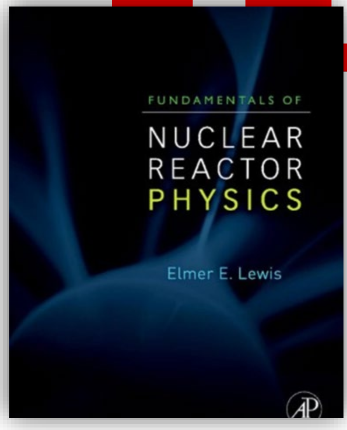

Montogue

Quiz NUC01

Reviewed Solutions to *Fundamentals of Nuclear Reactor Physics*, by E.E. Lewis

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Problem Distribution

Chapter	Problems Covered
1	1.6, 1.10, 1.11, 1.13, 1.14, 1.15, 1.17, 1.19
2	2.1, 2.3, 2.7, 2.8, 2.13, 2.15, 2.16
3	3.4, 3.7, 3.10
4	4.3, 4.4, 4.6, 4.8, 4.10, 4.11, 4.12
5	5.6, 5.9
6	6.6, 6.10, 6.14, 6.15, 6.16, 6.17, 6.18
7	7.2, 7.3, 7.6, 7.12, 7.13, 7.14, 7.16
8	8.1, 8.2, 8.6, 8.10, 8.11, 8.12, 8.13
9	9.4, 9.5
10	10.1, 10.3, 10.4, 10.6, 10.7, 10.9, 10.13

Problems

■ Chapter 1 – Nuclear Reactions

Problem 1.6

Consider the following nuclear and chemical reactions:

(a) A uranium-235 nucleus fissions as a result of being bombarded by a slow neutron. If the energy of fission is 200 MeV, approximately what fraction of the reactant's mass is converted to energy?

(b) A carbon-12 atom undergoes combustion following collision with an oxygen-16 molecule, forming carbon dioxide. If 4 eV of energy are released, approximately what fraction of the reactants' mass is converted to energy?

Problem 1.10

In Eq. (1.28), the uranium-239 and neptunium-239 both undergo beta decay with half-lives of 23.4 months and 2.36 days, respectively. If neutron bombardment in a reactor causes uranium-239 to be produced at a constant rate, how long will it take plutonium-239 to reach (assume that plutonium-239 undergoes no further reactions)

(a) 1/2 of its saturation activity?

(b) 90% of its saturation activity?

(c) 99% of its saturation activity?

Problem 1.11

Uranium-238 has a half-life of 4.51×10^{9} yr, whereas the half-life of uranium-235 is only 7.13×10^{8} yr. Thus, since the earth was formed 4.5 billion years ago, the isotopic abundance of uranium-235 has been steadily decreasing. At present, U-235 makes up only 0.7% of natural uranium.

(a) What was the enrichment of uranium when the earth was formed?

(b) How long ago was the enrichment 4%?

Problem 1.13

Suppose that a specimen is placed in a reactor, and a neutron bombardment causes a radioisotope to be produced at a rate of 2×10^{12} nuclei/s. The radioisotope has a half-life of 2 weeks. How long should the specimen be irradiated to produce 25 Ci of the radioisotope?

Problem 1.14

The decay constant for the radioactive antimony isotope $^{124}_{51}Sb$ is 1.33×10^{-7} s^{-1}.
(a) What is its half-life in years?
(b) How many years would it take for it to decay to 0.01% of its initial value?
(c) If it were produced at a constant rate, how many years would it take to reach 95% of its saturation value?

Problem 1.15

Approximately what mass of cobalt-60, which has a half-life of 5.26 yr, will have the same number of curies as 10 grams of strontium-90, which has a half-life of 28.8 yr?

Problem 1.17

A fission product "A" with a half-life of two weeks is produced at the rate of 5.0×10^{8} nuclei/sec in a reactor.
(a) What is the saturation activity in disintegrations/sec?
(b) What is the saturation activity in curies?
(c) How long after the startup of the reactor will 90 percent of the saturation activity be reached?
(d) If the fission product undergoes decay A → B → C, where B also has a two-week half-life, what will be the activity of B after two weeks?

Problem 1.19

Polonium-210 decays to lead-206 by emitting an alpha particle with a half-life of 138 days and an energy of 5.305 MeV.
(a) How many curies are there in one gram of pure polonium?
(b) How many watts of heat are produced by one gram of polonium?

■ Chapter 2 – Neutron Interactions

Problem 2.1

Neutrons impinge on a material with a cross-section of $\Sigma = 0.8$ cm^{-1}. How thick must the material be if no more than 5.0% of the neutrons are to penetrate the material without making a collision? What fraction of the neutrons make their first collision within the first 2.0 cm of the material?

Problem 2.3

A material has a neutron cross-section of 3.50×10^{-24} cm^2/nuclei, and contains 4.20×10^{23} nuclei/cm^3.
(a) What is the macroscopic cross-section?
(b) What is the mean free path?
(c) If neutrons impinge perpendicularly on a slab of the material, which is 3.0 cm thick, what fraction of them will penetrate the slab without making a collision?
(d) What fraction of the neutrons in part (c) will collide in the slab before penetrating a distance of 1.5 cm?

Problem 2.7

How many parts per million of boron must be dissolved in water at room temperature to double its absorption cross-section for thermal neutrons?

Problem 2.8

What is the total macroscopic thermal cross-section of uranium dioxide (UO_2) that has been enriched to 4%? Assume $\sigma^{25} = 607.5$ b, $\sigma^{28} = 11.8$ b, $\sigma^{O} = 3.8$ b, and that UO_2 has a density of 10.5 g/cm^3.

Problem 2.13

Equal volumes of graphite and iron are mixed together. Fifteen percent of the volume of the resulting mixture is occupied by air pockets. Find the total macroscopic cross-section given the following data: $\sigma_C = 4.75$ b, $\sigma_{Fe} = 10.9$ b, $\rho_C = 1.6$ g/cm^2, $\rho_{Fe} = 7.7$ g/cm^3. Is it reasonable to neglect the cross-section of air? Why?

