# Quiz VB101
# Free Vibration
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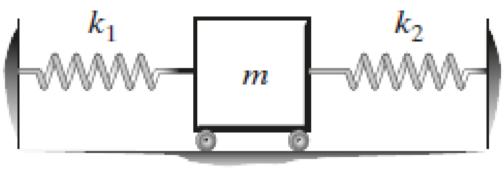
## PROBLEMS

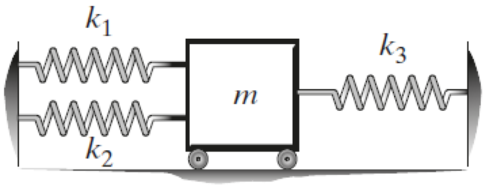
### Problem 1A (Inman, 2014, w/ permission)

Determine the natural frequency of the system illustrated below.



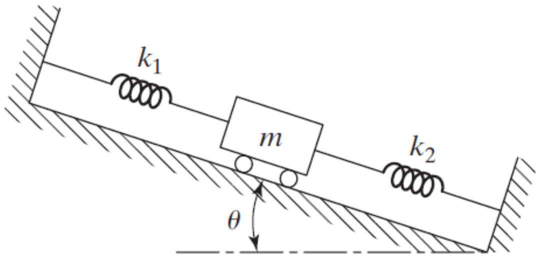

### Problem 1B

Determine the natural frequency of the system illustrated below.




### Problem 1C (Rao, 2011, w/ permission)

Find the natural frequency of vibration of a spring-mass system arranged on an inclined plane as shown.



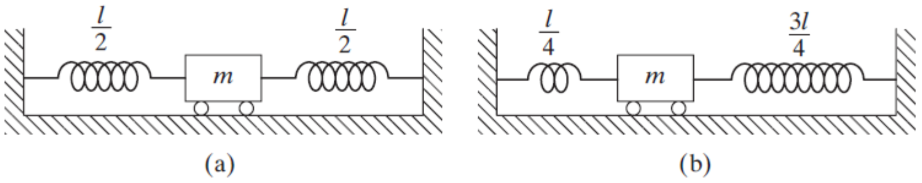

### Problem 2 (Inman, 2014, w/ permission)

When designing a linear mass-spring system it is often a matter of choosing a spring constant such that the resulting natural frequency has a specified value. Suppose that the mass of the system is 4 kg and the stiffness is 100 N/m. How much must the stiff stiffness be changed in order to increase the natural frequency by 10%?

**A)** The stiffness must be raised by 10%.
**B)** The stiffness must be raised by 15%.
**C)** The stiffness must be raised by 21%.
**D)** The stiffness must be raised by 26%.

## Problem 3 (Rao, 2014, w/ permission)

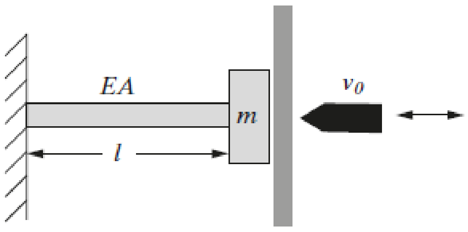
A helical spring of stiffness $k$ is cut into two halves and a mass $m$ is connected to the two halves as shown in Figure (a) below. The natural time period of this system is found to be 0.5 s. If an identical spring is cut so that one-fourth and the other part three-fourths of the original length, and the mass $m$ is connected to the parts as shown in Figure (b), what would be the natural period of the system?




**A)** $\tau_2$ = 0.30 s
**B)** $\tau_2$ = 0.43 s
**C)** $\tau_2$ = 0.57 s
**D)** $\tau_2$ = 0.69 s

## Problem 4A (Inman, 2014, w/ permission)

A bar of negligible mass fixed with a tip mass forms part of a machine used to punch holes in a sheet of metal as it passes the fixture as illustrated in the next figure. The impact to the mass and bar fixture causes the bar to vibrate. The impact of the mass and bar fixture causes the bar to vibrate and the speed of the process demands that frequency of vibration not interfere with the process. The static design yields a mass of 50 kg and that the bar be made of steel of length 0.25 m with a cross-sectional area of 0.01 m². Compute the system's natural frequency. Use $E$ = 200×10⁹ Pa.




**A)** $\omega_n$ = 12,650 rad/s
**B)** $\omega_n$ = 18,880 rad/s
**C)** $\omega_n$ = 24,200 rad/s
**D)** $\omega_n$ = 32,000 rad/s

## Problem 4B

Consider the punch fixture shown above. If the punch strikes the mass off center, it is possible that the steel bar may vibrate in torsion. The mass is 1000 kg and the bar 0.25-m long, with a square cross-section that is 0.1 m on a side. The mass polar moment of inertia of the tip mass is 10 kg/m². The polar moment of inertia for a square bar is $b^4/6$, where $b$ is the length of the side of the square. Compute both the torsion and longitudinal frequencies. Which one is larger? In addition to the Young's modulus specified in the previous problem, use $\rho$ = 7880 kg/m³ and $G$ = 80×10⁹ N/m².

**A)** The longitudinal natural frequency is about 145% greater than the torsional natural frequency.

**B)** The longitudinal natural frequency is about 290% greater than the torsional natural frequency.

**C)** The torsional natural frequency is about 145% greater than the longitudinal natural frequency.

**D)** The torsional natural frequency is about 290% greater than the longitudinal natural frequency.

## Problem 5 (Inman, 2014, w/ permission)

A manufacturer makes a cantilevered leaf spring from steel ($E = 2 \times 10^{11}$ N/m²) and sizes the spring so that the device has a specific frequency. Later, to save weight, the spring is made from aluminum ($E = 71 \times 10^{10}$ N/m²). Assuming that the mass of the spring is much less than that of the device the spring is attached to, determine if the frequency increases or decreases and by how much.

**A)** The natural frequency is increased by 40% with the use of aluminum.

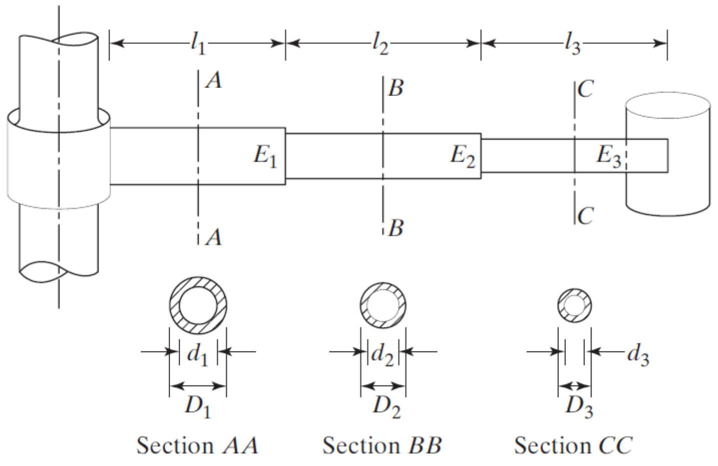
**B)** The natural frequency is increased by 20% with the use of aluminum.

