Quiz EL407
Fading Channels

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PROBLEMS

Problem 1 (Sklar, 2001, w/ permission)

For each of the fading-effect categories below, name an application that generally fits that category. Provide numerical justification.

Problem 1.1: Frequency-selective, fast-fading
Problem 1.2: Frequency-selective, slow-fading
Problem 1.3: Flat-fading, fast-fading
Problem 1.4: Flat-fading, slow-fading

Problem 2 (Madhow, 2008, w/ permission)

Consider the information-theoretic analysis of the coded Rayleigh fading channel described in Section 8.2.2 of Madhow (2008).

Problem 2.1: The ergodic capacity with Rayleigh fading is given by

$$C_{\text{Rayleigh}} = \mathbf{E}\left[\log\left(1+G\times SNR\right)\right] = \int_0^\infty \log\left(1+g\times SNR\right)e^{-g}\,dg$$

where bold E denotes expected value, G is the power gain seen by the transmitted signal (in the case of the equation above, an exponential random variable with mean one), SNR is signal-to-noise ratio, and g is the integrating variable. Using the result above and Jensen's inequality, show that the ergodic capacity with Rayleigh fading is strictly smaller than that of an AWGN channel with the same average SNR. That is, fading always reduces the capacity relative to a constant channel gain.

Problem 2.2: For a Rayleigh fading channel, the net channel gain $h[n] \sim CN(0,1)$, i.e., it is represented as a standard normal random variable. Compute the ergodic capacity $C_{fading}(S/N)$ and plot it as a function of SNR in dB. Also plot the AWGN capacity for comparison.

Problem 2.3: Find (analytically) the asymptotic penalty in dB for Rayleigh fading as $SNR \to \infty$.

Problem 2.4: Repeat part 3 for $SNR \to 0$.

Problem 2.5: If the transmitter also knew the fading coefficients $\{h[n]\}$ (e.g., this could be achieved by explicit channel feedback or, for slow fading, by the use of reciprocity), what is your intuition as to what the best strategy should be?

Problem 3 (Proakis and Salehi, 2008, w/ permission)

The scattering function $S(\tau;\lambda)$ for a fading multipath channel is nonzero for the range of values $0 \le \tau \le 1$ ms and $-0.1 \le \lambda \le 0.1$ Hz. Assume that the scattering function is approximately uniform in the two variables.

Problem 3.1: Give numerical values for the following parameters:
→ The multipath spread of the channel.
→ The Dopper spread of the channel.
→ The coherence time of the channel.
→ The coherence bandwidth of the channel.
→ The spread factor of the channel.

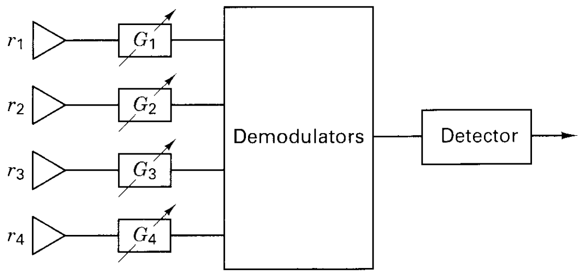
Problem 3.2: Explain the meaning of the following, taking into consideration the answers given in part 1:
→ The channel is frequency-nonselective.
→ The channel is slowly fading.
→ The channel is frequency-selective.

Problem 3.3: Suppose that the frequency allocation (bandwidth) is 10 kHz and we wish to transmit at a rate of 100 bits over this channel. Design a binary communication system with frequency diversity. In particular, specify

→ The type of modulation.
→ The number of subchannels.
→ The frequency separation between adjacent carriers.
→ The signaling interval used in your design.

Problem 4 (Sklar, 2001, w/ permission)

Consider a diversity scheme consisting of four branches as shown below. Each branch is responsible for processing an independent Rayleigh-faded signal *r(t)*. At a particular instant in time, the received signal can be expressed as a four branch vector $\boldsymbol{r} = [r_1\ r_2\ r_3\ r_4]$, where r_i represents the voltage signal at branch *i*. Also, the gain in the branches can be expressed with a four-branch vector $\boldsymbol{G} = [G_1\ G_2\ G_3\ G_4]$, where G_i represents the voltage gain at branch *i*. Consider a moment in time when the measured values of *r* equal [0.87, 1.21, 0.66, 1.90], and the associated gains of $\boldsymbol{G}$ equal [0.5, 0.8, 1.0, 0.8]. The average noise power of each branch *N* is equal to 0.25.



Problem 4.1: Calculate the signal-to-noise ratio applied to the detector.
Problem 4.2: It can be shown that the SNR is maximized when the value of each G_i is equal to r_i^2/N. Based on this condition, determine the maximum achievable SNR.

Problem 5 (Sklar, 2001, w/ permission)

A system makes use of branch diversity in order to improve the receiver signal-to-noise ratio. It is assumed that each branch receives an independently Rayleigh-faded signal. The receiver must meet a requirement that the probability of all branches being received with an SNR less than some threshold is equal to 10^{-4}, where the threshold is chosen to be 5 dB, and the average SNR is equal to 15 dB.
Problem 5.1: Calculate the number of diversity branches *M* required in the receiver to meet this requirement.
Problem 5.2: Based on the result from part 1, calculate the probability, up to seven decimal places, that any single branch will achieve an *SNR* > 5 dB.

Mathematical Interlude 1 (Madhow, 2008, w/ permission)

(Sum of i.i.d. exponential random variables) For a random variable *X*, set

$$K_X = \mathbf{E}\left[e^{sX}\right] = \int_{-\infty}^{\infty} e^{sx} p(x)\, dx$$

where *p* denotes the density of *X*. Note that $K_X(s)$ is the Laplace transform of the density *p(x)* (although *s* is replaced by –*s* in standard signals and systems texts). The present notation is more common when taking Laplace transforms of densities. The expression above is defined wherever the integral converges, and the range of *s* where the integral converges is called the region of convergence.
Problem MI1.1: For *X* exponential with mean one, show that $K_X(s) = 1/(1-s)$, with region of convergence $\mathrm{Re}(s) < 1$.
Problem MI1.2: If $Y = X_1 + X_2$, where X_1 and X_2 are independent, show that

$$K_Y = K_{X_1}(s)\, K_{X_2}(s)$$

Problem MI1.3 Differentiate the definition of K_X with respect to *s* to show that the Laplace transform of *xp(x)* is $dK_X(s)/ds$.
Problem MI1.4: Consider $Y = X_1 + X_2 + \ldots + X_N$ where $\{X_i\}$ are i.i.d., exponential with mean one. We term the random variable *Y* a standard Gamma random variable with dimension *N*. Show that

$$K_Y = \frac{1}{(1-s)^N} \ ; \ \mathrm{Re}(s) < 1$$

Problem MI1.5: Use part 4, and repeated applications of 3, to show that the density of *Y* in part 4 is given by

$$p(y) = \frac{y^{N-1}}{(N-1)!} e^{-y} \ ; \ y \geq 0$$

Problem MI1.6: Now, suppose that the exponential random variables in part 4 each have parameter μ, where $1/\mu$ is the mean. Show that the density of the sum Y is given by

$$p(y) = \mu \frac{(\mu y)^{N-1}}{(N-1)!} e^{-\mu y} \;\; ; \;\; y \geq 0$$

Hint: Scale the random variables in parts 4 and 5 by $1/\mu$.