Problem 2.15

What is the *minimum* number of elastic scattering collisions required to slow a neutron down from 1.0 MeV to 1.0 eV in
(a) Deuterium;
(b) Carbon-12;
(c) Iron-56; and
(d) Uranium-238?

Problem 2.16

Using the macroscopic scattering cross-sections in Appendix Table II-3, calculate the slowing down decrement for UO_2, where *U* is natural uranium. Does the presence of oxygen have a significant effect on the slowing down decrement?

■ Chapter 3 – Neutron Distributions in Energy

Problem 3.4

For thermal neutrons calculate $\bar{\eta}$ as a function of uranium enrichment and plot your results. Use the uranium data from the following table:

	ν	σ_f (b)	σ_a (b)
Uranium-235	2.43	505	591
Plutonium-239	2.90	698	973
Uranium-238	–	0	2.42

Problem 3.7

Lethargy defined as $u = \ln(E_0/E)$ is often used in neutron slowing down problems. Lethargy increases as energy decreases. Note the following transformations: $\varphi(E)dE = -\varphi(u)du$, $p(E \to E')dE' = -p(u \to u')du'$, and $\Sigma_x(E) = \Sigma_x(u)$.

(a) Show that $p(E \to E')$ given by Eq. (2.47) becomes

$$p(u \to u') = \begin{cases} \dfrac{1}{1-\alpha}\exp(u-u') \ , \ u \le u' \le u + \ln(1/\alpha) \\ 0 \ , \ \text{otherwise} \end{cases}$$

(b) Express Eq. (3.22) in terms of *u*. The equation is repeated below for convenience.

$$\Sigma_s(E)\varphi(E) = \int_E^{E/\alpha} \frac{1}{(1-\alpha)E'}\Sigma_s(E')\varphi(E')dE'$$

Problem 3.10 **[Parts (a) and (c) only]**

A power reactor is cooled by heavy water (D_2O) but a leak causes a 1.0 atom % contamination of the coolant with light water (H_2O). Determine the resulting percentage increase or decrease in the following characteristics of the of the coolant:

(a) Slowing down decrement.
(c) Slowing down ratio.

■ Chapter 4 – The Power Reactor Core

Problem 4.3

A sodium-cooled fast reactor is fueled with PuO_2, mixed with depleted UO_2. The structural material is iron. Averaged over the spectrum of fast neutrons, the microscopic cross-sections and densities are as follows:

	σ_f (b)	σ_a (b)	σ_t (b)	ρ (g/cm^3)
PuO_2	1.95	2.40	8.6	11.0
UO_2	0.05	0.404	8.2	11.0
Na	–	0.0018	3.7	0.97
Fe	–	0.0087	3.6	7.87

The fuel is 15% PuO_2 and 85% UO_2 by volume. The volumetric composition of the core is 30% fuel, 50% coolant, and 20% structural material. Calculate k_∞, assuming that the values of ν for plutonium and uranium in the fast spectrum are 2.98 and 2.47, respectively, and that the cross-sections of oxygen can be neglected. What fraction of the mass of the core does the fuel account for?

Problem 4.4

Suppose the nonleakage probability for a sodium-cooled fast reactor specified in Problem 4.3 is 0.90. Using the data from Problem 4.3, adjust the volume fractions of PuO_2 and UO_2 in the fuel so that k = 1.0. What is the % PuO_2 in the fuel by volume?

Problem 4.6

A pressurized water reactor has 3% enriched UO_2 fuel pins that are 1.0 cm in diameter and have a density of 11.0 gm/cm^3. The moderator to fuel volume ratio is 2:1. Calculate η_T, p, f, and k_∞ at room temperature under the assumptions that ε = 1.24, the thermal disadvantage factor is ς = 1.16, and the Dancoff correction increases the fuel diameter for the resonance integral calculation by 10%.

Problem 4.8

A reactor lattice consists of uranium rods in a heavy water moderator. The heavy water is replaced by light water.

(a) Would the resonance escape probability increase or decrease? Why?

(b) Would the thermal utilization increase or decrease? Why?

(c) What would you expect the net effect on k_∞ to be? Why?

Problem 4.10

Using the data from Problem 4.6, vary the coolant/fuel volume ratio between 0.5 and 2.5 and plot the following vs V_m/V_f:

(a) The resonance escape probability.

(b) The thermal utilization.

(c) k_∞.

(d) Determine the moderator-to-fuel volume ratio that yields the largest k_∞.

(e) What is the largest value of k_∞?

You may assume that changes in the fast fission factor and the thermal disadvantage factor are negligible.

Problem 4.11

A reactor designer decides to replace uranium with UO_2 fuel in a water cooled reactor, keeping the enrichment, fuel diameter and water to fuel volume ratios the same.

(a) Will the resonance escape probability *p* increase, decrease or remain unchanged? Why?

(b) Will the thermal utilization *f* increase, decrease or remain unchanged? Why?

(c) Will η_T increase, decrease or remain unchanged? Why?

Problem 4.12

The fuel for a thermal reactor has the following composition by atom ratio: 2% uranium-235, 1% plutonium-239, and 97% uranium-238. Calculate the value of η_T to be used for this fuel in the four-factor formula. (Use the data given for Problem 3.4).

■ Chapter 5 – Reactor Kinetics

Problem 5.6

A thermal reactor fueled with uranium operates at 1.0 W. The operator is to increase the power to 1.0 kW over a two-hour span of time.

(a) What reactor period would we put the reactor on?

(b) How many cents of reactivity must be present to achieve the period in part (a)?

Problem 5.9

Find the periods for reactors fueled by uranium-235, plutonium-239, and uranium-233 if

(a) One cent of reactivity is added to the critical systems.

(b) One cent of reactivity is withdrawn from the critical systems.

■ Chapter 6 – Spatial Diffusion of Neutrons

Problem 6.6

Neutrons impinge uniformly over the surface of a sphere made of graphite that has a diameter of 1.0 m. For the graphite D = 0.84 cm and $\Sigma_a = 2.1\times10^{-4}$ cm^{-1}.

(a) Determine the albedo of the graphite sphere.

(b) Determine the fraction of the impinging neutrons that are absorbed in the sphere.

