,120}{96,000} = 0.751 \text{ km/s}$$

The periods of orbits 3 and 4 are, respectively,

$$T_3 = \frac{2\pi}{\sqrt{\mu}}\left(\frac{r_A + r_D}{2}\right)^{3/2} = \frac{2\pi}{\sqrt{398,600}} \times \left(\frac{7000 + 96,000}{2}\right)^{3/2} = 116,310 \text{ s}$$

$$\therefore T_3 = 32.31 \text{ h}$$

$$T_4 = \frac{2\pi}{\sqrt{\mu}}\left(\frac{r_B + r_C}{2}\right)^{3/2} = \frac{2\pi}{\sqrt{398,600}} \times \left(\frac{13,000 + 32,000}{2}\right)^{3/2} = 33,590 \text{ s}$$

$$\therefore T_4 = 9.33 \text{ h}$$

Part (a): The minimum total Δv_1 required for a transfer from the perigee of the inner orbit to the apogee of the outer orbit is given by

$$\Delta v_{\text{total}} = |v_{D,2} - v_{D,3}| + |v_{A,3} - v_{A,1}| = |1.441 - 0.751| + |10.303 - 8.604|$$

$$\therefore \boxed{\Delta v_{\text{total}} = 2.389 \text{ km/s}}$$

The corresponding time of flight is $T_3/2 = 32.31/2 = 16.16$ h.

Part (b): The minimum total Δv_1 required for a transfer from the apogee of the inner orbit to the perigee of the outer orbit is given by

$$\Delta v_{\text{total}} = |v_{B,4} - v_{B,1}| + |v_{C,2} - v_{C,4}| = |6.604 - 4.633| + |4.323 - 2.683|$$

$$\therefore \boxed{\Delta v_{\text{total}} = 3.611 \text{ km/s}}$$

The corresponding time of flight is $T_4/2 = 9.33/2 = 4.67$ h.

■ **P6.14 (Precision: Minimum)**

We first compute the pertaining radial distances:

$$r_{\text{apogee},1} = 6378 + 302 = 6680 \text{ km}$$

$$r_{\text{perigee},1} = 6378 + 296 = 6674 \text{ km}$$

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$$r_{\text{apogee},2} = 6378 + 291 = 6669 \text{ km}$$

$$r_{\text{perigee},2} = 6378 + 259 = 6637 \text{ km}$$

$$r_3 = 6378 + 259 = 6637 \text{ km}$$

$$r_{\text{apogee},4} = 6378 + 255 = 6633 \text{ km}$$

$$r_{\text{perigee},4} = 6378 + 194 = 6572 \text{ km}$$

Then, we determine the momenta and the velocities:

$$h_1 = \sqrt{2\mu}\sqrt{\frac{r_{\text{apogee},1}r_{\text{perigee},1}}{r_{\text{apogee},1} + r_{\text{perigee},1}}} = \sqrt{2 \times 398,600 \times \frac{6680 \times 6674}{6680 + 6674}} = 51,590 \text{ km}^2/\text{s}$$

$$v_{\text{apogee},1} = \frac{h_1}{r_{\text{apogee},1}} = \frac{51,590}{6680} = 7.723 \text{ km/s}$$

$$v_{\text{perigee},1} = \frac{h_1}{r_{\text{perigee},1}} = \frac{51,590}{6674} = 7.730 \text{ km/s}$$

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_{\text{apogee},2}r_{\text{perigee},2}}{r_{\text{apogee},2} + r_{\text{perigee},2}}} = \sqrt{2 \times 398,600 \times \frac{6669 \times 6637}{6669 + 6637}} = 51,500 \text{ km}^2/\text{s}$$

$$v_{\text{apogee},2} = \frac{h_2}{r_{\text{apogee},2}} = \frac{51,500}{6669} = 7.722 \text{ km/s}$$

$$v_{\text{perigee},2} = \frac{h_2}{r_{\text{perigee},2}} = \frac{51,500}{6637} = 7.760 \text{ km/s}$$

$$v_3 = \sqrt{\frac{\mu}{r_3}} = \sqrt{\frac{396,800}{6637}} = 7.750 \text{ km/s}$$

$$h_4 = \sqrt{2\mu}\sqrt{\frac{r_{\text{apogee},4}r_{\text{perigee},4}}{r_{\text{apogee},4} + r_{\text{perigee},4}}} = \sqrt{2 \times 398,600 \times \frac{6633 \times 6572}{6633 + 6572}} = 51,300 \text{ km}^2/\text{s}$$

$$v_{\text{apogee},4} = \frac{h_4}{r_{\text{apogee},4}} = \frac{51,300}{6633} = 7.734 \text{ km/s}$$

$$v_{\text{perigee},4} = \frac{h_4}{r_{\text{perigee},4}} = \frac{51,300}{6572} = 7.806 \text{ km/s}$$

Now, for a transfer from the apogee of orbit 1 to the perigee of orbit 2, we must have

$$h_{12} = \sqrt{2\mu}\sqrt{\frac{r_{\text{apogee},1}r_{\text{perigee},2}}{r_{\text{apogee},1} + r_{\text{perigee},2}}} = \sqrt{2 \times 398,600 \times \frac{6680 \times 6637}{6680 + 6637}} = 51,520 \text{ km}^2/\text{s}$$

$$\therefore \Delta v_{12} = \left|\frac{h_{12}}{r_{\text{apogee},1}} - v_{\text{apogee},1}\right| + \left|\frac{h_{12}}{r_{\text{perigee},2}} - v_{\text{perigee},2}\right| = \left|\frac{51,520}{6680} - 7.723\right| + \left|\frac{51,520}{6637} - 7.760\right|$$

$$\therefore \Delta v_{12} = 0.0130 \text{ km/s}$$

For transition from the perigee of orbit 2 to circular orbit 3, assumed tangent, the delta-*v* is

$$\Delta v_{23} = v_{\text{perigee},2} - v_3 = 7.760 - 7.750 = 0.010 \text{ km/s}$$

For transition from circular orbit 3 to the perigee of orbit 4,

$$h_{34} = \sqrt{2\mu}\sqrt{\frac{r_3 r_{\text{perigee},4}}{r_3 + r_{\text{perigee},4}}} = \sqrt{2 \times 398,600 \times \frac{6637 \times 6572}{6637 + 6572}} = 51,310 \text{ km}^2/\text{s}$$