**C)** The natural frequency is decreased by 20% with the use of aluminum.

**D)** The natural frequency is decreased by 40% with the use of aluminum.

## Problem 6 (Rao, 2011, w/ permission)

A pick-and-place robot arm, shown in the next figure, carries an object weighing 10 lb. Find the natural frequency of the robot arm in the axial direction for the following data: $l_1$ = 12 in., $l_2$ = 10 in., $l_3$ = 8 in., $E_1 = E_2 = E_3 = 10^7$ psi, $D_1$ = 2 in., $D_2$ = 1.5 in., $D_3$ = 1 in., $d_1$ = 1.75 in., $d_2$ = 1.25 in., and $d_3$ = 0.75 in.




**A)** $\omega_n$ = 1040 rad/s

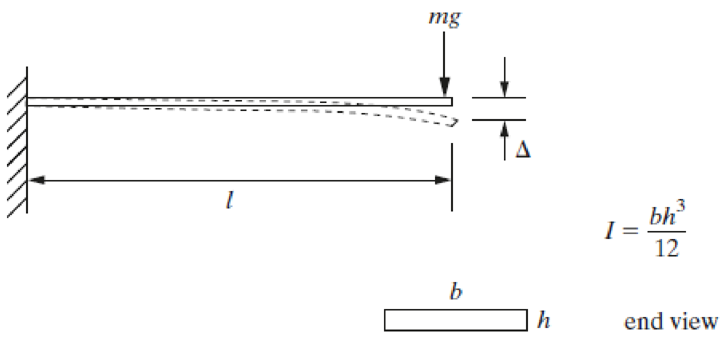
**B)** $\omega_n$ = 1565 rad/s

**C)** $\omega_n$ = 2032 rad/s

**D)** $\omega_n$ = 2578 rad/s

## Problem 7 (Inman, 2014, w/ permission)

Consider the diving board of the next figure. For divers, a certain level of static deflection is desirable, denoted here by Δ. Compute a design formula for the dimensions of the board ($b$, $h$, and $\ell$) in terms of the static deflection, the average diver's mass $m$, and the modulus of the board material.




## Problem 8 (Rao, 2011, w/ permission)

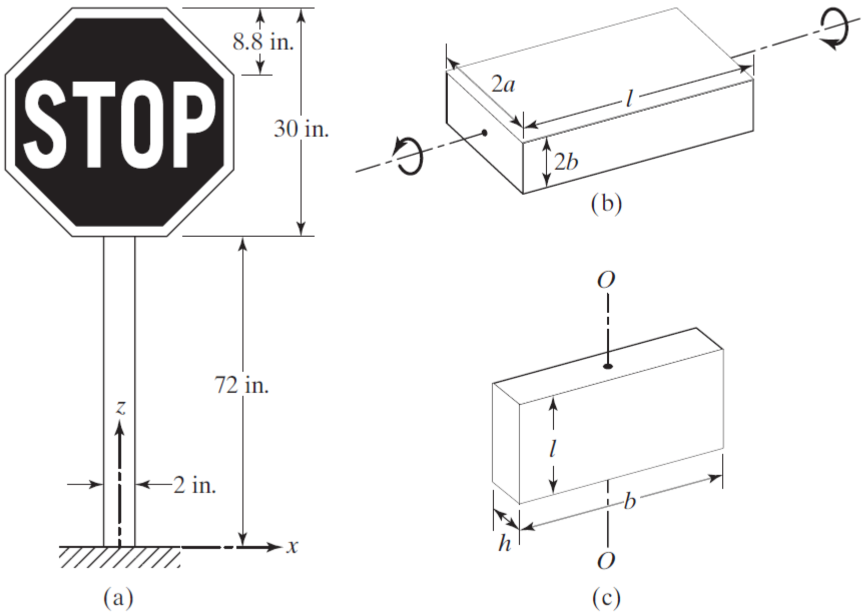
The next figure shows a steel traffic sign, of thickness 1/8 in. fixed to a steel post. The post is 72 in. high with a cross-section 2 in. × ¼ in., and it can undergo torsional vibration (about the $z$-axis) or bending vibration (either in the $zx$-plane or the $yz$-plane). Determine the mode of vibration of the post in a storm during which the wind velocity has a frequency component of 1.27 Hz. Neglect the weight of the post in finding the frequencies of vibration. Assume the torsional stiffness of a shaft with a rectangular section is given by

$$k_t = 5.33 \frac{ab^3 G}{\ell}\left[1 - 0.63\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$$

where $G$ is the shear modulus. In addition, the mass moment of inertia of a rectangular block about axis $OO$ (see figure (c)) is given by

$$I_{OO} = \frac{\rho\ell}{3}\left(b^3h + h^3b\right)$$

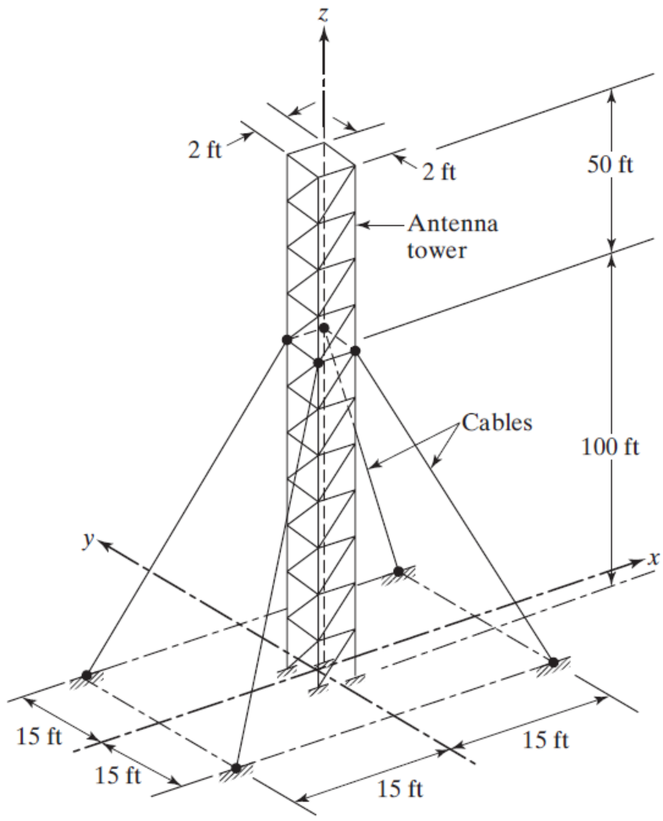
where $\rho$ is the density of the block. Use $\gamma$ = 0.283 lb/in.³ as the weight density of steel, $E = 30\times10^6$ psi., and $G = 12\times10^6$ psi.