Problem 6.10

A thin spherical shell of radius R emits s''_{pl} neutrons/cm^2/s in an infinite nonmultiplying medium with properties D and Σ_a.

(a) Determine the flux $\phi(r)$ for $0 \le r \le \infty$.

(b) Determine the flux ratio $\phi(0)/\phi(R)$.

Problem 6.14

Show that Eqs. (6.95) and (6.103) agree in the limit of $k_\infty \to 1$.

Problem 6.15

Suppose that the material in Problem 6.9 is fissionable with $k_\infty < 1$. Find the flux distribution in the sphere.

Problem 6.16

Suppose the material in Problem 6.9 is fissionable with $k_\infty > 1$:

(a) Find the flux distribution in the sphere.

(b) Show that the criticality condition is the same as Eq. (6.105).

Problem 6.17

Equations (6.95) and (6.103) give the flux distributions for a subcritical sphere with a uniform source for $k_\infty < 1$ and $k_\infty > 1$, respectively. Find the equivalent expression for $k_\infty = 1$.

Problem 6.18

Using Eqs. (6.95) and (6.103),

(a) Find expressions for the flux $\phi(0)$ at the center of the subcritical sphere.

(b) Using your results from part (a) make a plot of $\phi(0)$ for $0 \leq k_\infty < 1.154$ with $\tilde{R}/L = 8$.

(c) Using your results from part (a) make a plot of $\phi(0)$ for $0 < \tilde{R}/L < 8$ with $k_\infty = 1.154$.

(d) Compare the two curves and discuss their significance. [Normalize plots to S_o'''/Σ_a]

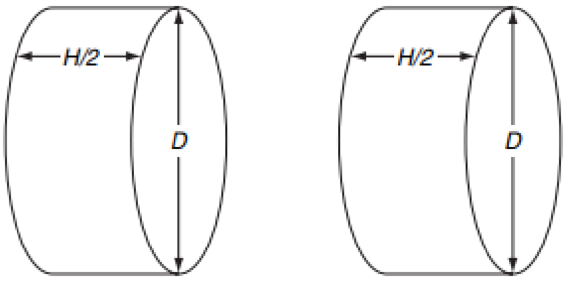
■ Chapter 7 – Neutron Distributions in Reactors

Problem 7.2

Determine the height-to-diameter ratio of a bare cylindrical reactor that will lead to the smallest critical mass.

Problem 7.3

Critical assemblies for studying the properties of fast reactors are sometimes built in halves as shown in the figure. The two halves are maintained in subcritical states by separating them with a sufficient distance that neutronic coupling between the two is negligible; they are then brought together to form a critical assembly. Suppose the core composition under investigation has an infinite medium multiplication of 1.36 and a migration length of 18.0 cm. The assembly is configured with a height-to-diameter ratio of one ($H = D$). Neglecting extrapolation distances,



(a) Determine the dimensions required to make the assembly exactly critical when the two halves are brought into contact.

(b) Determine the value of k for each of the halves when they are isolated from each other.

Problem 7.6

Consider a critical reactor that is a cube with extrapolated side length a:

(a) With the origin at the center, apply separation of variables in three-dimensional Cartesian geometry to show that the flux distribution is

$$\phi(x,y,z) = C\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\cos\left(\frac{\pi z}{a}\right)$$

(b) Find C in terms of the reactor power, volume, and $\gamma\Sigma_f$.

(c) Determine the reactor's buckling.

(d) Suppose that $a = 2.0$ m and $M = 20$ cm. Determine the value of k_∞ required to obtain criticality (i.e., $k = 1.0$).

Problem 7.12

Consider the situation when the spherical system discussed in Chapter 6.7 is critical. Determine the ratio of maximum to average flux in the sphere.

Problem 7.13

A spherical reactor of radius R is surrounded by a reflector that extends to $r = \infty$. L and D are the same for core and reflector. Find the criticality condition relating k_∞, R, L, and D.

Problem 7.14 **[Part (a) only]**

A spherical reactor is constructed with an internal reflector with parameters D and Σ_a^r and extending $0 \le r \le R$. The annular core, with parameters D, Σ_a, and k_∞ (>1), extends $R \le r \le 2R$. Find the criticality condition (neglecting the extrapolation distance).

Problem 7.16 **[Part (a) only]**

An infinite slab reactor (extending to infinity in the y and z directions) has a thickness of $2a$ with vacuum on either side. The properties for material 1 occupying $0 \le x \le a$ are $k_\infty^1 = k_\infty$, $D_1 = D$, and $\Sigma_a^1 = \Sigma_a$ and those for material 2 occupying $a \le x \le 2a$ are $k_\infty^2 = k_\infty$, $D_2 = D$, and $\Sigma_a^2 = 0$. Neglecting extrapolation distances, find a criticality equation relating a, k_∞, D, and Σ_a.

■ Chapter 8 – Energy Transport

Problem 8.1

The leakage probability of a power reactor is 0.0065. As a first approximation to a new reactor an engineer estimates that the same power density can be achieved if the power is to be increased by 20%. Assuming the height-to-diameter ratio of the cylindrical core remains the same:

(a) What will the leakage probability be in the new reactor with the power increased by 20%?

(b) If k_∞ is proportional to the fuel enrichment, by what percent will the enrichment of the core need to be changed to accommodate the 20% increase in power?

Problem 8.2

A sodium-cooled fast reactor lattice is designed to have a migration length of 20 cm and a maximum power density of 500 W/cm^3. Three bare cylindrical cores with height-to-diameter ratios of one are to be built, with power ratings of 300 MW(t), 1000 MW(t), and 3000 MW(t). For each of the three cores determine the following:

(a) The core height, H.

(b) The buckling B^2.

(c) The nonleakage probability P_{NL}.

Problem 8.6

You are to design a 3000 MW(t) pressurized water reactor. The reactor is a uniform bare cylinder with a height-to-diameter ratio of one. The coolant to fuel volume ratio is 2:1 in a square lattice. The volumes occupied by control and structural materials, as well as the extrapolation distances, can be neglected. The core inlet temperature is 290ºC. The reactor must operate under three thermal constraints: (1) maximum power density = 250 W/cm^3, (2) maximum cladding surface heat flux = 125 W/cm^2; and (3) maximum core outlet temperature = 330ºC. Determine the following:

(a) The reactor dimensions and volume.