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$$\Delta v_{34} = \left| \frac{h_{34}}{r_3} - v_3 \right| + \left| \frac{h_{34}}{r_{\text{perigee},4}} - v_{\text{perigee},4} \right| = \left| \frac{51,310}{6637} - 7.750 \right| + \left| \frac{51,310}{6572} - 7.806 \right|$$

$$\therefore \Delta v_{34} = 0.0205 \text{ km/s}$$

Gleaning our results, we obtain the total delta-*v*:

$$\Delta v_{\text{total}} = \Delta v_{12} + \Delta v_{23} + \Delta v_{34} = 0.013 + 0.010 + 0.0205 = 0.0435 \text{ km/s}$$

$$\therefore \boxed{\Delta v_{\text{total}} = 43.5 \text{ m/s}}$$

■ **P6.19 (Precision: Average)**

Part (a): For the circular orbit of the space station,

$$r = 6378 + 350 = 6728 \text{ km}$$

$$v_C = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398,600}{6728}} = 7.697 \text{ km/s}$$

$$T_C = \frac{2\pi}{\sqrt{\mu}} r^{3/2} = \frac{2\pi}{\sqrt{398,600}} \times 6728^{3/2} = 5492 \text{ s} = 91.53 \text{ min}$$

The time required for spacecraft *A* to reach the station is the time it takes for the space station to fly around the original position of spacecraft *A*:

$$t_{SA} = T_c \frac{2\pi r - 600}{2\pi r} = 5492 \times \frac{2\pi \times 6728 - 600}{2\pi \times 6728} = 5414 \text{ s} = \boxed{90.23 \text{ min}}$$

The time required for spacecraft *B* to reach the space station is the time it takes for the space station to fly around to the original position of spacecraft *B*:

$$t_{BS} = T_c \frac{2\pi r + 600}{2\pi r} = 5492 \times \frac{2\pi \times 6728 + 600}{2\pi \times 6728} = 5570 \text{ s} = \boxed{92.83 \text{ min}}$$

Part (b): The period of spacecraft *A*'s phasing orbit is t_{SA}, as determined in the previous part; the semimajor axis of that orbit follows as

$$t_{SA} = \frac{2\pi}{\sqrt{\mu}} a_A^{3/2} \rightarrow 5414 = \frac{2\pi}{\sqrt{398,600}} \times a_A^{3/2}$$

$$\therefore a_A = 6664 \text{ km}$$

Spacecraft *A* is at the *apogee* of its phasing orbit. Resorting to the energy equation, the corresponding speed is

$$v_A = \sqrt{\mu \left(\frac{2}{r} - \frac{1}{a_A} \right)} = \sqrt{398,600 \times \left(\frac{2}{6728} - \frac{1}{6664} \right)} = 7.660 \text{ km/s}$$

The delta-*v* required to drop into the phasing orbit is

$$\Delta v_A = v_A - v_C = 7.660 - 7.697 = -0.037 \text{ km/s}$$

Spacecraft *A* must therefore slow down in order to speed up (i.e., catch the space station). After one circuit of its phasing orbit, this delta-*v* must be *added* in order to rejoin the circular orbit. Thus,

$$\Delta v_{A,\text{total}} = 2 \left| \Delta v_A \right| = 2 \times \left| -0.037 \right| = \boxed{0.074 \text{ km/s}}$$

The period of spacecraft *B*'s phasing orbit is t_{BS}, which determines the semimajor axis of that orbit:

$$t_{BS} = \frac{2\pi}{\sqrt{\mu}} a_B^{3/2} \rightarrow 5570 = \frac{2\pi}{\sqrt{398,600}} a_B^{3/2}$$

$$\therefore a_B = 6792 \text{ km}$$

Spacecraft *B* is at the *perigee* of its phasing orbit. From the energy equation,

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$$v_B = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a_A}\right)} = \sqrt{398,600 \times \left(\frac{2}{6728} - \frac{1}{6792}\right)} = 7.733 \text{ km/s}$$

The prograde delta-v required to enter the phasing orbit is

$$\Delta v_B = v_B - v_C = 7.733 - 7.697 = 0.036 \text{ km/s}$$

Spacecraft B must therefore speed up in order to slow down (i.e., allow the space station to catch up). After one circuit of its phasing orbit, this delta-v must be *subtracted* in order to rejoin the circular orbit. Thus,

$$\Delta v_{B,\text{total}} = 2|\Delta v_B| = 2 \times |-0.036| = \boxed{0.072 \text{ km/s}}$$

■ **P6.21 (Precision: Average)**

The solution is started by finding the period of spacecraft 1:

$$a_1 = \frac{r_A + r_P}{2} = \frac{13,000 + 8000}{2} = 10,500 \text{ km}$$

$$e_1 = \frac{r_A - r_P}{r_A + r_P} = \frac{13,000 - 8000}{13,000 + 8000} = 0.238$$

$$h_1 = \sqrt{\mu(1+e)r_P} = \sqrt{398,600 \times (1+0.238) \times 8000} = 62,831 \text{ km}^2/\text{s}$$

$$T_1 = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398,600}} \times 10,500^{3/2} = 10,708 \text{ s}$$

Next, we compute the time of flight from P to C:

$$E_C = 2\tan^{-1}\left[\sqrt{\frac{1-e_1}{1+e_1}} \tan\left(\frac{\theta_C}{2}\right)\right] = \tan^{-1}\left[\sqrt{\frac{1-0.238}{1+0.238}} \times \tan\left(\frac{30°}{2}\right)\right] = 0.414 \text{ rad}$$

$$M_C = E_C - e_1 \sin(E_C) = 0.414 - 0.238 \times \sin(0.414) = 0.318$$

$$t_C = \frac{M_C}{2\pi} T = \frac{0.318}{2\pi} \times 10,708 = 542 \text{ s}$$

As for the time of flight from P to D:

$$E_D = 2\tan^{-1}\left[\sqrt{\frac{1-e_1}{1+e_1}} \tan\left(\frac{\theta_D}{2}\right)\right] = \tan^{-1}\left[\sqrt{\frac{1-0.238}{1+0.238}} \times \tan\left(\frac{90°}{2}\right)\right] = 1.33 \text{ rad}$$

$$M_D = E_D - e_1 \sin(E_D) = 1.33 - 0.238 \times \sin(1.33) = 1.10 \text{ rad}$$

$$t_D = \frac{M_D}{2\pi} T = \frac{1.10}{2\pi} \times 10,708 = 1875 \text{ s}$$

The time of flight from C to D is given by the difference

$$t_{CD} = t_D - t_C = 1875 - 542 = 1333 \text{ s}$$

We can determine the trajectory from P to D using Lambert's problem. Note first that radial distance r_D at point D is

$$r_D = \frac{h_1^2}{\mu} \frac{1}{1+e_1 \cos\theta_D} = \frac{62,831^2}{398,600} \times \frac{1}{1+0.238 \times \cos 90°} = 9904 \text{ km}$$

so that, in perifocal coordinates,

$$\mathbf{r}_P = 8000\mathbf{p} \text{ km} \text{ ; } \mathbf{r}_D = 9905\mathbf{q} \text{ km}$$