Mathematical Interlude 2 (Madhow, 2008, w/ permission)

(The Gamma function) For $x > 0$, the Gamma function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Problem MI2.1: For $x > 1$, use integration by parts to show that

$$\Gamma(x) = (x-1)\Gamma(x-1)$$

Problem MI2.2: Show that $\Gamma(1) = 1$.
Problem MI2.3: Use parts 1 and 2 to show that

$$\Gamma(N) = (N-1)!$$

Problem MI2.4: Show that $\Gamma(1/2) = \sqrt{\pi}$.

Hint: You can relate the corresponding density to a Gaussian density by substituting $t = z^2/2$.

Problem MI2.5: Suppose that Y is a standard Gamma random variable of dimension n, as in Problems MI1.4 – MI1.5. Find the expected value $\boldsymbol{E}[\sqrt{Y}]$.

Problem 6 (Madhow, 2008, w/ permission)

(Error probability bound for receive diversity with maximal ratio combining) Consider a coherent binary communication system over a Rayleigh fading channel with N-fold receive diversity and maximal ratio combining, where the fading gains are i.i.d. across elements, and the received signal at each element is corrupted by i.i.d. AWGN processes. Let e_b denote the average bit energy per diversity branch, and let E_b denote the average bit energy summed over diversity branches.

Problem 6.1: For a BPSK system, show that the error probability with maximal ratio combining can be written as

$$P_e = \mathbf{E}\left[Q\left(\sqrt{aY}\right)\right]$$

where a is a constant, and Y is a standard Gamma random variable of dimension N, as in problems MI1.4 and MI1.5. Express a in terms of both e_b/N_0 and E_b/N_0.

Problem 6.2: Using the bound $Q(x) \leq (1/2)e^{-x^2/2}$ in the equation for P_e above, show that (see Problem MI1.4)

$$P_e \leq \frac{1}{2} \times \frac{1}{\left(\frac{a}{2}+1\right)^N}$$

Problem 6.3: For N = 1, 2, 4, 8, plot the error probability bound in part 2 on a log scale versus E_b/N_0 (dB) for BPSK. Also plot for reference the error probability for BPSK transmission over the AWGN channel. Comment on how the error probability behaves with N.

Problem 7 (Madhow, 2008, w/ permission)

(Exact error probability for receive diversity with maximal ratio combining) Consider the setting of Problem 6. We illustrate the exact computation of the error probability $P_e = \boldsymbol{E}[Q(\sqrt{aY})]$ for N = 2 in this problem, using the Gamma function introduced in Problem MI1.2.

Problem 7.1: Calculations for error probability in Rayleigh fading can often be expressed in terms of the Gamma function. With N = 1, for example, we may write

$$P_e = \int_0^\infty Q\left(\sqrt{ay}\right)e^{-y}dy = -e^{-y}Q\left(\sqrt{ay}\right)\Big|_0^\infty + \int_0^\infty e^{-y}\frac{d}{dy}\left[Q\left(\sqrt{ay}\right)\right]dy$$

$$\therefore P_e = \frac{1}{2} - \frac{\sqrt{a}}{2\sqrt{2\pi}}\int_0^\infty y^{-\frac{1}{2}} e^{-\left(1+\frac{a}{2}\right)y} dy$$

$$\therefore P_e = \frac{1}{2} - \frac{\sqrt{a}}{2\sqrt{2\pi}}\Gamma\left(\frac{1}{2}\right)\left(1+\frac{a}{2}\right)^{-\frac{1}{2}}$$

$$\therefore P_e = \frac{1}{2}\left[1-\left(1+\frac{2}{a}\right)^{-\frac{1}{2}}\right]$$

Now, use the relationship derived in Problem *M* for $N = 2$ and integrate by parts to show that

$$P_e = \int_0^\infty Q\left(\sqrt{ay}\right) y e^{-y} dy = -e^{-y} y Q\left(\sqrt{ay}\right)\Big|_0^\infty + \int_0^\infty e^{-y}\frac{d}{dy}\left[yQ\left(\sqrt{ay}\right)\right]dy$$

Problem 7.2: Obtain the expression

$$P_e = \int_0^\infty e^{-y} Q\left(\sqrt{ay}\right) dy - \frac{\sqrt{a}}{2\sqrt{2\pi}}\int_0^\infty y^{\frac{1}{2}} e^{-\left(1+\frac{a}{2}\right)y} dy$$

The first term is exactly the same as in part 1 for *N*, and evaluates to the P_e given in part 1. The second term can be evaluated using Gamma functions. Simplify to obtain the expression

$$P_e = \frac{1}{2}\left[1-\left(1+\frac{2}{a}\right)^{-\frac{1}{2}} - \frac{1}{a}\left(1+\frac{2}{a}\right)^{-\frac{3}{2}}\right]$$

Problem 7.3: Plot the preceding expression (log scale) as a function of E_b/N_0 (dB) for BPSK. Plot for comparison the corresponding bound in Problem 6.3.

Problem 7.4: Show that the high SNR asymptotics for the error probability expressions developed in the introduction to part 1 and in part 2 are given by

$$P_e \approx \frac{1}{2a} \;;\; N=1$$

$$P_e \approx \frac{3}{2a^2} \;;\; N=2$$

Hint: For *x* small, $(1+x)^b \approx 1+bx$. Apply this for $x = 2/a$.

Problem 7.5: Mimic the development in previous parts for a general value of $N \geq 2$. That is, letting $P_e(N)$ denote the error probability with *N*-fold diversity, show that

$$P_e(N) = P_e(N-1) - f(N)$$

where you are to determine an expression for $f(N) > 0$.

Problem 8 (Proakis and Salehi, 2008, w/ permission)

A multipath fading channel has a multipath spread of $T_m = 1$ s and a Doppler spread $B_d = 0.01$ Hz. The total channel bandwidth at bandpass available for signal transmission is $W = 5$ Hz. To reduce the effects of intersymbol interference, the signal designer selects a pulse duration $T = 10$ s.

Problem 8.1: Determine the coherence bandwidth and the coherence time.

Problem 8.2: Is the channel frequency selective? Explain.

Problem 8.3: Is the channel fading slowly or rapidly? Explain.

Problem 8.4: Suppose that the channel is used to transmit binary data via (antipodal) coherently detected PSK in a frequency diversity mode. Explain how you would use the available channel bandwidth to obtain frequency diversity and determine how much diversity is available.

Problem 8.5: For the case in part 4, what is the approximate SNR required per diversity to achieve an error probability of 10^{-6}?

Problem 8.6: Suppose that a wideband signal is used for transmission and a RAKE-type receiver is used for demodulation. How many taps would you use in the RAKE receiver?

Problem 8.7: Explain whether or not the RAKE receiver can be implemented as a coherent receiver with maximal ratio combining.

Problem 8.8: If binary orthogonal signals are used for the wideband signal with square-law post-detection combining in the RAKE receiver, what is the *approximate* SNR required to achieve an error probability of 10^{-6}?