(b) The fuel element diameter and lattice pitch.

(c) The approximate number of fuel elements.

(d) The mass flow rate and average coolant velocity.

Problem 8.10

Consider the PWR design at the end of Section 8.3. Suppose that by varying the enrichment in the fuel assemblies and distributing the control poisons in a non-uniform pattern the designers are able to reduce the radial and axial peaking factors to $F_r = 1.30$ and $F_z = 1.46$. Redesign the reactor by solving parts *c* through *g* of the pressurized water reactor example using these peaking factors.

Problem 8.11

An unachievable ideal would be a reactor with a perfectly flat flux distributions: $F_r = 1.00$ and $F_z = 1.00$. Repeat problem 8.10 for such an idealized reactor.

Problem 8.12

Suppose that the designers of the pressurized water reactor treated in Section 8.3 conclude that the thermal-hydraulic design must have larger safety margins by reducing the coolant flow velocity by 10% and the maximum coolant temperature by 5ºC. The reactor physicists are asked to accommodate those changes by reducing the radial peaking factor. What percentage reduction would be required?

Problem 8.13

A reactor initially operating at a power P_o is put on a period T such that the power can be approximated as $P(t) = P_o exp(t/T)$. Assuming that the coolant temperature is maintained at its initial value of $T_c(0)$, solve equation (8.48) and show that the fuel temperature will be

$$T_f(t) = T_c(0) + \frac{P_o R_f}{1+\tau/T}\left[\exp(t/T) + (\tau/T)\exp(-t/\tau)\right]$$

■ Chapter 9 – Reactivity Feedback

Problem 9.4

At full power 1000 MW(t) sodium-cooled fast reactor has coolant inlet and outlet temperatures of 350 and 500°C, and an average fuel temperature of 1,150°C. The fuel and coolant temperature coefficients are $\alpha_f = -1.8\times10^{-5}$/°C and $\alpha_c =$ $+0.45\times10^{-5}$/°C.

(a) Estimate the core thermal resistance and the mass flow rate, taking for sodium a specific heat capacity c_P = 1250 J/kg°C.

(b) Estimate the temperature and power defects, assuming a "cold" temperature of 180°C.

Problem 9.5

A 3000 MW(t) pressurized water reactor has the following specifications: core thermal resistance 0.45 °C/MW(t), coolant flow 68×10^6 kg/hr, coolant specific heat 6.4×10^3 J/kg°C. The fuel temperature coefficient is

$$\frac{1}{k}\frac{\partial k}{\partial \overline{T}_f} = -\frac{7.2\times10^{-4}}{\sqrt{273+\overline{T}_f}}\left({}^{\circ}\mathrm{C}\right)^{-1}$$

and the coolant temperature coefficient is

$$\frac{1}{k}\frac{\partial k}{\partial \overline{T}_c} = \left(30 + 1.5\overline{T}_c - 0.0010\overline{T}_c^{2}\right)\cdot 10^{-6}\left({}^{\circ}\mathrm{C}\right)^{-1}$$

(a) Over what temperature range is the core overmoderated?

(b) What is the value of the temperature defect? Assume a room temperature of 21°C and an operating coolant inlet temperature of 290°C.

(c) What is the value of the power defect?

■ Chapter 10 – Long-Term Core Behavior

Problem 10.1

Prove that for a reactor operating at a very high flux level, the maximum xenon-135 concentration takes place at approximately 11.3 hours following shutdown.

Problem 10.3

A thermal reactor fueled with uranium has been operating at constant power for several days. Make a plot of concentration of xenon-135 to uranium-235 atoms in the reactor versus its average flux. Determine the maximum value that this ratio can take.

Problem 10.4

A pressurized water reactor at full power has an average power density of $\bar{P}''' = 80$ MW/m^3 and a peaking factor of $F_q = 2.0$. After the reactor has operated for several days and assuming a fission cross-section of $\bar{\Sigma}_f = 0.203$ cm^{-1}:

(a) What is the average xenon concentration?

(b) What is the maximum xenon concentration?

(c) What is the average samarium concentration?

(d) What is the maximum samarium concentration?

Problem 10.6

Samarium-157 is produced at a rate of 7.0×10^{-5} atoms/fission. It then undergoes decay:

$${}^{157}_{62}\mathrm{Sm} \xrightarrow[0.5\ \mathrm{min}]{\beta} {}^{157}_{63}\mathrm{Eu} \xrightarrow[15.2\ \mathrm{hr}]{\beta} {}^{157}_{64}\mathrm{Gd}$$

While the absorption cross-section of samarium and europium are negligible, the thermal absorption cross-section of samarium and europium are negligible, the thermal absorption cross-section of gadolinium is 240,000 b. Suppose that a reactor operates at a power density of 100 MW/m^3 and a flux level of 8.0×10^{12} n/cm^2/s.

(a) Solve the decay equations for *G(t),* the atom density of gadolinium, at a time *t* following reactor startup.
(b) Evaluate $G(\infty)$.
(c) If the reactor has been operated for several weeks and then is shut down, what is the concentration of gadolinium after the reactor has been shut down for several weeks? (Assume that the energy produced per fission is 3.1×10^{-11} W·s).

Problem 10.9

Under load following conditions a reactor operates each day at full power for 12 hours, followed by a shutdown of 12 hours. Calculate the iodine concentration, *I(t)*, over a 24 hour time span. Use periodic boundary conditions I(24 hr) = I(0).

Problem 10.13

Consider uranium fuel in a thermal reactor with an initial enrichment of 4%.
(a) What is the conversion ratio (CR) at the beginning of life.
(b) After 50% of the uranium-235 has been burned, what is the conversion ratio?
(c) After 50% of the uranium-235 has been burned, what fraction of the power is being produced from plutonium-239?