**A)** The mode of vibration will be transverse vibration about the *xz* plane.
**B)** The mode of vibration will be transverse vibration about the *yz* plane.
**C)** The mode of vibration will be torsional vibration about the *xy* plane.
**D)** The information is insufficient to determine the mode of vibration.

## Problem 9 (Rao, 2011, w/ permission)

A TV antenna tower is braced by four cables, as shown in the next figure. Each cable is under tension and is made of steel with a cross-sectional area of 0.5 in.² The antenna tower can be modeled as a steel beam of square section of side 1 in. for estimating its mass and stiffness. Find the tower's natural frequency of vibration about the *y*-axis. Use $E = 30\times10^6$ psi and $\rho$ = 0.283 lb/in.³ for steel.




**A)** $\omega_n$ = 10.2 rad/s
**B)** $\omega_n$ = 20.5 rad/s
**C)** $\omega_n$ = 30.7 rad/s
**D)** $\omega_n$ = 40.1 rad/s

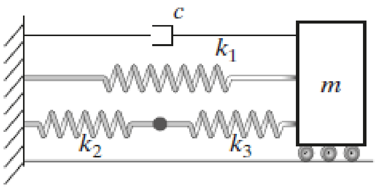
## Problem 10 (Kelly, 1996, w/ permission)

A 500-kg vehicle is mounted on springs such that its static deflection is 1.5 mm. What is the damping coefficient of a viscous damped to be added to the system in parallel with the springs such that the system is critically damped?

**A)** $c$ = 20,400 N · s/m
**B)** $c$ = 40,500 N · s/m
**C)** $c$ = 60,800 N · s/m
**D)** $c$ = 80,900 N · s/m

## Problem 11 (Inman, 2014, w/ permission)

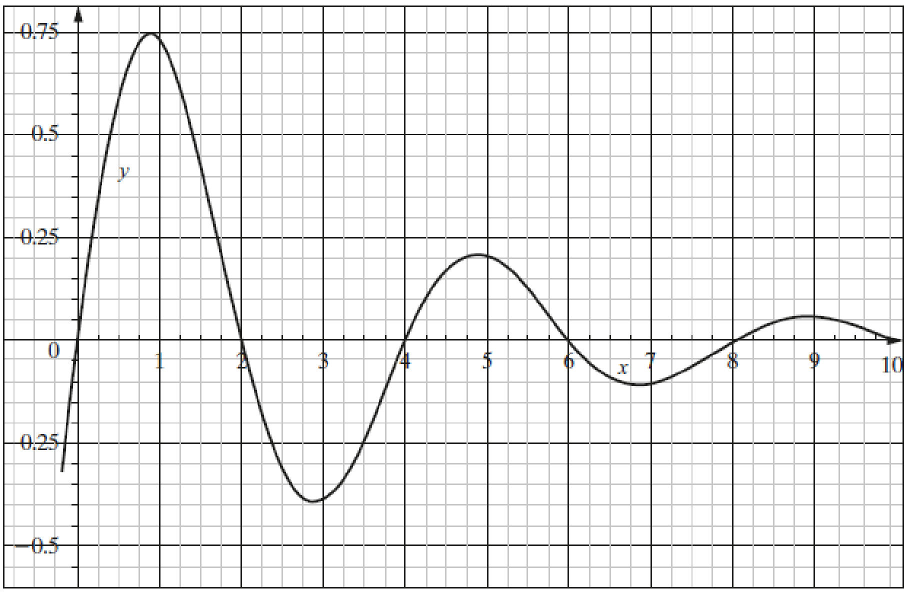
Calculate the natural frequency and damping ratio for the system in the figure below given the values $m$ = 10 kg, $c$ = 100 kg/s, $k_1$ = 4000 N/m, $k_2$ = 200 N/m, and $k_3$ = 1000 N/m. Assume that no friction acts on the rollers. Is the system overdamped, critically damped, or underdamped?




**A)** The system is underdamped.
**B)** The system is critically damped.
**C)** The system is overdamped.
**D)** There is not enough information to establish the type of mechanical damping.

## Problem 12 (Inman, 2014, w/ permission)

The displacement of a vibrating spring-mass-damper system is recorded on an $x - y$ plotter and reproduced in the next figure. The $y$-coordinate is the displacement in cm and the $x$-coordinate is time in seconds. True or false?



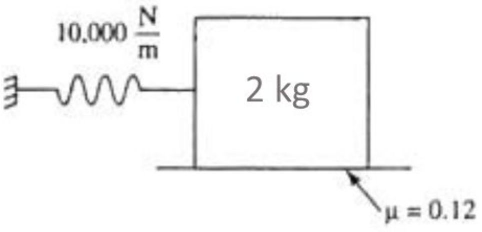

**1.** ( ) The system is underdamped.

**2.** ( ) The damped frequency of the system is greater than 1.75 rad/s.

**3.** ( ) The undamped natural frequency and the damped natural frequency of the system are within more than 10% of one another.

## Problem 13 (Kelly, 1996, w/ permission)

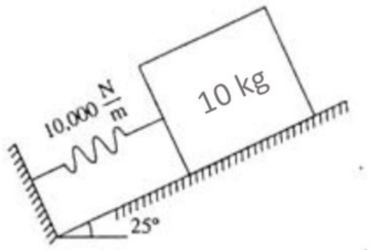
The block illustrated below is displaced 10 mm and released. How many cycles of motion will be executed before the block returns to rest?