Note further that, on orbit 1,

$$\mathbf{v}_{P,1} = \frac{\mu}{h_1}\left[-\sin\theta_P \mathbf{p} + (e + \cos\theta_P)\mathbf{q}\right] = \frac{398,600}{62,831} \times \left[-\sin 0\mathbf{p} + (0.238 + \cos 0)\mathbf{q}\right]$$

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$$\therefore \mathbf{v}_{P,1} = \frac{398,600}{62,831} \times (0\mathbf{p} + 1.238\mathbf{q})$$

$$\therefore \mathbf{v}_{P,1} = 7.854\mathbf{q} \text{ km/s}$$

and

$$\mathbf{v}_{D,1} = \frac{\mu}{h_1}\left[-\sin\theta_D\mathbf{p} + (e + \cos\theta_D)\mathbf{q}\right] = \frac{398,600}{62,831} \times \left[-\sin 90°\mathbf{p} + (0.238 + \cos 90°)\mathbf{q}\right]$$

$$\therefore \mathbf{v}_{D,1} = \frac{398,600}{62,831} \times (-\mathbf{p} + 0.238\mathbf{q})$$

$$\therefore \mathbf{v}_{D,1} = 6.344 \times (-\mathbf{p} + 0.238\mathbf{q}) = -6.344\mathbf{p} + 1.510\mathbf{q} \text{ km/s}$$

In turn, the velocities on orbit 2 at *P* and *D*, which we denote as **v**$_{D,2}$ and **v**$_{P,2}$ respectively, can be found using code *lambert.m* in Appendix D.25. Here's the input code:

```
global mu
mu = 398600;
deg = pi/180;

r1 = 8000;
r2 = 9905;
dt = 1333;
dtheta = 90;

R1 = [r1, 0, 0];
R2 = [r2*cos(dtheta*deg), r2*sin(dtheta*deg), 0];

string = 'pro';
[V1,V2] = lambert(R1, R2, dt, string);

fprintf('\n\n Solution: \n')
fprintf('\nVelocity vector V1 (km/s) = [%g, %g, %g]', V1(1), V1(2),
V1(3))
fprintf('\nVelocity vector V2 (km/s) = [%g, %g, %g]\n', V2(1), V2(2),
V2(3))
```

Here's the output:

Solution:

Velocity vector V1 (km/s) = [-2.51299, 9.56256, 0]
Velocity vector V2 (km/s) = [-7.72342, 4.35213, 0]

Clearly, the velocities we require are

$$\mathbf{v}_{P,2} = -2.513\mathbf{p} + 9.563\mathbf{q} \text{ km/s}$$
$$\mathbf{v}_{D,2} = -7.723\mathbf{p} + 4.352\mathbf{q} \text{ km/s}$$

We can proceed to determine the Δ*v*'s:

$$\Delta\mathbf{v}_P = \mathbf{v}_{P,2} - \mathbf{v}_{P,1} = (-2.513\mathbf{p} + 9.563\mathbf{q}) - 7.854\mathbf{q} = -2.513\mathbf{p} + 1.709\mathbf{q} \text{ km/s}$$

$$\therefore \|\Delta\mathbf{v}_P\| = \sqrt{2.513^2 + 1.709^2} = 3.039 \text{ km/s}$$

$$\Delta\mathbf{v}_D = \mathbf{v}_{D,2} - \mathbf{v}_{D,1} = (-7.723\mathbf{p} + 4.352\mathbf{q}) - (-6.344\mathbf{p} + 1.510\mathbf{q}) = -1.379\mathbf{p} + 2.842\mathbf{q} \text{ km/s}$$

$$\therefore \|\Delta\mathbf{v}_D\| = \sqrt{1.379^2 + 2.842^2} = 3.159 \text{ km/s}$$

Finally,

$$\Delta v_{\text{total}} = \|\Delta\mathbf{v}_P\| + \|\Delta\mathbf{v}_D\| = 3.039 + 3.159 = \boxed{6.198 \text{ km/s}}$$

■ **P6.23** (Precision: Average)

We begin by computing the period of orbit 1:

$$h_1 = \sqrt{2\pi}\sqrt{\frac{r_{p,1}r_{a,1}}{r_{p,1} + r_{a,1}}} = \sqrt{2 \times 398,600 \times \frac{8100 \times 18,900}{8100 + 18,900}} = 67,232 \text{ km}^2/\text{s}$$

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$$e_1 = \frac{r_{a,1} - r_{p,1}}{r_{a,1} + r_{p,1}} = \frac{18,900 - 8100}{18,900 + 8100} = 0.40$$

$$a_1 = \frac{r_{p,1} + r_{a,1}}{2} = \frac{8100 + 18,900}{2} = 13,500 \text{ km}$$

$$T_1 = \frac{2\pi}{\sqrt{\mu}} a_1^{3/2} = \frac{2\pi}{\sqrt{398,600}} \times 13,500^{3/2} = 15,610 \text{ s}$$

At point B on orbit 1:

$$r_B = \frac{h_1^2}{\mu} \frac{1}{1 + e_1 \cos \theta_B} = \frac{67,232^2}{396,800} \times \frac{1}{1 + 0.40 \times \cos(45°)} = 8840 \text{ km}$$

$$v_{\perp,B1} = \frac{h_1}{r_B} = \frac{67,232}{8840} = 7.605 \text{ km/s}$$

$$v_{r,B1} = \frac{\mu}{h_1} e_1 \sin(\theta_B) = \frac{398,600}{67,232} \times 0.40 \times \sin(45°) = 1.677 \text{ km/s}$$

$$v_{B,1} = \sqrt{\left(v_{\perp,B1}\right)^2 + \left(v_{r,B1}\right)^2} = \sqrt{7.605^2 + 1.677^2} = 7.788 \text{ km/s}$$

$$\gamma_{B,1} = \tan^{-1}\left(\frac{v_{r,B1}}{v_{\perp,B1}}\right) = \tan^{-1}\left(\frac{1.677}{7.605}\right) = 12.44°$$

The period of orbit 2 is the time t_{CB} it takes to fly from C to B on orbit 1.