Solutions

■ P1.6

Part (a): Since one atomic mass unit is equivalent to 931.5 MeV, the energy equivalent of the reactants is 236 × 931.5 = 219,800 MeV. The fraction χ of energy converted is then

$$\chi = \frac{200}{219{,}800}\times 100\% \approx \boxed{0.091\%}$$

Part (b): The energy of the reactants is (12 + 2 × 16) × 931.5 = 40,990 MeV = 4.099×10^{10} eV. Accordingly, the fraction χ of energy converted is

$$\chi = \frac{4}{4.099\times10^{10}}\times 100\% \approx \boxed{9.8\times10^{-9}\%}$$

The point of this simple exercise is to show that even a mild nuclear reaction greatly outstrips a chemical reaction in terms of the fraction of reactant mass converted to energy.

■ P1.10

Parts (a,b,c): Since the half-life of uranium-239 is very small compared to that of neptunium-239, we can assume that the uranium decay is instantaneous. The decay rate of neptunium is $\lambda = 0.693/t_{1/2} = 0.693/2.36 = 0.294\ \mathrm{d}^{-1}$. We can proceed to compute the time required to reach the specified saturation intensities:

$$\left[1-\exp(-\lambda t)\right] = 0.5 \;\rightarrow\; t = -\frac{1}{\lambda}\ln(0.5)$$

$$\therefore t = -\frac{1}{0.294}\ln(0.5) = \boxed{2.36\text{ days}}$$

(As expected, the time required to reach 50% of the saturation activity is simply the half-life of the isotope.) Proceeding similarly for (*b*),

$$\left[1-\exp(-\lambda t)\right] = 0.9 \;\rightarrow\; t = -\frac{1}{\lambda}\ln(0.1)$$

$$\therefore t = -\frac{1}{0.294}\ln(0.1) = \boxed{7.83\text{ days}}$$

Then, for (c),

$$\left[1-\exp(-\lambda t)\right] = 0.99 \;\rightarrow\; t = -\frac{1}{\lambda}\ln(0.01)$$

$$\therefore t = -\frac{1}{0.294}\ln(0.01) = \boxed{15.7\text{ days}}$$

■ P1.11

Part (a): At any time, we may define atom enrichment as

$$\tilde{e}(t) = \frac{1}{1+\dfrac{N^{28}(t)}{N^{25}(t)}} \quad (\mathrm{I.1})$$

Solving for the ratio of isotopes,

$$\frac{N^{28}(t)}{N^{25}(t)} = \left[\frac{1}{\tilde{e}(t)} - 1\right] \quad \text{(I.2)}$$

We can use the decay properties to restate the equation above as

$$\frac{N^{28}(0)\exp\left(-\lambda^{28}t\right)}{N^{25}(0)\exp\left(-\lambda^{25}t\right)} = \left[\frac{1}{\tilde{e}(t)} - 1\right]$$

$$\therefore \frac{N^{28}(0)}{N^{25}(0)} = \left[\frac{1}{\tilde{e}(t)} - 1\right]\exp\left[-\left(\lambda^{25} - \lambda^{28}\right)t\right] \quad \text{(II)}$$

Taking time in billions of years, we have the decay constants

$$\lambda^{28} = \frac{0.693}{t_{1/2}} = \frac{0.693}{4.51} = 0.154 \text{ Gyear}^{-1}$$

$$\lambda^{25} = \frac{0.693}{t_{1/2}} = \frac{0.693}{0.713} = 0.972 \text{ Gyear}^{-1}$$

Substituting into (II),

$$\frac{N^{28}(0)}{N^{25}(0)} = \left[\frac{1}{\tilde{e}(t)} - 1\right]\exp\left[-\left(0.972 - 0.154\right)t\right] = \left[\frac{1}{\tilde{e}(t)} - 1\right]\exp\left(-0.818t\right) \quad \text{(III)}$$

The present-day enrichment is 0.007 (0.7%); substituting t = 4.5 Gyr into the equation above,

$$\frac{N^{28}(0)}{N^{25}(0)} = \left[\frac{1}{0.007} - 1\right] \times \exp\left(-0.818 \times 4.5\right) = 3.57$$

Finally, the enrichment at the time the earth was formed is given by (I.1),

$$\tilde{e}(0) = \frac{1}{1 + \frac{N^{28}(0)}{N^{25}(0)}} = \frac{1}{1 + 3.57} = 0.219 = \boxed{21.9\%}$$

At the time the earth was formed, the enrichment of uranium was greater than 20%.

Part (b): Suppose we now let t = 0 correspond not to the formation of the earth but to the time when the enrichment was 4%. Combining equations (I.2) and (III), we find that

$$\left[\frac{1}{\tilde{e}(0)} - 1\right] = \left[\frac{1}{\tilde{e}(t)} - 1\right]\exp\left(-0.818t\right)$$

$$\therefore \left[\frac{1}{0.04} - 1\right] = \left[\frac{1}{0.007} - 1\right] \times \exp\left(-0.818t\right)$$

$$\therefore 24.0 = 142 \times \exp\left(-0.818t\right)$$

$$\therefore 0.169 = \exp\left(-0.818t\right)$$

$$\therefore -\ln 0.169 = 0.818t$$

$$\therefore t = -\frac{\ln 0.169}{0.818} = \boxed{2.17 \text{ Gyr}}$$

The enrichment approximately 2.17 billion years ago was 4%.

■ P1.13

To produce 25 Ci, we must have $\lambda N(t)$ = 25 × (3.7×10^{10}) = 9.25×10^{11} decays/s. Referring to equation (1.42),

$$\lambda N(t) = A_o\left[1 - \exp\left(-\lambda t\right)\right]$$

$$9.25\times10^{11}=\left(2\times10^{12}\right)\times\left[1-\exp\left(-\frac{0.693}{2.0}\times t\right)\right]$$

We can easily solve this equation with logarithms; alternatively, we speed things up with MATLAB's *fsolve* command:

```
>> fun = @(t) 9.25e11 - 2e12*(1-exp(-0.693/2*t));
t0 = 10;
fsolve(fun, t0)

ans =

    1.7917
```

That is, t = 1.79 weeks, or 12.5 days.