**A)** $N$ = 5 cycles
**B)** $N$ = 11 cycles
**C)** $N$ = 17 cycles
**D)** $N$ = 23 cycles

## Problem 14 (Kelly, 1996, w/ permission)

The block of the next figure is displaced 25 mm and released. It is observed that the amplitude decreases 1.2 mm each cycle. What is the coefficient of friction between the block and the surface?

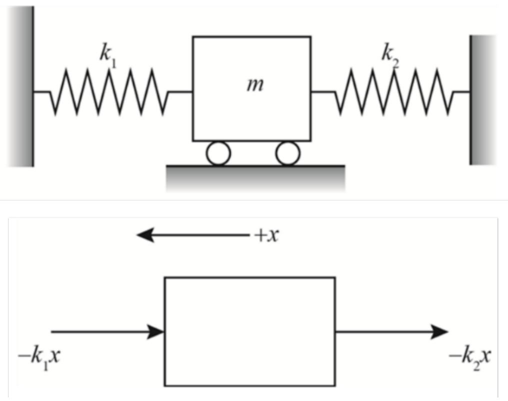



**A)** $\mu$ = 0.011
**B)** $\mu$ = 0.025
**C)** $\mu$ = 0.034
**D)** $\mu$ = 0.045

# SOLUTIONS

## P.1 › Solution

**Part A:** The spring-mass system and its corresponding free body diagram are illustrated in the next figure.




Applying Newton's second law of motion to the block, we have

$$m\ddot{x} = -k_1x - k_2x$$

$$\therefore m\ddot{x} + (k_1 + k_2)x = 0$$

$$\therefore \ddot{x} + \frac{(k_1 + k_2)}{m}x = 0$$

Let us compare this equation with the equation of motion for an undamped 1-DOF system, namely

$$m\ddot{x} + kx = 0$$

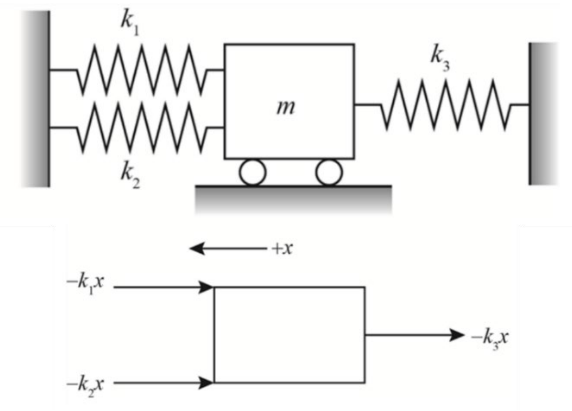
$$\therefore \ddot{x} + \frac{k}{m}x = 0$$

$$\therefore \ddot{x} + \omega_n^2 x = 0$$

Clearly, the natural frequency is given by

$$\omega_n^2 = \frac{k_1 + k_2}{m} \rightarrow \boxed{\omega_n = \sqrt{\frac{k_1 + k_2}{m}}}$$

**Part B:** The spring-mass system and its corresponding free body diagram are illustrated in the next figure.




The equation of motion of the system in question is

$$m\ddot{x} = -k_1x - k_2x - k_3x$$

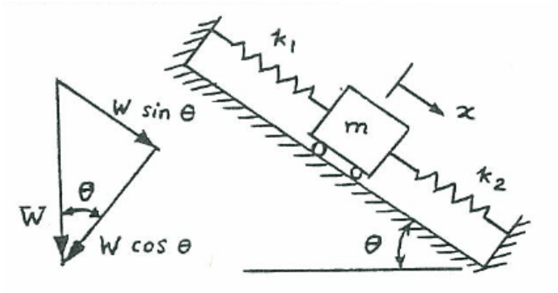
$$\therefore m\ddot{x} + (k_1 + k_2 + k_3)x = 0$$

$$\therefore \ddot{x} + \frac{(k_1 + k_2 + k_3)}{m}x = 0$$

Comparing this with the relation for a general 1-DOF system under free vibration, we see that

$$\omega_n^2 = \frac{k_1 + k_2 + k_3}{m} \rightarrow \boxed{\omega_n = \sqrt{\frac{k_1 + k_2 + k_3}{m}}}$$

**Part C:** The system in question and the forces acting upon it are illustrated below.




Let $x$ be measured from the position of mass at which the springs are unstretched, and $\delta$ be the elongation of the springs. The system's equation of motion is

$$m\ddot{x} = -k_1(x+\delta) - k_2(x+\delta) + W\sin\theta$$

From Hooke's law, we can write $W\sin\theta = (k_1 + k_2)\delta$. Substituting in the equation above gives

$$m\ddot{x} = -k_1(x+\delta) - k_2(x+\delta) + (k_1 + k_2)\delta$$

$$\therefore m\ddot{x} = -k_1x - \cancel{k_1\delta} - k_2x - \cancel{k_2\delta} + \cancel{k_1\delta} + \cancel{k_2\delta}$$

$$\therefore m\ddot{x} + (k_1 + k_2)x = 0$$

$$\therefore \ddot{x} + \frac{(k_1 + k_2)}{m}x = 0$$

Comparing this result with the equation of motion, the natural frequency is determined as

$$\omega_n^2 = \frac{k_1 + k_2}{m} \rightarrow \boxed{\omega_n = \sqrt{\frac{k_1 + k_2}{m}}}$$

Note that the result is the same as that of Part A; that is, the natural frequency is independent of the angle of the incline.

## P.2 › Solution

Initially, the natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{4}} = 5 \text{ rad/s}$$

Once the natural frequency is increased by 10%, its new value becomes $\omega_n'$ = $1.1\omega_n$ = 5.5 rad/s. Assume the new stiffness of the system to be $k'$. Its value is determined as

$$\omega_n' = \sqrt{\frac{k'}{m}} \rightarrow k' = m\omega_n'^2$$

$$\therefore k' = 4\times 5.5^2 = 121 \text{ N/m}$$

The percentage increase in stiffness is then

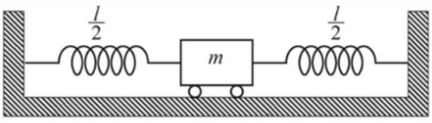
$$\Delta k = \frac{k'-k}{k}\times 100\% = \frac{121-100}{100} = \boxed{+21\%}$$

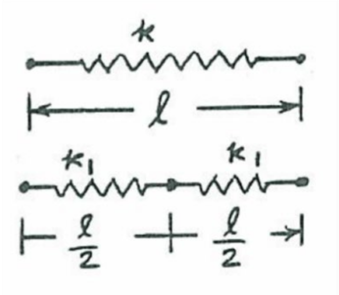
That is, in order for the natural frequency to be increased by one tenth, the stiffness must be raised by 21 percent.