$$\tan\left(\frac{E_C}{2}\right) = \sqrt{\frac{1 - e_1}{1 + e_1}} \tan\left(\frac{\theta_C}{2}\right) = \sqrt{\frac{1 - 0.4}{1 + 0.4}} \tan\left(\frac{150°}{2}\right) = 2.443$$

$$\therefore E_C = 2\tan^{-1}(2.443) = 2.365$$

$$M_C = E_C - e_1 \sin E_C = 2.365 - 0.4 \times \sin(2.365) = 2.085$$

$$t_C = \frac{M_C}{2\pi} T_1 = \frac{2.085}{2\pi} \times 15,610 = 5180 \text{ s}$$

$$\tan\left(\frac{E_B}{2}\right) = \sqrt{\frac{1 - e_1}{1 + e_1}} \tan\left(\frac{\theta_B}{2}\right) = \sqrt{\frac{1 - 0.4}{1 + 0.4}} \times \tan\left(\frac{45°}{2}\right) = 0.271$$

$$\therefore E_B = 2\tan^{-1}(0.271) = 0.529$$

$$M_B = E_B - e_1 \sin E_B = 0.529 - 0.4 \times \sin(0.529) = 0.327$$

$$t_B = \frac{M_B}{2\pi} T_1 = \frac{0.327}{2\pi} \times 15,610 = 812 \text{ s}$$

$$t_{CB} = T_1 - (t_C - t_B) = 15,610 - (5180 - 812) = 11,242 \text{ s}$$

$$\therefore T_2 = 11,242 \text{ s}$$

Then, we find the semimajor axis of orbit 2:

$$a_2 = \left(\frac{T_2\sqrt{\mu}}{2\pi}\right)^{\frac{2}{3}} = \left(\frac{11,242 \times \sqrt{398,600}}{2\pi}\right)^{\frac{2}{3}} = 10,847 \text{ km}$$

Since $a_2 = (r_B + r_{a,2})/2$, the apogee of orbit 2 is

$$r_{a,2} = 2a_2 - r_B = 2 \times 10,847 - 8840 = 12,854 \text{ km}$$

The specific angular momentum of orbit 2 is

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$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_B r_{a,2}}{r_B + r_{a,2}}} = \sqrt{2 \times 398{,}600 \times \frac{8840 \times 12{,}854}{8840 + 12{,}854}} = 64{,}619 \text{ km}^2/\text{s}$$

Since B is the perigee of orbit 2,

$$v_{B,2} = \frac{h_2}{r_B} = \frac{64{,}619}{8840} = 7.310 \text{ km/s}$$

and

$$\gamma_{B,2} = 0$$

The delta-v required to transfer from orbit 1 to orbit 2 at B is

$$\Delta v = \sqrt{v_{B,1}^2 + v_{B,2}^2 - 2 v_{B,1} v_{B,2} \cos\left(\gamma_{B,2} - \gamma_{B,1}\right)}$$

$$\Delta v = \sqrt{7.788^2 + 7.310^2 - 2 \times 7.788 \times 7.310 \times \cos\left(0 - 12.44^\circ\right)} = 1.703 \text{ km/s}$$

The delta-v required to transfer from orbit 2 back to orbit 1 at the end of the phasing maneuver is the same; therefore,

$$\Delta v_{\text{total}} = 2\Delta v_B = 2 \times 1.703 = \boxed{3.406 \text{ km/s}}$$

■ P6.33 (Precision: Average)

Let C denote the arrival point on spacecraft A as it embarks on the orbit of craft B. The time of flight from A to C is

$$t_{AC,3} = T_1 = \frac{2\pi}{\sqrt{\mu}} R_1^{3/2} = \frac{2\pi}{\sqrt{398{,}600}} \times 8000^{3/2} = 7121 \text{ s}$$

The true anomaly of C on orbit 2 is

$$\theta_{C,2} = 2\pi \frac{t_{AC,3}}{T_2} = 2\pi \left(\frac{R_1}{R_2}\right)^{\frac{3}{2}} = 2\pi \times \left(\frac{8000}{14{,}000}\right)^{\frac{3}{2}} = 2.714 \text{ rad} = 155.5^\circ$$

The perifocal position vectors of A and C are

$$\mathbf{r}_A = 8000\mathbf{p} \text{ km}$$

$$\mathbf{r}_C = R_2\left(\cos\theta_B\mathbf{p} + \sin\theta_B\mathbf{q}\right) = 14{,}000 \times \left(\cos 155.5^\circ \mathbf{p} + \sin 155.5^\circ \mathbf{q}\right)$$

$$\therefore \mathbf{r}_C = -12{,}740\mathbf{p} + 5806\mathbf{q} \text{ km}$$

Now, we can use *lambert.m* to determine orbit 3; the inputs are position vectors **r_A** and **r_C** and time $t_{AC,3}$ = 7121 s. The input code is shown below:

```
global mu
mu = 398600;
R1 = 8000;
R2 = 14000;
r1 = R1*[1, 0, 0];
dtheta = 2*pi*(R1/R2)^(3/2);
r2 = R2*[cos(dtheta), sin(dtheta), 0];
dt = 2*pi/sqrt(mu)*R1^(3/2);
string = 'pro';

[v1, v2] = lambert(r1, r2, dt, string);

fprintf('\n\n Solution: \n')
fprintf('\nVelocity vector V1 (km/s) = [%g, %g, %g]', v1(1), v1(2),
v1(3))
fprintf('\nVelocity vector V2 (km/s) = [%g, %g, %g]\n', v2(1), v2(2),
v2(3))
```

Here's the output:

Solution:

Velocity vector V1 (km/s) = [2.06162, 7.73163, 0]
Velocity vector V2 (km/s) = [-0.610235, -4.577, -0]

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That is,

$$\mathbf{v}_{A,3} = 2.062\mathbf{p} + 7.732\mathbf{q}$$
$$\mathbf{v}_{C,3} = -0.610\mathbf{p} - 4.577\mathbf{q}$$

Because orbits 1 and 2 are circular, equation (2.125) can be used to yield

$$\mathbf{v}_{A,1} = \sqrt{\frac{\mu}{R_1}}\mathbf{q} = \sqrt{\frac{398,600}{8000}}\mathbf{q} = 7.059\mathbf{q} \ \text{km/s}$$

$$\mathbf{v}_{C,2} = \sqrt{\frac{\mu}{R_2}}\left(-\sin\theta_B\mathbf{p} + \cos\theta_B\mathbf{q}\right) = \sqrt{\frac{398,600}{14,000}}\times\left(-\sin155.5^{\circ}\mathbf{p} + \cos155.5^{\circ}\mathbf{q}\right)$$

$$\therefore \mathbf{v}_{C,2} = -2.213\mathbf{p} - 4.855\mathbf{q} \ \text{km/s}$$

We can proceed to determine $\Delta\mathbf{v_A}$ and $\Delta\mathbf{v_C}$:

$$\Delta\mathbf{v}_A = \mathbf{v}_{A,3} - \mathbf{v}_{A,1} = \left(2.062\mathbf{p} + 7.732\mathbf{q}\right) - \left(7.059\mathbf{q}\right) = 2.062\mathbf{p} - 0.673\mathbf{q} \ \text{km/s}$$
$$\therefore \Delta v_A = \left\|\Delta\mathbf{v}_A\right\| = 2.169 \ \text{km/s}$$
$$\Delta\mathbf{v}_C = \mathbf{v}_{C,2} - \mathbf{v}_{C,3} = \left(-2.213\mathbf{p} - 4.855\mathbf{q}\right) - \left(-0.610\mathbf{p} - 4.577\mathbf{q}\right) = -1.603\mathbf{p} - 0.278\mathbf{q} \ \text{km/s}$$
$$\therefore \Delta v_C = \left\|\Delta\mathbf{v}_C\right\| = 1.627 \ \text{km/s}$$