■ P1.14

Part (a): The half-life of this antimony isotope is

$$t_{1/2}=\frac{0.693}{\lambda}=\frac{0.693}{1.33\times10^{-7}}=5.21\times10^{6}\text{ s}$$

Noting that 1 year = 365 × 86,400 sec, we have

$$t_{1/2}=\frac{5.21\times10^{6}}{365\times86,400}=\boxed{0.165\text{ yr}}$$

Part (b): We first convert the decay constant,

$$\lambda=\left(1.33\times10^{-7}\right)\times\left(86,400\times365\right)=4.19\text{ yr}^{-1}$$

The time t required for the isotope to reduce to 0.01% of its initial value is

$$0.01\times10^{-2}=\exp\left(-4.19t\right)$$

$$\therefore\ln\left(0.01\times10^{-2}\right)=-4.19t$$

$$\therefore t=\frac{-\ln\left(0.01\times10^{-2}\right)}{4.19}=\boxed{2.20\text{ yr}}$$

Part (c): Appealing to equation (1.42), we write

$$\lambda N\left(t\right)=A_o\left[1-\exp\left(-\lambda t\right)\right]$$

$$\therefore 0.95=1-\exp\left(-4.19t\right)$$

$$\therefore t=-\frac{\ln\left(0.05\right)}{4.19}=0.715\text{ yr}=\boxed{8.58\text{ months}}$$

■ P1.15

Let subscripts c and s denote cobalt and strontium, respectively. Assuming an equal number of curies, we may write that

$$\lambda_c N_c=\lambda_s N_s\quad\text{(I)}$$

But

$$N=V\times N'''$$

where V is volume and N''' is atom density. Atom density can be expressed as

$$N'''=\frac{\rho N_o}{A}$$

so that, substituting in (I),

$$\frac{\lambda_c\left(V_c\rho_c\right)N_o}{A_c}=\frac{\lambda_s\left(V_s\rho_s\right)N_o}{A_s}$$

Finally, noting that the product of density and volume equals mass, we substitute the given data to obtain

$$\frac{\lambda_c m_c N_o}{A_c} = \frac{\lambda_s m_s N_s}{A_s} \rightarrow m_c = \frac{\lambda_s A_c}{\lambda_c A_s} m_s$$

$$\therefore m_c = \frac{5.26 \times 60}{28.8 \times 90} \times 10 = \boxed{1.22 \text{ g}}$$

■ P1.17

Part (a): At saturation, the rate of production equals the rate of disintegration. Thus, the saturation activity is 5.0×10^8 nuclei/sec.

Part (b): Noting that 1 Ci = 3.7×10^{10} disintegrations per second, we may write

$$SA = \frac{5.0 \times 10^8}{3.7 \times 10^{10}} = \boxed{0.0135 \text{ Ci}}$$

Part (c): This is a straightforward application of equation (1.42) in the textbook:

$$\lambda N(t) = A_o \left[1 - \exp(-\lambda t)\right]$$

$$\therefore 0.90 = 1 - \exp(-\lambda t)$$

The decay constant is $\lambda = 0.693/2.0 = 0.347$ week^{-1}, so that

$$t = -\frac{\ln(0.10)}{\lambda} = -\frac{\ln(0.10)}{0.347} = \boxed{6.64 \text{ weeks}}$$

Part (d): The equation that describes the decay of isotope A is (1.42). The differential equation governing B, in turn, is

$$\frac{d}{dt} N_B(t) = \lambda N_A(t) - \lambda N_B(t)$$

Combining this with equation (1.42) yields

$$\frac{d}{dt} N_B(t) = A_0 \left[1 - \exp(-\lambda t)\right] - \lambda N_B(t)$$

Multiplying both sides by *exp(λt)*,

$$\frac{d}{dt}\left[N_B(t)\exp(\lambda t)\right] = A_0\left[\exp(\lambda t) - 1\right]$$

Integrating between 0 and *t*,

$$N_B(t)\exp(\lambda t) - N_B(0)\exp(\lambda \times 0) = A_0 \left\{\frac{1}{\lambda}\left[\exp(\lambda t) - \exp(\lambda \times 0)\right] - t\right\}$$

But $N_B(0) = 0$ and $\exp(\lambda \times 0) = 1$, so that

$$\lambda N_B(t) = A_0 \left[1 - (1 + \lambda t)\exp(-\lambda t)\right]$$

After 2 weeks, $\lambda \times t = (0.693/2) \times 2 = 0.693$, giving

$$\lambda N_B(2) = \left(5.0 \times 10^8\right) \times \left[1 - (1 + 0.693)\exp(-0.693)\right] = 7.67 \times 10^7 \text{ dis./s}$$

or equivalently,

$$\lambda N_B(2) = \frac{7.67 \times 10^7}{3.7 \times 10^{10}} = \boxed{0.00207 \text{ Ci}}$$

■ P1.19

Part (a): The number of atoms in one gram of an isotope is N_0/A, and therefore the number of Ci in one gram is

$$\chi = \frac{(\lambda N_0/A)}{3.7 \times 10^{10}} = \frac{\left(0.693 N_0/t_{1/2} A\right)}{3.7 \times 10^{10}}$$

The half-life expressed in seconds is $138 \times 86{,}400 = 1.19 \times 10^7$ sec; substituting above brings to

$$\chi = \frac{\frac{0.693\times\left(6.02\times10^{23}\right)}{\left(1.19\times10^{7}\right)\times210}}{3.7\times10^{10}} = \boxed{4512\ \text{Ci}}$$

Part (b): The number of disintegrations per second is 4512 × (3.7×10^{10}) = 1.67×10^{14} dis./sec. Each alpha particle dissipates 5.305 MeV, and 1 MeV = 1.6×10^{-13} J. The heat produced by one gram of Po-210 then becomes

$$Q = \left(1.67\times10^{14}\right)\times5.305\times\left(1.6\times10^{-13}\right) = 141.8\ \text{J/s} = \boxed{141.8\ \text{W}}$$

■ P2.1

The equation to use is (2.4) on the textbook,

$$I(x) = I(0)\exp(-\Sigma x)$$

so that, solving for *x*,

$$x = -\frac{\ln(0.05)}{0.8} = \boxed{3.745\ \text{cm}}$$

The fraction colliding is one minus the fraction penetrating without making a collision,

$$1-\exp(-0.8\times2.0) = \boxed{0.798}$$

Approximately 80% of the neutrons make their first collision within the first 2 cm of the material.