▸ The correct answer is **C**.

## P.3 › Solution

The initial arrangement and a schematic of the spring are shown below.




From the illustration of the spring system, we have

$$\frac{1}{k_{\text{total}}} = \frac{1}{k_1} + \frac{1}{k_1} = \frac{2}{k_1} \rightarrow k_1 = 2k$$

where we have replaced $k_{\text{total}}$ with $k$. We then establish the equivalent stiffness for the initial arrangement,

$$k_{\text{eq},1} = 2k_1 \rightarrow k_{\text{eq},1} = 2\times 2k = 4k$$

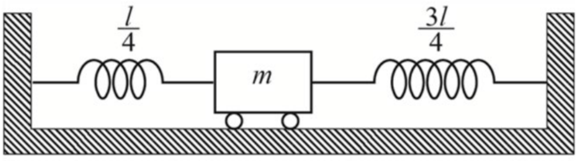
The period of vibrational motion is given by the usual relation

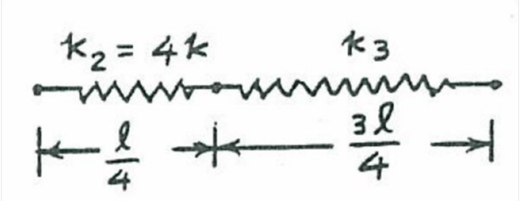
$$\tau_1 = 2\pi\sqrt{\frac{m}{k_{\text{eq},1}}} = 2\pi\sqrt{\frac{m}{4k}} \rightarrow \tau_1 = \pi\sqrt{\frac{m}{k}}$$

However, we know that the period is 0.5 s for the initial arrangement. Therefore,

$$0.5 = \pi\sqrt{\frac{m}{k}} \rightarrow \sqrt{\frac{m}{k}} = \frac{1}{2\pi} \quad \text{(I)}$$

The second arrangement and the schematic of the modified spring are shown in continuation.




In this case, we can state that

$$\frac{1}{k_{\text{total}}}=\frac{1}{k_2}+\frac{1}{k_3}$$

$$\therefore\frac{1}{k}=\frac{1}{4k}+\frac{1}{k_3}$$

$$\therefore\frac{1}{k_3}=\frac{4}{4k}-\frac{1}{4k}=\frac{3}{4k}$$

$$\therefore k_3=\frac{4}{3}k$$

The equivalent stiffness for the second arrangement is then

$$k_{\text{eq},2}=k_2+k_3=4k+\frac{4}{3}k=\frac{16}{3}k$$

The period of oscillation for the second arrangement is given by

$$\tau_2=2\pi\sqrt{\frac{m}{k_{\text{eq},2}}}=2\pi\sqrt{\frac{3m}{16k}}$$

$$\therefore\tau_2=\frac{\pi\sqrt{3}}{2}\sqrt{\frac{m}{k}}$$

In view of equation (I), we know that $\sqrt{m/k}=1/2\pi$. Accordingly,

$$\tau_{n,2}=\frac{\pi\sqrt{3}}{2}\sqrt{\frac{m}{k}}\rightarrow\tau_2=\frac{\cancel{\pi}\sqrt{3}}{2}\times\frac{1}{2\cancel{\pi}}=\frac{\sqrt{3}}{4}=\boxed{0.43\text{ s}}$$

The natural period of oscillation of the second arrangement is 0.43 seconds.

▶ The correct answer is **B**.

## P.4 › Solution

**Part A:** The stiffness of the bar is given by

$$k=\frac{EA}{\ell}=\frac{\left(200\times10^9\right)\times0.01}{0.25}=8\times10^9\text{ N/m}^2$$

The natural frequency of the bar follows as

$$\omega_n=\sqrt{\frac{k}{m}}=\sqrt{\frac{\left(8\times10^9\right)}{50}}=\boxed{12{,}650\text{ rad/s}}$$

▶ The correct answer is **A**.

**Part B:** We begin by determining the longitudinal stiffness of the system,

$$k=\frac{EA}{\ell}=\frac{Eb^2}{\ell}=\frac{\left(200\times10^9\right)\times0.1^2}{0.25}=8\times10^9\text{ N/m}$$

The longitudinal natural frequency of the system is then

$$\omega_{n,l}=\sqrt{\frac{k}{m+\frac{\rho b^2L}{3}}}=\sqrt{\frac{\left(8\times10^9\right)}{1000+\frac{7880\times0.1^2\times0.25}{3}}}=2819\text{ rad/s}$$

We now turn to the torsional motion of the system. The torsional stiffness is determined as

$$k_t=\frac{GI_o}{L}=\frac{Gb^4}{6L}=\frac{\left(80\times10^9\right)\times0.1^4}{6\times0.25}=5.33\times10^6\text{ N}\cdot\text{m/rad}$$

The torsional natural frequency of the system is then

$$\omega_{n,t}=\sqrt{\frac{k_t}{J}}=\sqrt{\frac{5.33\times10^6}{10}}=730\text{ rad/s}$$

Clearly, $\omega_{n,l} > \omega_{n,t}$. The longitudinal frequency of the system is greater than the torsional stiffness – in numerical terms, nearly 290% greater.

The correct answer is **B**.

## P.5 Solution

We begin by writing the equation for the natural frequency of a cantilevered leaf spring made from steel,

$$\omega_{\text{steel}} = \sqrt{\frac{3E_{\text{st}}}{m\ell^3}} = \sqrt{\frac{3\times\left(2\times10^{11}\right)}{m\ell^3}} = \sqrt{\frac{6\times10^{11}}{m\ell^3}} \quad \text{(I)}$$

Note that we have not substituted variables $m$ and $\ell$, as the mass of the device and the length of the spring have not been established. We proceed to determine the natural frequency of the leaf spring made of aluminum,

$$\omega_{\text{Al}} = \sqrt{\frac{3E_{\text{Al}}}{m\ell^3}} = \sqrt{\frac{3\times\left(7.1\times10^{10}\right)}{m\ell^3}} = \sqrt{\frac{2.13\times10^{11}}{m\ell^3}} \quad \text{(II)}$$

where once again we have left the mass $m$ and length $\ell$ unspecified. We can determine the change in the natural frequency, however, if we divide one equation by the other. Dividing (II) by (I) gives

$$\frac{\omega_{\text{Al}}}{\omega_{\text{steel}}} = \frac{\sqrt{\frac{2.13\times10^{11}}{\cancel{m\ell^3}}}}{\sqrt{\frac{6\times10^{11}}{\cancel{m\ell^3}}}} = \sqrt{\frac{2.13\times10^{11}}{6\times10^{11}}} = \boxed{0.596}$$

That is, the frequency is decreased by approximately 40% with the use of aluminum.