Finally,

$$\Delta v_{\text{total}} = \Delta v_A + \Delta v_C = 2.169 + 1.627 = \boxed{3.796 \ \text{km/s}}$$

■ **P6.34 (Precision: Average)**

Firstly, we obtain velocity vector $\mathbf{v_B}$ for craft B as it outlines elliptical orbit 2:

$$\mathbf{r}_B = 16,000\mathbf{J} \ \text{(km)}$$

$$h_2 = \sqrt{\mu r_B \left(1 + e_2\right)} = \sqrt{398,600 \times 16,000 \times \left(1 + 0.5\right)} = 97,808 \ \text{km}^2/\text{s}$$

$$\mathbf{v}_B = \frac{h_2}{r_B}\mathbf{K} = \frac{97,808}{16,000}\mathbf{K} = 6.113\mathbf{K} \ \text{km/s}$$

Equipped with $\mathbf{r_B}$ and $\mathbf{v_B}$, we can determine the corresponding $\mathbf{r_C}$ and $\mathbf{v_{C,2}}$ at time t_{BC} = 1 hr using code *rv_from_r0v0.m*. Here's the input:

```
global mu
mu = 398600;

r0 = [0, 16000, 0]; %Initial position vector (km)
v0 = [0, 0, 6.113]; %Initial velocity vector (km/s)
t = 3600; %Elapsed time (in this case, 1 hour)

[r, v] = rv_from_r0v0(r0, v0, t); %Algorithm 3.4

fprintf('\n Final position vector: ')
fprintf('\n r = [%12.5e %12.5e, %12.5e] (km)\n', r(1), r(2), r(3))
fprintf('\n Final velocity vector: ')
fprintf('\n v = [%12.5e %12.5e,I %12.5e] (km/s)\n', v(1), v(2), v(3))
```

Here's the output:

```
Final position vector:
 r = [ 0.00000e+00  7.82906e+03,  1.84968e+04] (km)

Final velocity vector:
 v = [ 0.00000e+00 -3.75299e+00,  3.62618e+00] (km/s)
```

The state vector is then

$$\mathbf{r}_C = 7829\mathbf{J} + 18,497\mathbf{K} \ \text{(km)}$$
$$\mathbf{v}_{C,2} = -3.753\mathbf{J} + 3.626\mathbf{K} \ \text{(km/s)}$$

We also have $\mathbf{r_A}$ = 10,000$\mathbf{I}$ km. Knowing $\mathbf{r_A}$, $\mathbf{r_C}$, and the flight time Δt_{AC} = 1 h, we may use *lambert.m* (Algorithm 5.2) to obtain $\mathbf{v_{A,3}}$ and $\mathbf{v_{C,3}}$. Here's the input code:

```
global mu
mu = 398600;
rA = [10000, 0, 0];
rC = [0, 7829, 18497];
tAC = 3600;
```

```
string = 'pro';

[vA, vC] = lambert(rA, rC, tAC, string);

fprintf('\n\n Solution of Lambert''s problem: \n')
fprintf('\n Velocity vector vA (km/s) =
[%12.5e, %12.5e, %12.5e]', ...
    vA(1), vA(2), vA(3))
fprintf('\n Velocity vector vC (km/s) =
[%12.5e, %12.5e, %12.5e]\n', ...
    vC(1), vC(2), vC(3))
```

Here's the output:

```
Solution of Lambert's problem:

 Velocity vector vA (km/s) = [ 9.48526e-01,  3.13605e+00,
7.40932e+00]
 Velocity vector vC (km/s) = [-4.00569e+00,  1.20499e+00,
2.84695e+00]
```

That is,

$$\mathbf{v}_{A,3} = 0.949\mathbf{I} + 3.136\mathbf{J} + 7.409\mathbf{K} \quad \text{km/s}$$
$$\mathbf{v}_{C,3} = -4.006\mathbf{I} + 1.205\mathbf{J} + 2.847\mathbf{K} \quad \text{km/s}$$

Note that $\mathbf{v}_{A,1}$ for craft A in the circular orbit 1 is

$$\mathbf{v}_{A,1} = \sqrt{\frac{\mu}{r_{A,1}}}\mathbf{J} = \sqrt{\frac{398,600}{10,000}}\mathbf{J} = 6.313\mathbf{J} \quad \text{km/s}$$

It remains to determine $\Delta\mathbf{v}_A$,

$$\Delta\mathbf{v}_A = \mathbf{v}_{A,3} - \mathbf{v}_{A,1} = (0.949\mathbf{I} + 3.136\mathbf{J} + 7.409\mathbf{K}) - 6.313\mathbf{J}$$
$$\therefore \Delta\mathbf{v}_A = 0.949\mathbf{I} - 3.177\mathbf{J} + 7.409\mathbf{K} \quad \text{km/s}$$
$$\therefore \Delta v_A = \|\Delta\mathbf{v}_A\| = \boxed{8.117 \text{ km/s}}$$

■ P6.35 (Precision: Average)

We begin by computing parameters of orbit 1:

$$e_1 = \frac{r_{a,1} - r_{p,1}}{r_{a,1} + r_{p,1}} = \frac{10,000 - 7000}{10,000 + 7000} = 0.176$$

$$h_1 = \sqrt{\mu r_{p,1}(1 + e_1)} = \sqrt{398,600 \times 7000 \times (1 + 0.176)} = 57,282 \text{ km}^2/\text{s}$$

$$a_1 = \frac{r_{a,1} + r_{p,1}}{2} = \frac{10,000 + 7000}{2} = 8500 \text{ km}$$

$$T_1 = \frac{2\pi}{\sqrt{\mu}} a_1^{3/2} = \frac{2\pi}{\sqrt{398,600}} \times 8500^{3/2} = 7799 \text{ s}$$

Next, we establish the time since perigee at C:

$$E_C = 2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_C}{2}\right) = 2\tan^{-1}\left(\sqrt{\frac{1-0.176}{1+0.176}}\tan\frac{120°}{2}\right) = 1.934 \text{ rad}$$

$$M_C = E_C - e\sin E_C = 1.934 - 0.176 \times \sin(1.934) = 1.769 \text{ rad}$$

$$t_C = \frac{M_C}{2\pi}T_1 = \frac{1.769}{2\pi} \times 7799 = 2196 \text{ s}$$

The time of flight from C to A on orbit 1 then becomes

$$t_{CA,1} = \frac{T_1}{2} - t_C = \frac{7799}{2} - 2196 = 1704 \text{ s}$$

The time of flight from B to A on orbit 2 is the same as $t_{CA,1}$, that is,

$$t_{BA,2} = t_{CA,1} = 1704 \text{ s}$$

The state vector of point A on orbit 1 is

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$$\mathbf{r}_A = -10,000\mathbf{p} \ \text{km}$$

$$\mathbf{v}_A = -\frac{h_1}{r_A}\mathbf{q} = -\frac{57,282}{10,000}\mathbf{q} = -5.728\mathbf{q} \ \text{km/s}$$