■ P2.3

Part (a): The macroscopic cross-section Σ is given by the product

$$\Sigma = \sigma N''' = \left(3.50\times10^{-24}\right)\times\left(4.20\times10^{23}\right) = \boxed{1.47\ \text{cm}^{-1}}$$

Part (b): The mean free path λ is the reciprocal of the macro cross-section:

$$\lambda = \frac{1}{\Sigma} = \frac{1}{1.47} = \boxed{0.680\ \text{cm}}$$

Part (c): The fraction χ of neutrons that will penetrate a 3.0-cm slab without making a collision is

$$\chi = \exp(-3\Sigma) = \exp(-3\times1.47) = 0.0122 = \boxed{1.22\%}$$

Part (d): The fraction of neutrons that will collide in the slab before penetrating a distance of 1.5 cm is

$$\chi = 1-\exp(-1.5\times\Sigma) = 1-\exp(-1.5\times1.47) = 0.890 = \boxed{89.0\%}$$

■ P2.7

If the absorption cross-section for thermal neutrons in water is to be doubled, we may write

$$N^{B}\sigma_a^{B} + N^{H2O}\sigma_a^{H2O} = 2N^{H2O}\sigma_a^{H2O}$$

This can be restated as

$$\frac{N^{B}}{N^{H2O}} = \frac{\sigma_a^{H2O}}{\sigma_a^{B}}$$

Using the absorption cross-section data from Appendix E, we obtain

$$\frac{N^{B}}{N^{H2O}} = \frac{\sigma_a^{H2O}}{\sigma_a^{B}} = \frac{0.2948}{767} = 3.84\times10^{-4}$$

or, equivalently, 384×10^{-6}, which amounts to 384 ppma, or parts per million by atom. However, ppm is normally measured as a mass ratio. Accordingly, if boron weighs 10.811 amu and water has 18 amu, we may write

$$\frac{m^{B}}{m^{H2O}} = 3.84\times10^{-4}\times\frac{10.811}{18} = 230.6 \approx \boxed{231\ \text{ppm}}$$

■ P2.8

We first use equation (2.26) to compute the composite microscopic cross-section of the uranium:

$$\sigma^U = \tilde{e}\sigma^{25} + (1-\tilde{e})\sigma^{28} = 0.04\times 607.5 + (1-0.04)\times 11.8 = 35.6 \text{ b}$$

The microscopic cross-section of UO_2 is then

$$\sigma^{UO_2} = \sigma^U + 2\sigma^O = 35.6 + 2\times 3.8 = 43.2 \text{ b}$$

The macroscopic cross-section of UO_2 follows as

$$\Sigma^{UO_2} = \frac{\rho N_o}{A}\sigma^{UO_2} = \frac{10.5\times(6.02\times 10^{23})}{238+2\times 16}\times(43.2\times 10^{-24}) = \boxed{1.01 \text{ cm}^{-1}}$$

Note that we have rounded the molar mass of the enriched uranium to 238.

■ P2.13

The density of air is too small to contribute significantly to the cross-section. Given that 85% of the mixture is occupied by graphite and iron, we may write

$$\frac{V_{\text{Fe}} + V_C}{V} = 0.85$$

But $V_{Fe} = V_C$, hence

$$\frac{V_{\text{Fe}}}{V} = \frac{V_{\text{C}}}{V} = 0.425$$

The macroscopic cross-section is then

$$\Sigma = \frac{V_{Fe}}{V}\frac{\rho_{Fe}N_o}{A_{Fe}}\sigma_{Fe} + \frac{V_C}{V}\frac{\rho_C N_o}{A_C}\sigma_C$$

$$\therefore \Sigma = 0.425\times\frac{7.87\times(6.02\times 10^{23})}{55.85}\times(10.9\times 10^{-24}) + 0.425\times\frac{1.6\times(6.02\times 10^{23})}{12.01}\times(4.75\times 10^{-24})$$

$$\therefore \boxed{\Sigma = 0.555 \text{ cm}^{-1}}$$

■ P2.15

Part (a): The minimum number of collisions will result if the neutron loses the maximum amount of energy in each collision; thus, appealing to equation (2.46),

$$E = \underbrace{\left(\frac{A-1}{A+1}\right)^2}_{=\alpha} E' = \alpha E'$$

Accordingly, we have 1.0 eV = $\alpha^N \times 10^6$ eV, where N is the minimum number of collisions. Solving for N,

$$N = -\frac{\ln(10^6)}{\ln(\alpha)}$$

For deuterium,

$$\alpha = \left(\frac{2-1}{2+1}\right)^2 = 0.111$$

so that

$$N = -\frac{\ln(10^6)}{\ln(0.111)} = 6.28$$

$$\therefore \lceil N \rceil = \lceil 6.28 \rceil = \boxed{7}$$

For deuterium, the minimum number of collisions required to slow down a neutron by the desired amount is 7.

Part (b): Proceeding similarly with carbon-12,

$$\alpha = \left(\frac{12-1}{12+1}\right)^2 = 0.716$$

$$\therefore N = -\frac{\ln(10^6)}{\ln(0.716)} = 41.4$$

$$\therefore \lceil N \rceil = \lceil 41.4 \rceil = \boxed{42}$$

For carbon-12, the minimum number of collisions required to slow down a neutron by the desired amount is 42.
Part (c): Proceeding similarly with iron-56,

$$\alpha = \left(\frac{56-1}{56+1}\right)^2 = 0.931$$

$$\therefore N = -\frac{\ln\left(10^6\right)}{\ln\left(0.931\right)} = 193.2$$

$$\therefore \lceil N \rceil = \lceil 193.2 \rceil = \boxed{194}$$

For iron-56, the minimum number of collisions required to slow down a neutron by the desired amount is 194.
Part (d): Finally, for uranium-238, we write

$$\alpha = \left(\frac{238-1}{238+1}\right)^2 = 0.983$$

$$\therefore N = -\frac{\ln\left(10^6\right)}{\ln\left(0.983\right)} = 805.8$$

$$\therefore \lceil N \rceil = \lceil 805.8 \rceil = \boxed{806}$$

For uranium-238, the minimum number of collisions required to slow down a neutron by the desired amount is 806.