Problem 9 (Proakis and Salehi, 2008, w/ permission)

The Chernov bound for the probability of error for binary FSK with diversity L in Rayleigh fading can be shown to be

$$P_2(L) < \left[4p(1-p)\right]^L = \left[4\frac{(1+\bar{\gamma}_c)}{(2+\bar{\gamma}_c)^2}\right]^L < 2^{-\bar{\gamma}_b g(\bar{\gamma}_c)}$$

where

$$g(\bar{\gamma}_c) = \frac{1}{\bar{\gamma}_c}\log_2\left[\frac{(2+\bar{\gamma}_c)^2}{4(1+\bar{\gamma}_c)}\right]$$

Problem 9.1: Plot $g(\bar{\gamma}_c)$ and determine its approximate maximum value and the value of $\bar{\gamma}_c$ where the maximum occurs.
Problem 9.2: For a given $\bar{\gamma}_b$, determine the optimal order of diversity.
Problem 9.3: Compare $P_2(L)$, under the condition that $g(\bar{\gamma}_c)$ is maximized (optimal diversity), with the error probability for binary FSK and AWGN with no fading, which is

$$P_2 = \frac{1}{2}e^{-\gamma_b/2}$$

and determine the penalty in SNR due to fading and noncoherent (square-law) combining.

Problem 10 (Proakis and Salehi, 2008, w/ permission)

Consider a binary communication system for transmitting a binary sequence over a fading channel. The modulation is orthogonal FSK with third-other frequency diversity ($L = 3$). The demodulator consists of matched filters followed by square-law detectors. Assume that the FSK carriers fade independently and identically according to a Rayleigh envelope distribution. The additive noises on the diversity signals are zero-mean Gaussian with autocorrelation functions $E[z_k^*(t)z_k(t+\tau)] = 2N_0\delta(\tau)$. The noise processes are mutually statistically independent.
Problem 10.1: The transmitted signal may be viewed as binary FSK with square-law detection, generated by a repetition code of the form

1 $\rightarrow c_1 = [1\,1\,1]$
0 $\rightarrow c_2 = [0\,0\,0]$

Determine the error rate performance P_{bh} for a hard-decision decoder following the square-law-detected signals.
Problem 10.2: Evaluate P_{bh} for code SNRs $\bar{\gamma}_c = 100$ and 1000.
Problem 10.3: Evaluate the error rate P_{bs} for $\bar{\gamma}_c = 100$ and 1000 if the decoder employs soft-decision decoding.
Problem 10.4: Consider the generalization of the result in part 1. If a repetition code of block length L (L odd) is used, determine the error probability P_{bh} of the hard-decision decoder and compare that with P_{bs}, the error rate of the soft-decision decoder. Assume $\bar{\gamma} \gg 1$.

Problem 11 (Sklar, 2001, w/ permission)

Consider a TDMA mobile wireless system having a carrier frequency of 1900 MHz that operates on a high-speed train at a speed of 180 km/hr. For learning the channel's impulse response in order to provide equalization, each user's transmission consists of training bits in addition to data bits. It is required that the training sequence should consist of 20 bits, should not occupy more than 20% of the total bits, and that these training bits should be embedded in the data, at least every $T_0/4$ secs, where T_0 denotes coherence time. Assuming binary modulation, what is the slowest transmission rate that would support these requirements without fast fading?

Problem 12 (Sklar, 2001, w/ permission)

Consider a mobile wireless system, using QPSK modulation at 24.3 ksymbols/sec, and a carrier frequency of 1900 MHz. What is the fastest speed in km/hr that is permissible for a vehicle using such a system, if it is required that the change in phase, $\Delta\theta$, due to spectral broadening (Doppler spread) does not exceed 5°/symbol?

Problem 13 (Sklar, 2001, w/ permission)

A mobile wireless system is configured to support a data rate of 200 kbits/sec using QPSK modulation and a carrier frequency of 1900 MHz. It is intended for use in a vehicle that typically travels at a speed of 96 km/hr.

Problem 13.1: What change in phase angle value, $\Delta\theta$ per symbol, can be expected?

Problem 13.2: What is the value of $\Delta\theta$ per symbol, if the data rate is decreased to 100 kbits/sec?

Problem 13.3: Repeat part 2 for a speed of 48 km/hr.

Problem 13.4: Draw some general conclusions.

SOLUTIONS

P.1 ➡ *Solution*

Problem 1.1: Frequency-selective and fast-fading are characterized by channels having a signal bandwidth that exceeds the channel coherence bandwidth, and a fading rapidity that exceeds the symbol rate. Historically, this was first seen in low-data rate telegraphy channels sent over high-frequency (HF) channels having a narrow coherence bandwidth. Since then, we are interested in the fading rapidity as related to the symbol rate, as it should be clear that too slow a signaling rate can be the root cause of fast fading degradation.

Problem 1.2: Frequency-selective and slow-fading are characterized by channels having a signal bandwidth that exceeds the channel coherence bandwidth and a symbol rate that exceeds the fading rapidity. An application that generally fits this category is a cellular telephone channel. For example, in the GSM system, signaling is at the rate of 271 ksymbols/s, and a typical value for the channel coherence bandwidth is under 100 kHz. The symbol duration is 3.69 µs, and for a carrier frequency of 900 MHz and a velocity of 100 km/h, the coherence time is in the order of about 5 – 6 ms. Thus, there are over a thousand symbols transmitted during the coherence-time interval.

Problem 1.3: Flat-fading and fast-fading are characterized by channels having a channel coherence bandwidth that exceeds the signal bandwidth, and a fading rapidity that exceeds the symbol rate. An application that can fit this category is a low-data rate system operating in an environment having small multipath delay spread (large channel coherence bandwidth), where the speed of movement results in fast fading. This might be represented by a low-data rate system operating in a fast-moving vehicle in a desert environment, or a low-data rate radio on a rapidly moving indoor conveyor belt.

Problem 1.4: Flat-fading and slow-fading are characterized by channels having a coherence bandwidth that exceeds the signal bandwidth, and a symbol rate that exceeds the fading rapidity. An application that fits this category is an indoor (low-multipath delay spread) high-data rate system. Here, the data rate need not be very large, if we presume that the fastest speed of movement is represented by a person walking.

P.2 ➡ *Solution*

Problem 2.1: The ergodic capacity is given as

$$C_{\text{erg}} = \mathbf{E}\left[\log_2\left(1+G\times SNR\right)\right]$$

By Jensen's inequality,

$$C_{\text{erg}} = \mathbf{E}\left[\log_2\left(1+G\times SNR\right)\right] \le \log_2\left[\mathbf{E}\left(1+G\times SNR\right)\right]$$

Noting that $\boldsymbol{E}(G)=1$,

$$C_{\text{erg}} = \mathbf{E}\left[\log_2\left(1+G\times SNR\right)\right] \le \log_2\left(1+SNR\right)$$