The state vector of point *B* on orbit 1 is

$$\mathbf{r}_B = \frac{h_1^2}{\mu}\frac{1}{1+e_1\cos(\theta_B)}\left(\cos\theta_B\mathbf{p}+\sin\theta_B\mathbf{q}\right) = \frac{57,282^2}{398,600}\times\frac{1}{1+0.176\times\cos 90^\circ}\left(\cos 90^\circ\,\mathbf{p}+\sin 90^\circ\,\mathbf{q}\right)$$

$$\therefore \mathbf{r}_B = 8232\mathbf{q} \ \text{km}$$

$$\mathbf{v}_{B,1} = \frac{\mu}{h_1}\left[-\sin\theta_B\mathbf{p}+\left(e_1+\cos\theta_B\right)\mathbf{q}\right] = \frac{398,600}{57,282}\times\left[-\sin 90^\circ\,\mathbf{p}+\left(0.176+\cos 90^\circ\right)\mathbf{q}\right]$$

$$\therefore \mathbf{v}_{B,1} = -6.959\mathbf{p}+1.225\mathbf{q} \ \text{km/s}$$

Equipped with $\mathbf{r_A}$, $\mathbf{r_B}$, and $t_{BA,2}$, we can use *lambert.m* (Appendix D.25) to determine orbit 2. Here's the input MATLAB code:

```
global mu
mu = 398600;
rA = [-10000, 0, 0];
rB = [0, 8232, 0];
tBA = 1704;
string = 'pro';

[vB, vA] = lambert(rB, rA, tBA, string);

fprintf('\n\n Solution of Lambert''s Problem:\n')
fprintf('\n Velocity vector vA (km/s) =
[%12.5e, %12.5e, %12.5e]', ...
    vA(1), vA(2), vA(3))
fprintf('\n Velocity vector vB (km/s) =
[%12.5e, %12.5e, %12.5e]\n', ...
    vB(1), vB(2), vB(3))
```

Here's the output:

```
Solution of Lambert's Problem:

 Velocity vector vA (km/s) = [-2.35756e+00, -6.78022e+00,
0.00000e+00]

 Velocity vector vB (km/s) = [-8.23642e+00, -9.01357e-01,
0.00000e+00]
```

That is,

$$\mathbf{v}_{A,2} = -2.358\mathbf{p}-6.780\mathbf{q} \ \text{km/s}$$

$$\mathbf{v}_{B,2} = -8.236\mathbf{p}-0.901\mathbf{q} \ \text{km/s}$$

The corresponding Δ*v*'s and their magnitudes are calculated next.

$$\Delta\mathbf{v}_A = \mathbf{v}_{A,1}-\mathbf{v}_{A,2} = -5.728\mathbf{q}-\left(-2.358\mathbf{p}-6.708\mathbf{q}\right)$$

$$\therefore \Delta\mathbf{v}_A = 2.358\mathbf{p}+0.980\mathbf{q}$$

$$\therefore \Delta v_A = \left\|\Delta\mathbf{v}_A\right\| = 2.554 \ \text{km/s}$$

$$\Delta\mathbf{v}_B = \mathbf{v}_{B,2}-\mathbf{v}_{B,1} = \left(-8.236\mathbf{p}-0.901\mathbf{q}\right)-\left(-6.959\mathbf{p}+1.225\mathbf{q}\right)$$

$$\therefore \Delta\mathbf{v}_B = -1.277\mathbf{p}-2.126\mathbf{q}$$

$$\therefore \Delta v_B = \left\|\Delta\mathbf{v}_B\right\| = 2.480 \ \text{km/s}$$

Finally,

$$\Delta v_{\text{total}} = \Delta v_A + \Delta v_B = 2.554 + 2.480 = \boxed{5.034 \ \text{km/s}}$$

■ **P13.1** (Precision: Minimum)

The effective exhaust velocity is $c = I_{sp}g = 235 \times 9.81 = 2310$ m/s. The burnout velocity v_{bo} is given by equation (*f*) in Example 13.1:

$$v_{bo} = c \ln\left(\frac{m_0}{m_f}\right) - \frac{g_0}{\dot{m}_e}\left(m_0 - m_f\right)$$

$$\therefore v_{bo} = 2310 \times \ln\left(\frac{249.5}{170.1}\right) - \frac{9.81}{10.61} \times \left(249.5 - 170.1\right) = 811 \text{ m/s}$$

Likewise, the burnout altitude is given by equation (e) in the same example,

$$h_{bo} = \frac{c}{\dot{m}_e}\left(m_f \ln\frac{m_f}{m_0} + m_0 - m_f\right) - \frac{1}{2}\left(\frac{m_0 - m_f}{\dot{m}_e}\right)^2 g_0$$

$$\therefore h_{bo} = \frac{2310}{10.61} \times \left(170.1 \ln\frac{170.1}{249.5} + 249.5 - 170.1\right) - \frac{1}{2}\left(\frac{249.5 - 170.1}{10.61}\right)^2 \times 9.81$$

$$\therefore h_{bo} = 2830 \text{ m}$$

To find the starting parameters for the second stage, we must account for the 3-second staging delay:

$$v = v_{bo} - g\Delta t_s = 811 - 9.81 \times 3 = 782 \text{ m/s}$$

$$h = h_{bo} + v_{bo}\Delta t_s - \frac{1}{2}g\Delta t_s^2 = 2830 + 811 \times 3 - \frac{1}{2} \times 9.81 \times 3^2 = 5220 \text{ m}$$

Accordingly, the starting parameters for the second stage are v_0 = 782 m/s and h = 5220 m. Since the specific impulse of the second stage is the same as that of the first, the exhaust velocity continues to be c = 2310 m/s. The burnout velocity is now

$$v_{bo} = 782 + 2310 \times \ln\left(\frac{113.4}{58.97}\right) - \frac{9.81}{4.053} \times \left(113.4 - 58.97\right) = 2160 \text{ m/s}$$

The burnout altitude is, in turn,

$$h_{bo} = h_0 + v_0\left(\frac{m_0 - m_f}{\dot{m}_e}\right) + \frac{c}{\dot{m}_e}\left[\ln\left(\frac{m_f}{m_0}\right)m_f + m_0 - m_f\right] - \frac{1}{2}\left(\frac{m_0 - m_f}{\dot{m}_e}\right)g_0$$

$$\therefore h_{bo} = 5220 + 782 \times \left(\frac{113.4 - 58.97}{4.053}\right) + \frac{2310}{4.053}\left[\ln\left(\frac{58.97}{113.4}\right) \times 58.97 + 113.4 - 58.97\right]$$

$$- \frac{1}{2}\left(\frac{113.4 - 58.97}{4.053}\right) \times 9.81 = 24,700 \text{ m}$$

As the rocket coasts to apogee, we have as initial conditions v_0 = 2160 m/s and h_0 = 24,700 m. The maximum time of flight, assuming the craft is subjected to a constant gravity $g \approx 9.81$ m/s², is

$$0 = v_0 - gt_{max} \rightarrow t_{max} = \frac{v_0}{g}$$

$$\therefore t_{max} = \frac{2160}{9.81} = 220 \text{ s}$$

so that

$$h_{max} = h_0 + v_0 t_{max} - \frac{1}{2}gt_{max}^2 = 24,700 + 2160 \times 220 - \frac{1}{2} \times 9.81 \times 220^2 = 262,500 \text{ m}$$

$$\therefore \boxed{h_{max} = 262.5 \text{ km}}$$

This result diverges from the answer mentioned in the textbook, which is 322 km.