The correct answer is **D**.

## P.6 Solution

The areas of cross-sections AA, BB, and CC are

$$A_1 = \frac{\pi}{4}\times\left(D_1^2 - d_1^2\right) = \frac{\pi}{4}\times\left(2^2 - 1.75^2\right) = 0.74\ \text{in.}^2$$

$$A_2 = \frac{\pi}{4}\times\left(D_2^2 - d_2^2\right) = \frac{\pi}{4}\times\left(1.5^2 - 1.25^2\right) = 0.54\ \text{in.}^2$$

$$A_3 = \frac{\pi}{4}\times\left(D_3^2 - d_3^2\right) = \frac{\pi}{4}\times\left(1.0^2 - 0.75^2\right) = 0.34\ \text{in.}^2$$

We can then determine the transverse stiffness of segments 1, 2, and 3,

$$k_1 = \frac{A_1E_1}{l_1} = \frac{0.74\times10^7}{12} = 6.17\times10^5\ \text{lb/in.}$$

$$k_2 = \frac{A_2E_2}{l_2} = \frac{0.54\times10^7}{10} = 5.4\times10^5\ \text{lb/in.}$$

$$k_3 = \frac{A_3E_3}{l_3} = \frac{0.34\times10^7}{8} = 4.25\times10^5\ \text{lb/in.}$$

From the geometry of the system, we see that the springs (i.e., the arm segments) are connected in series. The equivalent stiffness is then

$$\frac{1}{k_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \rightarrow \frac{1}{k_{\text{eq}}} = \frac{1}{6.17\times10^5} + \frac{1}{5.4\times10^5} + \frac{1}{4.25\times10^5}$$

$$\therefore k_{\text{eq}} = 1.72\times10^5\ \text{lb/in.}$$

It remains to calculate the natural frequency of the robot arm, which is given by

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m}} = \sqrt{\frac{\left(1.72\times10^5\right)}{\left(10/386.4\right)}} = \boxed{2578\ \text{rad/s}}$$

The correct answer is **D**.

## P.7 › Solution

Deflection Δ can be determined with Hooke's law,

$$k\Delta = mg \rightarrow \Delta = \frac{mg}{k}$$

We know that the longitudinal stiffness of the board is given by

$$k = \frac{3EI}{\ell}$$

so that, backsubstituting in the first equation, we obtain

$$\Delta = \frac{mg}{\left(\frac{3EI}{\ell}\right)} = \frac{mg\ell}{3EI}$$

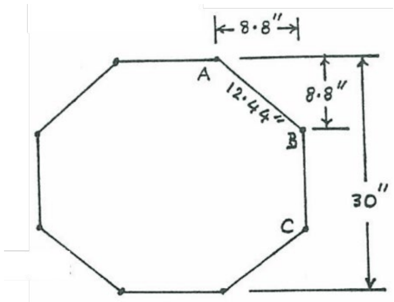
The moment of inertia of a rectangular section such as the present one is $I = bh^3/12$, giving

$$\Delta = \frac{mg\ell}{3EI} = \frac{4mg\ell}{Ebh^3} \rightarrow \boxed{\frac{\ell}{bh^3} = \frac{E\Delta}{4mg}}$$

The expression above relates the dimensions of the board to the deflection and other available parameters.

## P.8 › Solution

The circular frequency of wind is $\omega_w$ = 1.27×2$\pi$ = 8 rad/s. The cross-section of the stop sign is illustrated below.




Segment *AB* has length

$$AB = \sqrt{8.8^2 + 8.8^2} = 12.44 \text{ in.}$$

The area of the octagon is then

$$\text{Area} = 30 \times 30 - 4 \times \left(\frac{1}{2} \times 8.8 \times 8.8\right) = 745 \text{ in.}^2$$

Knowing that the thickness of the sign is 1/8 in. and the weight density of steel is 0.283 lb/in.$^3$, the weight of the sign is determined as

$$\text{Weight of sign} = 0.283 \times 0.125 \times 745 = 26.35 \text{ lb}$$

The weight of the sign post is

$$\text{Weight of sign post} = 72 \times 2 \times \frac{1}{4} \times 0.283 = 10.19 \text{ lb}$$

The transverse stiffness of the sign post, taken as a cantilever beam, is $k = 3EI/\ell$. The moments of inertia of the cross-section of the sign post are

$$I_{xx} = \frac{2 \times \left(\frac{1}{4}\right)^3}{12} = 0.00260 \text{ in.}^4$$

$$I_{yy} = \frac{2^3 \times \frac{1}{4}}{12} = 0.17 \text{ in.}^4$$

We can then calculate the bending stiffnesses of the sign post,

$$k_{xz} = \frac{3EI_{yy}}{\ell^3} = \frac{3\times(30\times10^6)\times0.17}{72^3} = 41 \text{ lb/in.}$$

$$k_{yz} = \frac{3EI_{xx}}{\ell^3} = \frac{3\times(30\times10^6)\times0.00260}{72^3} = 0.63 \text{ lb/in.}$$

The torsional stiffness of the post can be determined with the formula we were given,

$$k_t = 5.33\times\frac{1\times0.125^3\times(12\times10^6)}{72}\times\left[1-0.63\times\frac{0.125}{1}\times\left(1-\frac{0.125^4}{12\times1^4}\right)\right] = 1598 \text{ lb}\cdot\text{in/rad}$$

Thence, the natural frequency for bending in the *xz*-plane is

$$\omega_{xz} = \sqrt{\frac{k_{xz}}{m}} = \sqrt{\frac{41}{(26.35/386.4)}} = 24.52 \text{ rad/s}$$

The natural frequency for bending in the *yz*-plane is

$$\omega_{yz} = \sqrt{\frac{k_{yz}}{m}} = \sqrt{\frac{0.63}{(26.35/386.4)}} = 3.04 \text{ rad/s}$$