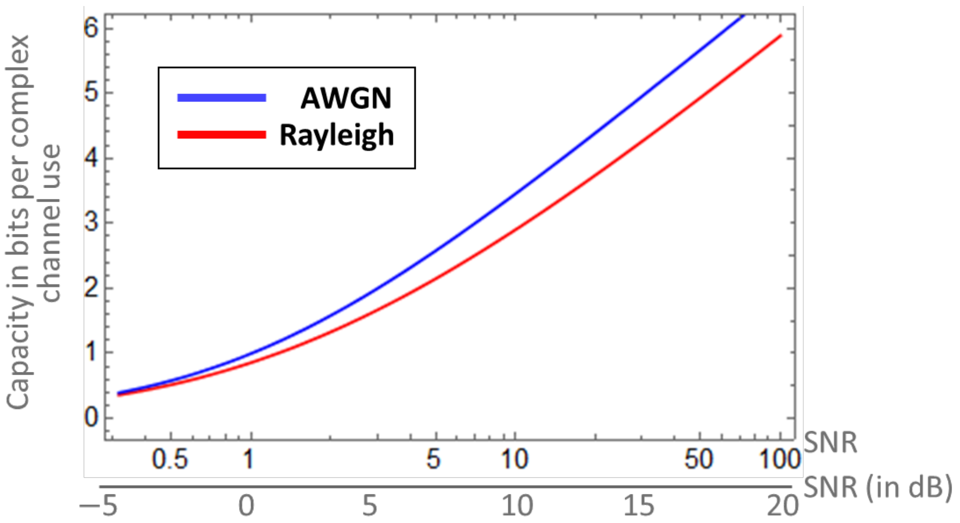
The right-hand side turns out to be the ergodic capacity of an AWGN channel. It follows that the ergodic capacity of a channel with Rayleigh fading is strictly smaller than that of an AWGN channel with the same average SNR.

Problem 2.2: The ergodic capacities of Rayleigh and AWGN channels are given by, respectively,

$$C_{\text{ray}} = \int_0^\infty \log_2\left(1+g\times SNR\right)e^{-g}\,dg$$

$$C_{AWGN} = \log_2\left(1+SNR\right)$$

The two capacity curves are plotted below.



Problem 2.3: Suppose that the Rayleigh fading capacity at some signal-to-noise ratio SNR_1 equals the AWGN capacity at some signal-to-noise ratio SNR_2. We wish to find the ratio SNR_1/SNR_2 as the SNRs get large in order to determine the asymptotic performance loss due to fading. We write

$$\mathbf{E}\left[\log_2\left(1+G\times SNR\right)\right]-\log_2\left(1+SNR_2\right)=0$$

which implies

$$\mathbf{E}\left[\log_2\left(\frac{1+G\times SNR}{1+SNR}\right)\right]-\log_2\left(\frac{1+SNR_2}{1+SNR_1}\right)=0$$

Taking the limit of large SNR brings to

$$\mathbf{E}\left[\log_2 G\right]=\log_2\left(\frac{SNR_2}{SNR_1}\right)$$

$$-\gamma=\mathbf{E}\left[\log_2 G\right]=\int_0^\infty \log_2\left(g\right)e^{-g}dg=-0.577$$

so that the penalty or SNR difference in decibels is $10\gamma/\log_2 10$ = 2.5 dB.

Problem 2.4: As the signal-to-noise ratio tends to zero, the parameter $1+g\times SNR$ tends to unity, leading to the approximation

$$C_{\text{ray}}=\int_0^\infty \log_2\left(\underbrace{1+g\times SNR}_{\to 1}\right)e^{-g}dg=\int_0^\infty\left(\underbrace{\log_2 1}_{=0}\right)e^{-g}dg=0$$

Hence, as the SNR tends to zero, the penalty in dB also tends to zero.

Problem 2.5: Waterfilling across time can be used if the transmitter knows the fading coefficients.

P.3 ➡ *Solution*

Problem 3.1: Based on the information about the scattering function, we glean a multipath spread T_m = 1 ms and a Doppler spread B_d = 0.2 Hz. This gives the first two parameters we were asked to compute. The coherence time, in turn, can be taken as the reciprocal of B_d, that is, $(\Delta t)_c \approx 1/B_d = 1/0.2$ = 5 sec. The coherence bandwidth can be taken as the reciprocal of T_m, namely $(\Delta f)_c \approx 1/T_m = 1/10^{-3}$ = 1 kHz. Finally, the spread factor is the product of T_m and B_d,

$$T_m B_d=\left(1.0\times 10^{-3}\right)\times 0.2=\boxed{2.0\times 10^{-4}}$$

Problem 3.2: → To have a frequency non-selective channel means that the signal transmitted over the channel has a bandwidth lower than the coherence bandwidth of the channel – in the present case, ~1000 Hz.

→ The channel is slowly fading if the signaling interval T is much less than the channel's coherence time – in the case at hand, ~5 sec.

→ Conversely to the first explanation, a channel will be frequency-selective if the signal transmitted over the channel has a bandwidth greater than the coherence bandwidth of the channel.

Problem 3.3: The signal design problem does not have a unique solution. We should use orthogonal M = 4 FSK with a symbol rate of 50 symbols/sec. The signaling interval would then be T = 1/50 sec. For signal orthogonality, we select the frequencies with relative separation $\Delta f = 1/T$ =

50 Hz. With this separation and the stated frequency allocation of 10 kHz, we obtain 10,000/50 = 200 frequencies. Since four frequencies are required to transmit 2 bits, this scheme affords us up to 50th-order diversity. We may use simple repetition-type diversity or a more efficient block or convolutional code of rate ≥ 1/50. The demodulator may use square-law combining.

P.4 ➡ *Solution*

Problem 4.1: Noting that M = 4, we first calculate the total signal envelope in terms of the voltage gains, G_i, which is given by

$$r_M = \sum_{i=1}^{M} G_i r_i \to r_4 = \sum_{i=1}^{4} G_i r_i$$

$$\therefore r_4 = 0.5\times 0.87 + 0.8\times 1.21 + 1.0\times 0.66 + 0.8\times 1.90 = 3.58$$

Since each signal is received with its own demodulator, the total noise power can be computed as

$$N_T = \sum_{i=1}^{M} G_i^2 = 0.5^2 + 0.8^2 + 1.0^2 + 0.8^2 = 2.53$$

For the signal-to-noise ratio $\bar{\gamma}$, we write

$$\gamma_M = \frac{r_M^2}{2N_T} = \frac{3.58^2}{2\times 2.53} = \boxed{2.53}$$

The factor 1/2 was included because the total average normalized power of a bandpass waveform can be shown to equal ½ of the average of the envelope magnitude-square.

Problem 4.2: For the case where $G_i = r_i^2/N$ and the SNR out of the diversity combiner is the sum of the SNRs in each branch, the sum of the individual SNRs is

$$\sum_{i=1}^{M} \frac{r_i^2}{2N} = \frac{1}{2\times 0.25}\times\left(0.87^2 + 1.21^2 + 0.66^2 + 1.90^2\right) = \boxed{12.5}$$

P.5 ➡ *Solution*

Problem 5.1: In Rayleigh fading, the probability that all M independent signal diversity branches are received simultaneously with an SNR less than some threshold value $\bar{\gamma}$ is

$$P\left(\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_M \le \bar{\gamma}\right) = \left[1 - \exp\left(-\frac{\bar{\gamma}}{\Gamma}\right)\right]^M \quad \text{(I)}$$

where Γ is the SNR averaged through the "ups and downs" of fading. Using the relation above, the probability that any single branch achieves SNR > $\bar{\gamma}$ is

$$P\left(\gamma_i > \gamma\right) = 1 - \left[1 - \exp\left(-\frac{\bar{\gamma}}{\Gamma}\right)\right]^M \quad \text{(II)}$$

Appealling to (I), we set the equation to 10^{-4}, substitute $\bar{\gamma}$ = 5 dB = 3.16, Γ = 15 dB = 31.6 and solve for M, giving

$$\left[1 - \exp\left(-\frac{3.16}{31.6}\right)\right]^M = 10^{-4} \to 0.0952^M = 10^{-4}$$

$$\therefore M \times \log_{10} 0.0952 = \log_{10} 10^{-4}$$

$$\therefore M = \frac{-4}{\log_{10} 0.0952} = 3.92$$

Rounding up to the nearest integer, we conclude that at least 4 branches are needed to meet the above specification.