■ **P13.2** (Precision: Minimum)
   The latitude at the KSC is about 28°. Thus,

$$v_0 = \omega_{earth} R_{earth} \cos\phi = \left(7.29 \times 10^{-5}\right) \times 6378 \times \cos 28° = 0.411 \text{ km/s}$$

and the corresponding delta-v is, given that the craft must achieve an altitude of 300 km,

$$\Delta v = \sqrt{\frac{398,600}{6378+300}} + 2 - 0.411 = 9.315 \text{ km/s}$$

Substituting into equation (13.28) and accounting for both stages,

$$v_{bo} = v_{bo,1} + v_{bo,2} = I_{sp,1}g_0 \ln\left(\frac{m_{0,1}}{m_{f,1}}\right) + I_{sp,2}g_0 \ln\left(\frac{m_{0,2}}{m_{f,2}}\right) = 9315$$

$$\therefore 290 \times 9.81 \times \ln\left(\frac{2 \times 525,000 + 30,000 + 600,000 + m_{PL}}{2 \times (525,000 - 450,000) + 30,000 + 600,000 + m_{PL}}\right)$$

$$+450 \times 9.81 \times \ln\left(\frac{30,000 + 600,000 + m_{PL}}{30,000 + m_{PL}}\right) = 9315$$

$$\therefore 2845 \times \ln\left(\frac{1.68 \times 10^6 + m_{PL}}{780,000 + m_{PL}}\right) + 4415 \times \ln\left(\frac{630,000 + m_{PL}}{30,000 + m_{PL}}\right) = 9315$$

It remains to solve the transcendental equation above for the payload mass m_{PL}. One way to go is MATLAB's *fsolve* command:

```
rocket = @(mPL) 2845*log((1.68e6 + mPL)/(7.8e5 + mPL)) + ...
    4415*log((6.3e5 + mPL)/(3e4 + mPL)) - 9315;
m0 = 1000;
fsolve(rocket, m0)
```

The code above outputs $m_{PL} \approx 110,880$ kg; this is the payload mass we were asked to determine.

■ **P13.3** (Precision: Minimum)

Let $m_{P,\text{out}}$ and $m_{P,\text{in}}$ denote the masses of propellant in the outbound and inbound legs of the trajectory, respectively. From the problem statement,

$$m_p = m_{p,\text{out}} + m_{p,\text{in}} = m_{p,\text{out}} + \frac{m_{p,\text{out}}}{4} = \frac{5m_{p,\text{out}}}{4}$$

Further, let m_e and m_{PL} denote the empty and payload masses, respectively. From the rocket delta-*v* equation, we may write

$$\Delta v = I_{sp}g_0 \ln\left(\frac{m_e + m_p + m_{PL}}{m_e + m_p - m_{p,\text{out}} + m_{PL}}\right) = 4220 \text{ m/s}$$

$$\therefore \Delta v = 450 \times 9.81 \times \ln\left(\frac{m_e + \frac{5}{4}m_{p,\text{out}} + 3500}{m_e + \frac{5}{4}m_{p,\text{out}} - m_{p,\text{out}} + m_{PL}}\right) = 4220$$

$$\therefore \Delta v = 4218 \times \ln\left(\frac{m_e + 1.25m_{p,\text{out}} + 3500}{m_e + 0.25m_{p,\text{out}} + 3500}\right) = 4220 \text{ (I)}$$

Considering now the return of the craft from GEO to LEO,

$$\Delta v = I_{sp}g_0 \ln\left(\frac{m_e + \frac{m_{p,\text{out}}}{4}}{m_e}\right) = 4220$$

$$\therefore \Delta v = 430 \times 9.81 \times \ln\left(\frac{m_e + \frac{m_{p,\text{out}}}{4}}{m_e}\right) = 4220$$

$$\therefore \frac{m_e + \frac{m_{p,\text{out}}}{4}}{m_e} = \exp\left(\frac{4220}{430 \times 9.81}\right)$$

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$$\therefore \frac{m_e + \dfrac{m_{p,\text{out}}}{4}}{m_e} = 2.72$$

$$\therefore m_e + \frac{m_{p,\text{out}}}{4} = 2.72 m_e$$

$$\therefore \frac{m_{p,\text{out}}}{4} = 1.72 m_e$$

$$\therefore m_{p,\text{out}} = 6.88 m_e$$

Substituting $m_{p,\text{out}}$ into (I) and manipulating,

$$\therefore \Delta v = 4218 \times \ln\left(\frac{m_e + 1.25 \times 6.88 m_e + 3500}{m_e + 0.25 \times 6.88 m_e + 3500}\right) = 4220$$

$$\therefore \ln\left(\frac{m_e + 8.6 m_e + 3500}{m_e + 1.72 m_e + 3500}\right) = \frac{4220}{4218} = 1.00$$

$$\therefore \ln\left(\frac{9.6 m_e + 3500}{2.72 m_e + 3500}\right) = 1.0$$

$$\therefore \frac{9.6 m_e + 3500}{2.72 m_e + 3500} = 2.72$$

$$\therefore 9.6 m_e + 3500 = 2.72 \times (2.72 m_e + 3500)$$

$$\therefore 9.6 m_e + 3500 = 7.40 m_e + 9520$$

$$\therefore 2.2 m_e = 6020$$

$$\therefore m_e = \frac{6020}{2.2} = \boxed{2736 \text{ kg}}$$

■ **P13.4 (Precision: Minimum)**

Part (a): The payload mass fraction π_{PL} is

$$\pi_{PL} = \frac{m_{PL}}{m_0} = \frac{10,000}{150,000} = 0.0667$$

The common payload ratio for a restricted three-stage rocket is given on page 724:

$$\lambda = \frac{\pi_{PL}^{1/3}}{1 - \pi_{PL}^{1/3}} = \frac{0.0667^{1/3}}{1 - 0.0667^{1/3}} = 0.682$$