By approximating the shape of the sign as a square of side 30 in. in lieu of an octagon, we can determine its mass moment of inertia utilizing the formula proposed in the problem statement,

$$I_{OO} = \frac{\left(\frac{0.283}{386.3}\right)\times30}{3}\times\left(30^3\times0.125+0.125^3\times30\right) = 24.73 \text{ lb}\cdot\text{in.}^2$$

The natural torsional frequency then follows as

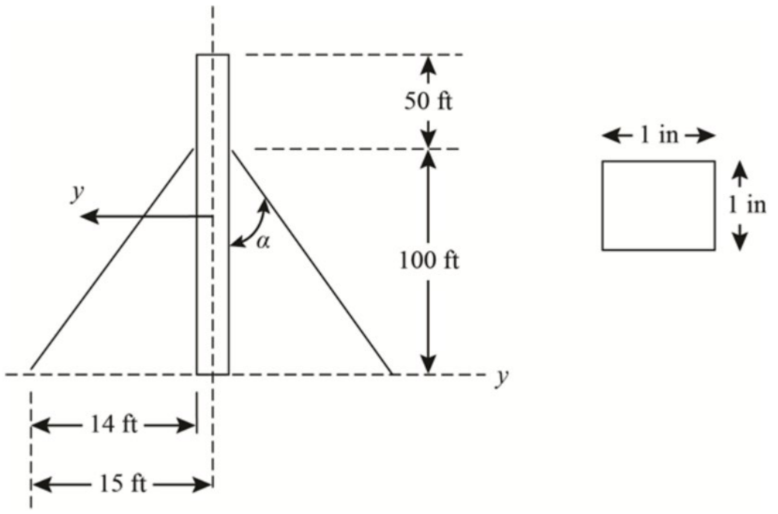
$$\omega_t = \sqrt{\frac{k_t}{I_{OO}}} = \sqrt{\frac{1598}{24.73}} = 8.04 \text{ rad/s}$$

The mode of vibration of the post sign will be that which has a natural frequency closest to the frequency of wind, $\omega_W$ = 8 rad/s. Clearly, the natural frequency of torsional vibration is close to this value. We conclude, then, that the motion of the sign will be one of torsion in the *xy*-plane.

▸ The correct answer is **C**.

## P.9 › Solution

The tower is illustrated below.




The length $l_c$ of the cables is calculated as

$$\ell_c = \sqrt{14^2+14^2+100^2} = 101.9 \text{ ft} = 1223 \text{ in.}$$

The angle outlined between the vertical (*z*-axis) and the cables is such that

$$\sin\alpha = \frac{(14\times12)}{1223} = 0.137$$

We also require the mass *m* of the tower,

$$m = \rho A_t h_t$$

where $\rho$ = 0.283 lb/m³, $A_t$ = 1² = 1 in.² is the cross-sectional area of the tower, and $h_t$ = 150×12 = 1800 in. is its height. Accordingly,

$$m = 0.283 \times 1 \times 1800 = 509.4 \text{ lb} = 15.83 \text{ slugs}$$

The moment of inertia of the tower is $I = (1/12)b^4 = (1/12)\times 1^4 = 0.0833$ in.⁴. We proceed to compute the transverse stiffness of the antenna tower, which is given by

$$k_t = \frac{3EI}{\left(\frac{1}{2}h\right)^3} = \frac{3\times\left(30\times 10^6\right)\times 0.0833}{\left[\frac{1}{2}\times\left(150\times 12\right)\right]^3} = 0.0103 \text{ lb/in.}$$

The four cables are identical and hence have the same transverse stiffness, which is determined as

$$k_c = \frac{A_c E}{\ell_c} = \frac{0.5\times\left(30\times 10^6\right)}{1223} = 12{,}265 \text{ lb/in.}$$

At this point, note that the potential energy of the tower can be expressed as

$$U_1 = \frac{1}{2}k_t y^2$$

where *y* is the deflection of the center of mass of the tower. Similarly, the potential energy of the cables can be expressed as

$$U_2 = \frac{1}{2}k_c \times \left(y_c \sin\alpha\right)^2$$

where $y_c$ is the displacement of the cable and $\alpha$ is the angle between the cable and the vertical. From the geometry of the system, we can state that

$$\frac{y_c}{y} = \frac{h_c}{\frac{1}{2}h_t} \rightarrow y_c = \frac{2h_c}{h_t}y$$

Substituting this result into the expression for $U_2$ gives

$$U_2 = \frac{1}{2}\times k_c \times \left(\frac{2h_c}{h_t}y\sin\alpha\right)^2 = \frac{1}{2}k_c\left(\frac{2h_c}{h_t}\sin\alpha\right)^2 y^2$$

Accounting for the potential energy of all 4 cables, we have

$$U_{\text{tot},2} = 4\times\frac{1}{2}k_c\left(\frac{2h_c}{h_t}\sin\alpha\right)^2 y^2 = 2k_c\left(\frac{2h_c}{h_t}\sin\alpha\right)^2 y^2$$

We can then posit that the total potential energy of the system, considering the 4 cables and the tower itself, is

$$U = U_{\text{tot},2} + U_1 = 2k_c\left(\frac{2h_c}{h_t}\sin\alpha\right)^2 y^2 + \frac{1}{2}k_t y^2$$

$$\therefore U = \left[2k_c\left(\frac{2h_c}{h_t}\sin\alpha\right)^2 + \frac{1}{2}k_t\right]y^2$$

However, the total potential energy *U* of the system can also be determined by the simple relation

$$U = \frac{1}{2}k_{\text{eq}}y^2$$

where $k_{\text{eq}}$ is the equivalent stiffness of the tower system. We can equate the two previous expressions, giving

$$\frac{1}{2}k_{\text{eq}}\cancel{y^2} = \left[2k_c\left(\frac{2h_c}{h_t}\sin\alpha\right)^2 + \frac{1}{2}k_t\right]\cancel{y^2}$$

$$\therefore k_{\text{eq}} = 2\left[2k_c\left(\frac{2h_c}{h_t}\sin\alpha\right)^2 + \frac{1}{2}k_t\right]$$

$$\therefore k_{\text{eq}} = 4k_c\left(\frac{2h_c}{h_t}\sin\alpha\right)^2 + k_t$$

Substituting the pertaining variables, we determine the equivalent stiffness $k_{eq}$,

$$k_{\text{eq}} = 4\times 12{,}265\times\left(\frac{2\times 1200}{1800}\times 0.137\right)^2 + 0.0103 = 1637 \text{ lb/in.}$$

It remains to compute the natural frequency of the tower,

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m}} = \sqrt{\frac{1637}{15.83}} = \boxed{10.2 \text{ rad/s}}$$

The natural frequency of the tower is about 10 radians per second.