Problem 5.2: Setting M = 4 and $\bar{\gamma}/\Gamma$ = 3.16/31.6 = 0.1 in equation (II), the probability we aim for is

$$P\left(\gamma_i > \gamma\right) = 1 - \left[1 - \exp(-0.1)\right]^4 = \boxed{0.9999180}$$

or 99.9918%.

P.MI1 ➡ *Solution*

Problem MI1.1:

$$K_X = \int_0^{\infty} e^{(s-1)x} dx$$

The integral converges to $1/(1-s)$ only when $\mathrm{Re}(s-1) < 0$ or $\mathrm{Re}(s) < 1$.

Problem MI1.2: Using the definition of Y, we see that

$$K_Y(s) = \mathbf{E}\left[e^{sY}\right] = \mathbf{E}\left[e^{s(X_1+X_2)}\right] = \mathbf{E}\left[e^{sX_1}e^{sX_2}\right]$$

Since (X_1, X_2) are independent, (e^{sX_1}, e^{sX_2}) are also independent. It follows that $K_Y(s) = K_{X_1}(s)\,K_{X_2}(s)$.

Problem MI1.3: Differentiating with respect to s,

$$K'_X(s) = \int e^{sx} x p(x)\, dx$$

So, $K'_X(s)$ is the transform of $xp(x)$.

Problem MI1.4: The result of part 2 is repeatedly applied along with the result in part 1 to arrive at this result.

$$K_Y(s) = K_{X_1+X_2+\ldots+X_N}$$

$$\therefore K_Y(s) = \frac{1}{1-s} \times K_{X_2+\ldots+X_N} \quad \mathrm{Re}(s) < 1$$

$$\therefore K_Y(s) = \frac{1}{(1-s)^2} \times K_{X_3+\ldots+X_N} \quad \mathrm{Re}(s) < 1$$

The pattern is continued to yield the answer.

Problem MI1.5: Let $p(y,N)$ represent the probability density function of $Y = X_1 + X_2 + \ldots + X_N$. Then, from part 4,

$$p(y, N-1) \leftrightarrow \frac{1}{(1-s)^{N-1}} \quad \mathrm{Re}(s) < 1$$

Using part 3,

$$yp(y, N-1) \leftrightarrow \frac{N-1}{(1-s)^{N}} \quad \mathrm{Re}(s) < 1$$

Thus, $yp(y, N-1)$ has the same transform and region of convergence as $(N-1)p(y,N)$. Therefore, $p(y,N) = y/(N-1) \times p(y, N-1)$. This implies

$$p(y,N) = \frac{y}{N-1} \times \frac{y}{N-2} \times \ldots \times \frac{y}{1} \times p(y,1)$$

Since $p(y)$ is just $p(y) = e^{-y}$, the relation above becomes

$$p(y) = \frac{y^{N-1}}{(N-1)!} e^{-y} \quad ; \ y \geq 0$$

Problem MI1.6: We have $Y = X_1 + X_2 + \ldots + X_N = (X'_1 + X'_2 + \ldots + X'_N)/\mu$, where $X'_i = \mu X_i$ is exponential with mean 1. Let $Y' = (X'_1 + X'_2 + \ldots + X'_N)$. Then, $Y = Y'/\mu$. Since Y' has density $p(y)$ as given in part 5, Y has density $p_Y(y) = \mu p(\mu y)$. Substituting the expression for $p(y)$ from part 5 ultimately yields

$$p(y) = \mu \frac{(\mu y)^{N-1}}{(N-1)!} e^{-\mu y} \quad ; \ y \geq 0$$

P.MI2 ➡ *Solution*

Problem MI2.1: The indefinite integral can be integrated by parts as

$$\Gamma(x) = \int t^{x-1} e^{-t} dt = -t^{x-1}e^{-t} + (x-1)\int t^{x-2} e^{-t} dt$$

Substituting the bounds 0 and ∞ gives the general relationship $\Gamma(x) = (x-1)\Gamma(x-1)$.

Problem MI22: Substituting $x = 1$ simplifies the integral to

$$\Gamma(1) = \int_0^{\infty} \underbrace{t^{1-1}}_{=1} e^{-t} dt = \int_0^{\infty} e^{-t} dt = \boxed{1}$$

Problem MI2.3: Using the expression devised in parts 1 and 2, we write the recursive relationship

$$\Gamma(N)=(N-1)\Gamma(N-1)=(N-1)(N-2)\Gamma(N-2)$$
$$\therefore \Gamma(N)=(N-1)(N-2)\times\ldots\times 1\,\underbrace{\Gamma(1)}_{=1}$$
$$\therefore \boxed{\Gamma(N)=(N-1)!}$$

That is, when the variable entry N is an integer, the Gamma function output is the factorial of N minus one unit. Indeed, the Gamma function is basically a generalized version of the factorial operation to noninteger and complex numbers.

Problem MI2.4: Substituting x = 1/2 in the definition of the Gamma function, we write

$$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty t^{\frac{1}{2}-1}e^{-t}dt=\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt$$

Substituting $t=z^2/2$ and manipulating, we obtain

$$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt\rightarrow\Gamma\left(\frac{1}{2}\right)=\sqrt{2}\int_0^\infty e^{-\frac{z^2}{2}}dz$$
$$\therefore \Gamma\left(\frac{1}{2}\right)=2\sqrt{\pi}\underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz}_{=1/2}$$
$$\therefore \Gamma\left(\frac{1}{2}\right)=2\sqrt{\pi}\times\frac{1}{2}=\boxed{\sqrt{\pi}}$$

as we intended to show.