The structural ratio ε is, in turn,

$$\varepsilon = \frac{m_E}{m_0 - m_{PL}} = \frac{20,000}{150,000 - 10,000} = 0.143$$

The common mass ratio is given by equation (13.55), that is,

$$n_{\text{three-stage}} = \frac{1}{\pi_{PL}^{1/3}(1 - \varepsilon) + \varepsilon} = \frac{1}{0.0667^{1/3} \times (1 - 0.143) + 0.143} = \boxed{2.04}$$

The total delta-v afforded by the three stages follows from equation (13.56):

$$\Delta v = I_{sp} g_0 \ln n^3 = 3 I_{sp} g_0 \ln n = 3 \times 310 \times 9.81 \times \ln(2.04) = 6505 \text{ m/s}$$

$$\therefore \boxed{\Delta v = 6.51 \text{ km/s}}$$

Part (b): The stage propellant masses are given by eqs. (13.58) on page 724:

$$m_{P,1} = \frac{\left(1 - \pi_{PL}^{1/3}\right)(1 - \varepsilon)}{\pi_{PL}} m_{PL} = \frac{\left(1 - 0.0667^{1/3}\right) \times (1 - 0.143)}{0.0667} \times 10,000 = 76,400 \text{ kg}$$

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$$m_{P,2} = \frac{\left(1-\pi_{PL}^{1/3}\right)\left(1-\varepsilon\right)}{\pi_{PL}^{2/3}} m_{PL} = \frac{\left(1-0.0667^{1/3}\right)\times\left(1-0.143\right)}{0.0667^{2/3}}\times 10{,}000 = 31{,}000 \text{ kg}$$

$$m_{P,3} = \frac{\left(1-\pi_{PL}^{1/3}\right)\left(1-\varepsilon\right)}{\pi_{PL}^{2/3}} m_{PL} = \frac{\left(1-0.0667^{1/3}\right)\times\left(1-0.143\right)}{0.0667^{1/3}}\times 10{,}000 = 12{,}600 \text{ kg}$$

Part (c): The empty stage masses are given by eqs. (13.57):

$$m_{E,1} = \frac{\left(1-\pi_{PL}^{1/3}\right)\varepsilon}{\pi_{PL}} m_{PL} = \frac{\left(1-0.0667^{1/3}\right)\times 0.143}{0.0667}\times 10{,}000 = 12{,}740 \text{ kg}$$

$$m_{E,2} = \frac{\left(1-\pi_{PL}^{1/3}\right)\varepsilon}{\pi_{PL}^{2/3}} m_{PL} = \frac{\left(1-0.0667^{1/3}\right)\times 0.143}{0.0667^{2/3}}\times 10{,}000 = 5170 \text{ kg}$$

$$m_{E,3} = \frac{\left(1-\pi_{PL}^{1/3}\right)\varepsilon}{\pi_{PL}^{1/3}} m_{PL} = \frac{\left(1-0.0667^{1/3}\right)\times 0.143}{0.0667^{1/3}}\times 10{,}000 = 2096 \text{ kg}$$

Part (d): Gleaning results from previous parts, the total masses m_0 are calculated to be

$$m_{0,3} = m_{E,3} + m_{P,3} + m_{PL} = 2096 + 12{,}600 + 10{,}000 = 24{,}700 \text{ kg}$$

$$m_{0,2} = m_{E,2} + m_{P,2} + m_{0,3} = 5170 + 31{,}000 + 24{,}700 = 60{,}870 \text{ kg}$$

$$m_{0,1} = m_{E,1} + m_{P,1} + m_{0,2} = 12{,}740 + 76{,}400 + 60{,}870 = 150{,}010 \text{ kg}$$

The small difference in the final result, which should've been 150,000 kg, is due to roundoff.

■ **P13.6 (Precision: Minimum)**

The first step is to determine the exhaust velocities for the two stages:

$$c_1 = I_{sp,1}g_0 = 300\times 9.81 = 2943 \text{ m/s} = 2.943 \text{ km/s}$$

$$c_2 = I_{sp,2}g_0 = 235\times 9.81 = 2305 \text{ m/s} = 2.305 \text{ km/s}$$

The burnout velocity v_{bo} was given as 6.2 km/s; substituting into equation (13.84) brings to

$$c_1 \ln\left(\frac{c_1\eta - 1}{c_1\varepsilon_1\eta}\right) + c_2 \ln\left(\frac{c_2\eta - 1}{c_2\varepsilon_2\eta}\right) = v_{bo}$$

$$\therefore 2.943\ln\left(\frac{2.943\times\eta - 1}{2.943\times 0.2\times\eta}\right) + 2.305\ln\left(\frac{2.305\times\eta - 1}{2.305\times 0.3\times\eta}\right) = 6.2$$

$$\therefore 2.943\ln\left(\frac{2.943\eta - 1}{0.589\eta}\right) + 2.305\ln\left(\frac{2.305\eta - 1}{0.692\eta}\right) = 6.2$$

This transcendental equation can be solved for the Lagrange multiplier η:

```
lagrange = @(eta) 2.943*log((2.943*eta - 1)/(0.589*eta)) + ...
    2.305*log((2.305*eta - 1)/(0.692*eta)) - 6.2;
eta0 = 1;
fsolve(lagrange, eta0)
```

This MATLAB snippet returns $\eta = 1.731$. We can proceed to calculate the optimum mass ratios using equation (13.87):

$$n_1 = \frac{c_1\eta - 1}{c_1\varepsilon_1\eta} = \frac{2.943\times 1.731 - 1}{2.943\times 0.2\times 1.731} = 4.02$$

$$n_2 = \frac{c_2\eta - 1}{c_2\varepsilon_2\eta} = \frac{2.305\times 1.731 - 1}{2.305\times 0.3\times 1.731} = 2.50$$

The corresponding stage masses follow as

$$m_2 = \frac{n_2 - 1}{1 - \varepsilon_2 n_2} m_{PL} = \frac{2.50 - 1}{1 - 0.3 \times 2.50} \times 10 = 60 \text{ kg}$$

$$m_1 = \frac{n_1 - 1}{1 - \varepsilon_1 n_1}\left(m_2 + m_{PL}\right) = \frac{4.02 - 1}{1 - 0.2 \times 4.02} \times \left(60 + 10\right) = 1079 \text{ kg}$$

Finally, the optimum mass *M* of the vehicle becomes

$$M = m_1 + m_2 = 1079 + 60 = \boxed{1139 \text{ kg}}$$

REFERENCES

- CURTIS, H.D. (2014). *Orbital Mechanics for Engineering Students*. 3rd edition. Oxford: Butterworth-Heinemann.
- CURTIS, H.D. (2020). *Orbital Mechanics for Engineering Students*. 4th edition. Oxford: Butterworth-Heinemann.

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