▸ The correct answer is **A**.

## P.10 › Solution

The static deflection is related to the natural frequency by

$$\omega_n = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.0015}} = 80.9 \text{ rad/s}$$

The addition of a viscous damper of damping coefficient *c* leads to a damping ratio of

$$\zeta = \frac{c}{2m\omega_n} \to c = 2\zeta m\omega_n$$

In order for the system to be critically damped, the damping ratio must be $\zeta = 1.0$. Substituting this quantity and other data in the equation above, we find that

$$c = 2\times 1\times 500\times 80.9 = \boxed{80{,}900 \text{ N}\cdot\text{s/m}}$$

▸ The correct answer is **D**.

## P.11 › Solution

The springs with stiffness $k_2$ and $k_3$ are connected in series, and in turn are connected in parallel to spring $k_1$. Therefore, the equivalent stiffness is given by

$$k_{\text{eq}} = \frac{k_2k_3}{k_2+k_3} + k_1 = 4000 + \frac{200\times 1000}{200+1000} = 4167 \text{ N/m}$$

The natural frequency of the system is then

$$\omega_n = \sqrt{\frac{4167}{10}} = 20.4 \text{ rad/s}$$

We can now determine the damping ratio $\zeta$,

$$\zeta = \frac{c}{2m\omega_n} = \frac{100}{2\times 10\times 20.4} = 0.25$$

Since $0 < \zeta < 1$, we conclude that the system is underdamped.

▸ The correct answer is **A**.

## P.12 › Solution

**1. True.** Inspecting the graph, we see that the first two peaks occur at times $t_1$ = 0.875 s and $t_2$ = 4.875 s; the corresponding positions are $y_1$ = 0.75 cm and $y_2$ = 0.22 cm. Using these coordinates, the logarithmic decrement is

$$\delta = \ln\left(\frac{y_1}{y_2}\right) = \ln\left(\frac{0.75}{0.22}\right) = 1.23$$

The damping ratio, in turn, is calculated as

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} = \frac{1.23}{\sqrt{4\pi^2+1.23^2}} = 0.192$$

Since $\zeta \in (0;1)$, we conclude that the system is underdamped.

**2. False.** The damped time period is given by

$$T_d = t_2 - t_1 = 4.875 - 0.875 = 4.0 \text{ s}$$

The damped natural frequency is then

$$\omega_d = \frac{2\pi}{T_d} = \frac{2\pi}{4} = 1.57 \text{ rad/s}$$

**3. True.** The natural frequency is calculated by means of the relation

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{1.57}{\sqrt{1-0.192^2}} = 1.60 \text{ rad/s}$$

Since 1.60/1.57 = 1.019, we conclude that the damped frequency and the natural frequency are within less than 10% of one another.

## P.13 › Solution

The differential equation that governs the motion of the block can be shown to be

$$m\ddot{x} + kx = \begin{cases} \mu mg \; ; \; \dot{x} > 0 \\ \mu mg \; ; \; \dot{x} < 0 \end{cases}$$

Comparing this with the general equation of motion for a 1-DOF system under Coulomb damping,

$$\dot{x} + \omega_n^2 x = \begin{cases} -\dfrac{F_f}{m_{\text{eq}}} \; ; \; \dot{x} > 0 \\ \dfrac{F_f}{m_{\text{eq}}} \; ; \; \dot{x} < 0 \end{cases}$$

we see that

$$F_f = \mu mg$$

and the decrease in the amplitude of response in each cycle follows as

$$\Delta A = \frac{4\mu mg}{k} = \frac{4\times 0.12\times 2\times 9.81}{10{,}000} = 0.94 \text{ mm}$$

Motion will cease when the amplitude is such that the spring force cannot overcome the friction force. We should also account for the permanent displacement of the mass-spring system, which is given by

$$kx_f = \mu mg \rightarrow x_f = \frac{\mu mg}{k} = \frac{0.12\times 2\times 9.81}{10{,}000} = 0.235 \text{ mm}$$

Hence, the number of cycles that the system will undergo before returning to rest is

$$n = \frac{x_o - x_f}{\Delta A} = \left\lceil \frac{10-0.235}{0.94} \right\rceil = \lceil 10.39 \rceil = \boxed{11 \text{ cycles}}$$

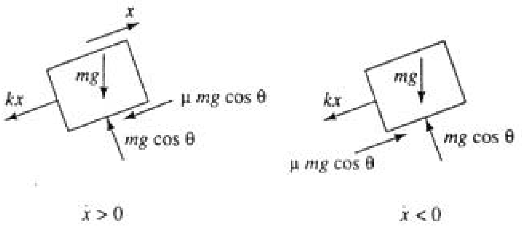
The system will oscillate for 11 cycles before its motion stops.

▸ The correct answer is **B**.

## P.14 › Solution

Application of Newton's law to the free body diagrams of the next figure leads to

$$m\ddot{x}+kx=\begin{cases}-\mu mg\cos\theta,\ \dot{x}>0\\ \mu mg\cos\theta,\ \dot{x}<0\end{cases}$$




Accordingly, the friction force has the form $F_f=\mu mg\cos\theta$ and the decrease in amplitude per cycle follows as

$$\Delta A=\frac{4\mu mg\cos\theta}{k}\rightarrow\mu=\frac{k\Delta A}{4mg\cos\theta}$$

Substituting the appropriate variables, we obtain

$$\mu=\frac{10,000\times\left(1.2\times10^{-3}\right)}{4\times10\times9.81\times\cos25^{\circ}}=\boxed{0.034}$$

▸ The correct answer is **C**.

## ANSWER SUMMARY

| | | |
|---|---|---|
| **Problem 1** | **1A** | *Open-ended pb.* |
| | **1B** | *Open-ended pb.* |
| | **1C** | *Open-ended pb.* |
| **Problem 2** | | **C** |
| **Problem 3** | | **B** |
| **Problem 4** | **4A** | **A** |
| | **4B** | **B** |
| **Problem 5** | | **D** |
| **Problem 6** | | **D** |
| **Problem 7** | | *Open-ended pb.* |
| **Problem 8** | | **C** |
| **Problem 9** | | **A** |
| **Problem 10** | | **D** |
| **Problem 11** | | **A** |
| **Problem 12** | | T/F |
| **Problem 13** | | **B** |
| **Problem 14** | | **C** |

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