Problem MI2.5: Using the expression of the density function Y from Problem MI1, we have

$$\mathbf{E}\left[\sqrt{Y}\right]=\mu\int_0^\infty \sqrt{y}\frac{(\mu y)^{N-1}}{(N-1)!}e^{-\mu y}dy$$
$$\therefore \mathbf{E}\left[\sqrt{Y}\right]=\frac{\mu^N}{(N-1)!}\int_0^\infty y^{N-\frac{1}{2}}e^{-\mu y}dy$$
$$\therefore \boxed{\mathbf{E}\left[\sqrt{Y}\right]=\frac{\mu^N}{(N-1)!}\Gamma\left(N+\frac{1}{2}\right)}$$

P.6 ➡ *Solution*

Problem 6.1: The received output at each element is given as

$$r_i=h_i x+n_i$$

where parameters n_i and h_i are assumed to be represented as $n_i \sim CN(0,N_0)$ and $h_i \sim CN(0,1)$. Accordingly, the average bit energy per diversity branch can be stated as

$$e_b=\mathbf{E}\left[|h_i|^2\|x\|^2\right]=\|x\|^2$$

Similarly, the average bit energy summed over the diversity branches is $E_b = N||x||^2$. On maximal ratio combining, we get

$$Z=\left(\sum_{i=1}^N|h_i|^2\right)x+\sum_{i=1}^N h_i^* n_i$$

where the real part of Z represents the decision statistic. Taking the real parts, this can be simplified as

$$\mathrm{Re}(Z)=Yx+n_1$$

where $n_1 \sim N(0, N_0Y/2)$ and

$$Y=\sum_{i=1}^N|h_i|^2$$

It follows that the probability of error for BPSK alphabet x can be written as

$$P_e = \mathbf{E}\left[Q\left(\sqrt{\frac{Y^2\|x\|^2}{N_0 Y/2}}\right)\right] = \boxed{\mathbf{E}\left[Q\sqrt{aY}\right]}$$

with $a = 2e_b/N_0 = 2E_b/(N_0 N)$.

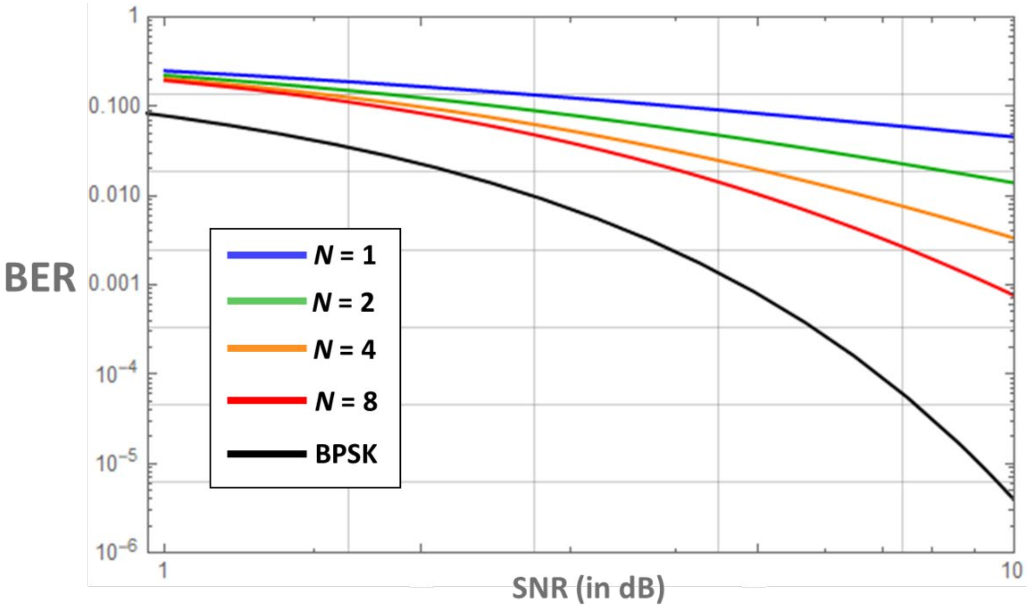
Problem 6.2: Using the bound, we have

$$P_e = \mathbf{E}\left[Q(aY)\right] \le \frac{1}{2}\mathbf{E}\left[e^{-aY/2}\right]$$

The expectation can now be evaluated using the Laplace transform of the density function of Y derived in Problem MI1.4 as

$$P_e \le \frac{1}{2}\mathbf{E}\left[e^{-aY/2}\right] = \boxed{\frac{1}{2\left(1+\frac{a}{2}\right)^N}}$$

Problem 6.3: The BER is plotted against E_b/N_0 for Rayleigh fading with various receive diversity orders. Also shown is the exact BER of BPSK (black curve).



It can be seen that as receive diversity, N, increases, BER decreases and the graph becomes closer to the graph of BPSK. It must be noted, however, that the comparison is not fair because graphs for Rayleigh fading correspond to bounds on BER while the graph of BPSK represents the exact BER.

P.7 ➡ *Solution*

Problem 7.1: It is easy to see that the first term in the expression for P_e evaluates to zero. To evaluate the second term we use the definition of $Q(x)$,

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

which brings to

$$\frac{d}{dy}\left[yQ\left(\sqrt{ay}\right)\right] = Q\left(\sqrt{ay}\right) - \frac{\sqrt{ay}}{2\sqrt{2\pi}}e^{-ay/2}$$

Using this expression, the result given in the problem statement, and expanding the integral with integration by parts, we obtain the expression we aim for.

Problem 7.2: The first integral on the right-hand side evaluates to the result given in the beginning of the problem statement. To obtain the second integral,

$$S = \frac{\sqrt{a}}{2\sqrt{2\pi}}\int_0^{\infty} y^{\frac{1}{2}} e^{-\left(1+\frac{a}{2}\right)y}\,dy$$

we make the substitution $z = (1 + a/2)y$ and manipulate, giving

$$S = \frac{\sqrt{a}}{2\sqrt{2\pi}} \int_0^\infty y^{\frac{1}{2}} e^{-\left(1+\frac{a}{2}\right)y} dy \to S = \frac{\sqrt{a}}{2\sqrt{2\pi}} \left(1+\frac{a}{2}\right)^{-3/2} \int_0^\infty z^{\frac{1}{2}} e^{-z} dz$$

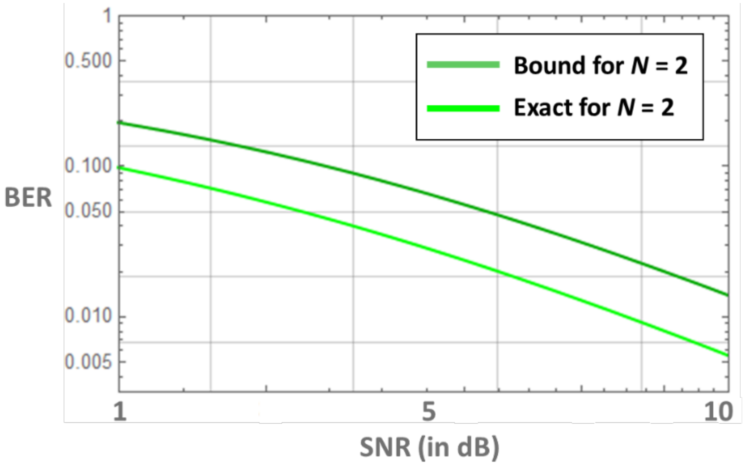
$$\therefore S = \frac{\sqrt{a}}{2\sqrt{2\pi}} \left(1+\frac{a}{2}\right)^{-\frac{3}{2}} \Gamma\left(\frac{3}{2}\right)$$

$$\therefore S = \frac{1}{2a}\left(1+\frac{a}{2}\right)^{-\frac{3}{2}}$$

Substituting in the expression for P_e gives the expression we intended to show,

$$\boxed{P_e = \frac{1}{2}\left[1-\left(1+\frac{2}{a}\right)^{-\frac{1}{2}} - \frac{1}{a}\left(1+\frac{2}{a}\right)^{-\frac{3}{2}}\right]}$$

Problem 7.3: A plot of the exact BER and the bound on BER is shown in continuation.



Problem 7.4: Noting that $(1+x)^b \approx 1+bx$, we write, for $N = 1$,

$$P_e = \frac{1}{2}\left[1-\left(1+\frac{2}{a}\right)^{-\frac{1}{2}}\right] \approx \frac{1}{2}\left[1-\left(1-\frac{1}{a}\right)\right]$$

$$\therefore \boxed{P_e = \frac{1}{2a}}$$

Likewise, for $N = 2$,

$$P_e = \frac{1}{2}\left[1-\left(1+\frac{2}{a}\right)^{-\frac{1}{2}} - \frac{1}{a}\left(1+\frac{2}{a}\right)^{-\frac{3}{2}}\right] \approx \frac{1}{2}\left[1-\left(1-\frac{1}{a}\right)-\frac{1}{a}\left(1-\frac{3}{a}\right)\right]$$

$$\therefore P_e = \frac{1}{2}\left(1-1+\frac{1}{a}-\frac{1}{a}+\frac{3}{a^2}\right)$$

$$\therefore \boxed{P_e = \frac{3}{2a^2}}$$

Problem 7.5: Using the density function of Y, we have

$$P_e(N) = \int_0^\infty Q\left(\sqrt{ay}\right)\frac{y^{N-1}e^{-y}}{(N-1)!}dy$$

$$\therefore P_e(N) = -e^{y} Q\left(\sqrt{ay}\right)\frac{y^{N-1}}{(N-1)!}\bigg|_0^\infty + \int_0^\infty \left\{Q\left(\sqrt{ay}\right)\frac{y^{N-2}}{(N-2)!} - \frac{\sqrt{a}e^{-ay/2}y^{\left(N-\frac{3}{2}\right)}}{2\sqrt{2\pi}}\right\}e^{-y}dy$$

$$\therefore P_e(N) = P(N-1) - \int_0^\infty \frac{\sqrt{a}e^{-ay/2}y^{N-\frac{3}{2}}}{2\sqrt{2\pi}}e^{-y}dy$$

Now, $f(N)$ (the second term in the expression for $P_e(N)$) can be simplified if we make the substitution $z = y(1+a/2)$, giving

$$f(N)=\frac{\sqrt{a}}{2\sqrt{2\pi}}\left(1+\frac{a}{2}\right)^{N-\frac{1}{2}}\int_0^{\infty}e^{-z}z^{N-\frac{3}{2}}dz$$

$$\therefore \boxed{f(N)=\frac{\sqrt{a}}{2\sqrt{2\pi}}\left(1+\frac{a}{2}\right)^{N-\frac{1}{2}}\Gamma\left(N-\frac{1}{2}\right)}$$

P.8 ➡ *Solution*

Problem 8.1: Since the multipath spread is 1 sec, the coherence bandwidth is $(\Delta f)_c \approx 1/T_m = 1$ Hz. The coherence time, in turn, is approximated as the reciprocal of the Doppler spread B_d, that is, $(\Delta t)_c \approx 1/B_d = 1/0.01 = 100$ sec.

Problem 8.2: A fading channel is frequency selective when signal bandwidth is larger than channel bandwidth. Since $W = 5$ Hz and $(\Delta f)_c = 1$ Hz, the channel at hand is frequency selective.

Problem 8.3: A fading channel is fading rapidly when the coherence time is less than the signal pulse duration. Since $T = 10$ s < $(\Delta t)_c$, the channel in question is slowly fading.

Problem 8.4: The desired data rate is not specified in this problem, and hence must be assumed. Note that with a pulse duration of $T = 10$ sec, the binary PSK signals can be spaced at $1/T = 1/10 = 0.1$ Hz apart. With a bandwidth of 5 Hz, we can form 50 subchannels or carrier frequencies. On the other hand, the amount of diversity available in the channel is $W/(\Delta f)_c = 5/1 = 5$. Suppose the desired data rate is 1 bit/sec. In such a case, 10 adjacent carriers can be used to transmit the data in parallel and the information is repeated five times using the total number of 50 subcarriers to achieve 5-th order diversity. A subcarrier separation of 1 Hz is maintained to achieve independent fading of subcarriers carrying the same information.

Problem 8.5: For $\bar{\gamma}_c \gg 1$, we may use the approximation

$$P_b \approx \binom{2L-1}{L}\left(\frac{1}{4\bar{\gamma}_c}\right)^L$$

Setting the probability of error to 10^{-5} as stated and using $L = 5$ as established in part 4, we solve for the average SNR per channel, $\bar{\gamma}_c$, giving

$$P_b = \underbrace{\binom{2\times 5-1}{5}}_{=126}\left(\frac{1}{4\bar{\gamma}_c}\right)^5 < 10^{-6}$$

$$\therefore \left(\frac{1}{4\bar{\gamma}_c}\right)^5 < \frac{10^{-6}}{126}$$

$$\therefore \frac{1}{4\bar{\gamma}_c} < \sqrt[5]{\frac{10^{-6}}{126}}$$

$$\therefore 4\bar{\gamma}_c > \sqrt[5]{\frac{126}{10^{-6}}}$$

$$\therefore \bar{\gamma}_c > \frac{1}{4}\sqrt[5]{\frac{126}{10^{-6}}} = 10.4 = \boxed{10.2\,\text{dB}}$$

Problem 8.6: The tap spacing between adjacent taps is 1/5 = 0.2 seconds. The total multipath spread is $T_m = 1$ sec. Accordingly, we employ a RAKE receiver with at least 5 taps.

Problem 8.7: Since the fading is slow relative to the pulse duration, in principle we can employ a coherent receiver with pre-detection combining.

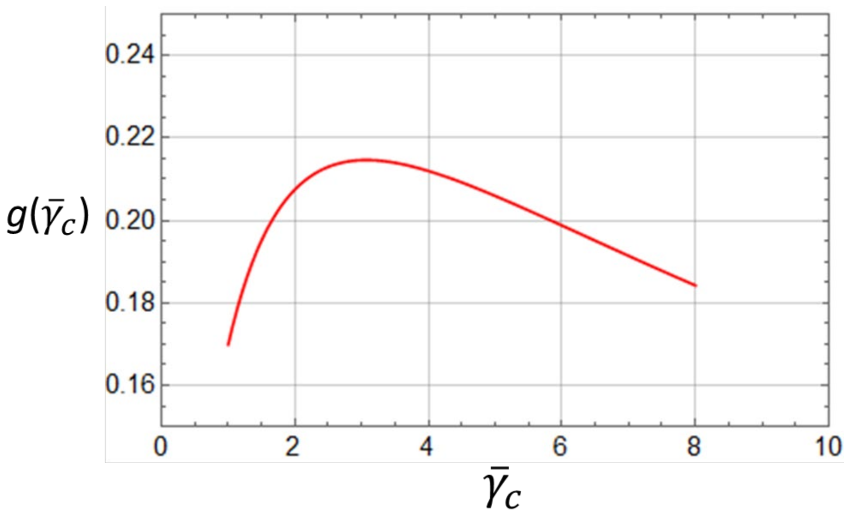
Problem 8.8: The same formula as part 5 still applies. Using Mathematica to speed things up a bit, we have

```
In[21]:= Solve[Binomial[9, 5] * (1 / γc)^5 == 10^-6., γc]
   ··· Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a
       corresponding exact system and numericizing the result.
Out[21]= {{γc → -33.7312 - 24.5071 i}, {γc → -33.7312 + 24.5071 i},
   {γc → 12.8842 - 39.6534 i}, {γc → 12.8842 + 39.6534 i}, {γc → 41.6941}}
```

That is, $\bar{\gamma}_c = 41.7 = 16.2$ dB.

P.9 ➡ *Solution*

Problem 9.1: $g(\bar{\gamma}_c)$ is plotted against $\bar{\gamma}_c$ below.



Using the *Get Coordinates* function in Mathematica, we see that g reaches a maximum of approximately 0.213 at $\bar{\gamma}_c = 2.94 \approx 3$.

Problem 9.2: The SNR per bit and the average SNR per channel are related by the simple expression $\bar{\gamma}_c = \bar{\gamma}_b/L$. It follows that for a given $\bar{\gamma}_b$ the optimum diversity is $L = \bar{\gamma}_b/\bar{\gamma}_c = \bar{\gamma}_b/3$.

Problem 9.3: For the optimum diversity, we have

$$P_2\left(L_{\text{opt}}\right) < 2^{-0.213\bar{\gamma}_b} = e^{-\ln 2\times 0.213\bar{\gamma}_b} = e^{-0.148\bar{\gamma}_b} = \frac{1}{2}e^{-0.148\bar{\gamma}_b+\ln 2}$$

For the non-fading channel,

$$P_2 = \frac{1}{2}e^{-0.5\bar{\gamma}_b}$$

$$10\log_{10}\left(\frac{0.5}{0.148}\right) = \boxed{5.29\,\text{dB}}$$

P.10 ➡ *Solution*

Problem 10.1: P_{2h} can be stated as

$$P_{2h} = p^3 + 3p^2(1-p)$$

where $p = 1/(2+\bar{\gamma}_c)$ and $\bar{\gamma}_c$ is the received SNR per cell.

Problem 10.2: We can easily evaluate P_{2h} for the two given channel SNRs with Mathematica's *Table* function,

```
In[33]:= p = 1/(2 + γc)
Out[33]= 1/(2 + γc)

In[37]:= ScientificForm[Table[p^3 + 3 * p^2 * (1 - p), {γc, {100., 1000.}}]]
Out[37]//ScientificForm=
{2.86466 × 10^-4, 2.98605 × 10^-6}
```

That is, $P_{2h} \approx 2.86\times10^{-4}$ for $\bar{\gamma}_c = 100$ and $P_{2h} \approx 2.99\times10^{-6}$ for $\bar{\gamma}_c = 1000$.

Problem 10.3: Since $\bar{\gamma}_c \gg 1$, we may use the approximation

$$P_{2s} \approx \binom{2L-1}{L}\left(\frac{1}{\bar{\gamma}_c}\right)^L$$

where L is the order of diversity. For $L = 3$, the expression simplifies to

$$P_{2s} \approx \binom{2\times 3-1}{3}\left(\frac{1}{\bar{\gamma}_c}\right)^3 = 10\left(\frac{1}{\bar{\gamma}_c}\right)^3$$

With $\bar{\gamma}_c = 100$,

$$P_{2s} \approx 10\left(\frac{1}{100}\right)^3 = \frac{10}{10^6} = \boxed{10^{-5}}$$

With $\bar{\gamma}_c = 1000$,

$$P_{2s} \approx 10\left(\frac{1}{1000}\right)^3 = \frac{10}{10^9} = \boxed{10^{-8}}$$

Problem 10.4: For hard-decision decoding, we may write

$$P_{2h} = \sum_{k=(L+1)/2}^{L} \binom{L}{k} p^k (1-p)^{L-k} \le \left[4p(1-p)\right]^{L/2}$$

where the latter is the Chernoff bound, L is odd, and $p = 1/(2 + \bar{\gamma}_c)$. For soft-decision decoding, the approximation to use is the one adopted in part 3,

$$P_{2s} \approx \binom{2L-1}{L}\left(\frac{1}{\bar{\gamma}_c}\right)^L$$

P.11 ➡ *Solution*

The coherence time can be related to the Doppler spread/fading rate by the simple expression

$$T_0 \approx \frac{0.5}{f_d} = \frac{0.5}{V/\lambda} \rightarrow T_0 = \frac{0.5\lambda}{V}$$

where $\lambda = c/f = (3\times10^8)/(1.9\times10^9) = 0.158$ m and $V = 180/3.6 = 50$ m/s, which brings to

$$T_0 = \frac{0.5\times0.158}{50} = 1.58\times10^{-3}\ \text{s}$$

Thus, the training sequence must be received every $T_0/4 = 3.95\times10^{-4}$ s. Since the training sequence consists of 20 bits and should not occupy more than 20% of the total bits, then the slowest data rate corresponds to delivering 100 bits in $T_0/4$ bits, leading to a rate $R = 100/(3.95\times10^{-4}) = 253$ kbits/sec. If the bit rate were any lower, it would require more time than $T_0/4$ s to receive the 20-bit training sequence.

P.12 ➡ *Solution*

The wavelength is $\lambda = (3\times10^8)/(1.9\times10^9) = 0.158$ m. The spread frequency $f_d = V/\lambda$ is related to the symbol rate and the spectral broadening $\Delta\theta$ by the simple expression

$$\Delta\theta = \frac{f_d}{R_S}\times360^\circ$$

so that

$$\Delta\theta = \frac{f_d}{R_S}\times360^\circ = \frac{V/\lambda}{R_S}\times360^\circ$$

$$\therefore V = \frac{\Delta\theta R_S\lambda}{360^\circ} = \frac{5^\circ\times24{,}300\times0.158}{360^\circ} = 53.3\,\text{m/s} = \boxed{192\,\text{km/h}}$$

P.13 ➡ *Solution*

Problem 13.1: The wavelength is 0.158 m. Noting that the speed of the vehicle is equivalent to 26.7 m/s, the Doppler spread is calculated to be $f_d = V/\lambda = 26.7/0.158 = 169$ Hz. The signaling is QPSK, which means that a data rate of 200 kbits/s corresponds to a signaling rate of 100 ksymbols/sec. Finally, the phase angle value $\Delta\theta$ per symbol is determined as

$$\Delta\theta = \frac{f_d}{R_S}\times360^\circ = \frac{169}{100{,}000}\times360^\circ = \boxed{0.608^\circ}$$

Problem 13.2: For a data rate of 100 kbits/sec, the QPSK symbol rate is 50 ksymbols/s, leading to a $\Delta\theta$ per symbol such that

$$\Delta\theta = \frac{f_d}{R_S}\times360^\circ = \frac{169}{50{,}000}\times360^\circ = \boxed{1.22^\circ}$$

Problem 13.3: A speed of 48 km/h is equivalent to 13.3 m/s, and leads to a Doppler spread frequency $f_d = 13.3/0.158 = 84.2$ Hz. The associated $\Delta\theta$/symbol is

$$\Delta\theta = \frac{f_d}{R_S} \times 360^{\circ} = \frac{84.2}{50,000} \times 360^{\circ} = \boxed{0.606^{\circ}}$$

Problem 13.4: $\Delta\theta$ per symbol is directly proportional to velocity and inversely proportional to symbol rate.

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