■ P2.16

We first compute the alpha values for both uranium isotopes:

$$\alpha^{25} = \left(\frac{235-1}{235+1}\right)^2 = 0.98312$$

$$\alpha^{28} = \left(\frac{238-1}{238+1}\right)^2 = 0.98333$$

Assuming the α for the uranium isotopes is that of uranium-238, we compute the slowing down decrement for uranium (eq. (2.56)):

$$\xi_U = 1 + \frac{\alpha^{28}}{1-\alpha^{28}} \ln \alpha^{28} = 1 + \frac{0.9833}{1-0.9833} \times \ln\left(0.9833\right) = 0.00840$$

Proceeding similarly with oxygen,

$$\alpha^O = \left(\frac{16-1}{16+1}\right)^2 = 0.7785$$

$$\therefore \xi_O = 1 + \frac{0.7785}{1-0.7785} \times \ln\left(0.7785\right) = 0.120$$

The average slowing down decrement for uranium dioxide can be established with equation (2.61),

$$\xi_{UO_2} = \frac{\xi_U N_U \sigma_S^U + \xi_O N_O \sigma_S^O}{N_U \sigma_S^U + N_O \sigma_S^O}$$

Here, $N_O = 2N_U$; also, σ_S^U = 9.146 b and σ_S^O = 3.761 b can be read from Appendix Table II-3. It follows that

$$\xi_{UO_2} = \frac{\xi_U \sigma_S^U + 2\xi_O \sigma_S^O}{\sigma_S^U + 2\sigma_S^O} = \frac{0.0084 \times 9.146 + 2 \times 0.120 \times 3.761}{9.146 + 2 \times 3.761} = 0.0588$$

Clearly, ξ_{UO_2} = 0.0588 > ξ_U = 0.00840, hence the oxygen atoms have an appreciable effect on the slowing down decrement.

■ P3.4

The eta value is given by

$$\bar{\eta} = \frac{\left(\nu N\sigma_f\right)^{25}}{\left(N\sigma_a\right)^{25} + \left(N\sigma_a\right)^{28}}$$

This can be restated as

$$\bar{\eta} = \frac{\nu^{25}\sigma_f^{25}}{\sigma_a^{25}} \frac{1}{1 + \left(\frac{N^{28}}{N^{28}}\frac{\sigma_a^{28}}{\sigma_a^{25}}\right)}$$

But, given the enrichment $\tilde{e} = N^{25}/(N^{25} + N^{28})$, we have

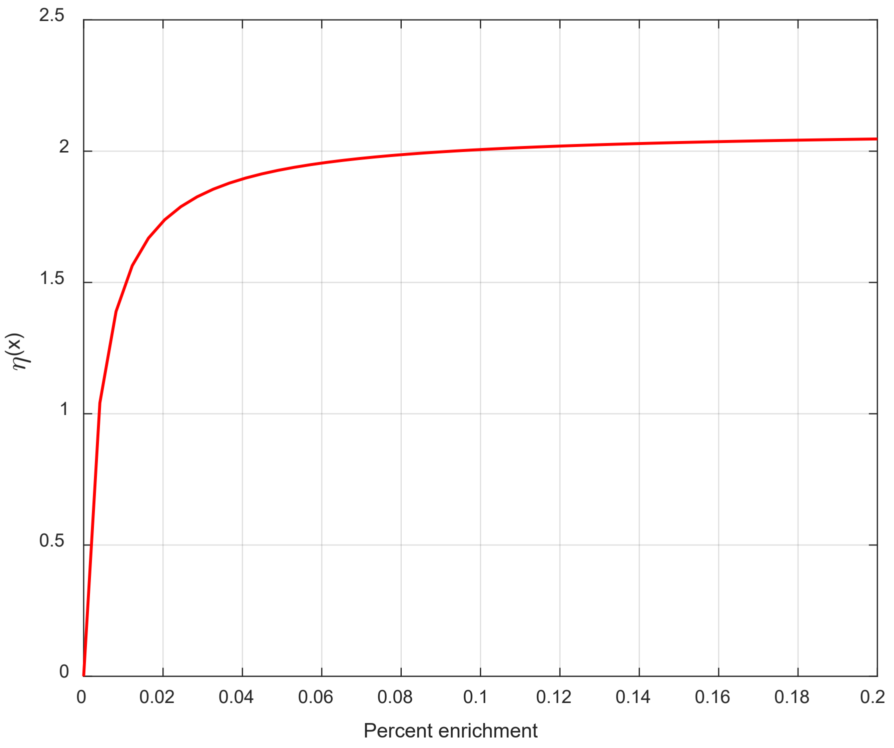
$$\bar{\eta} = \frac{\nu^{25}\sigma_f^{25}}{\sigma_a^{25}} \frac{1}{1 + \left(\tilde{e}^{-1} - 1\right)\frac{\sigma_a^{28}}{\sigma_a^{25}}}$$

Substituting values from the given table,

$$\bar{\eta} = \frac{2.43 \times 505}{591} \times \frac{1}{1 + \left(\tilde{e}^{-1} - 1\right) \times \frac{2.42}{591}}$$

$$\therefore \bar{\eta} = 2.076 \frac{1}{1 + 0.00409\left(\tilde{e}^{-1} - 1\right)}$$

This equation is plotted below as a function of enrichment $\tilde{e}$. As can be seen, η rises rapidly for enrichments ranging from 0 to ~6%; enrichments greater than 6% or so yield only marginal improvements in η.



■ P3.7

Part (a): Note first that $E = E_o exp(-u)$ and thus $dE/du = -E_o exp(-u)$, so that

$$p\left(u \to u'\right) = -p\left(E \to E'\right) dE'/du' = p\left(E \to E'\right) E_o \exp\left(-u'\right)$$

But, from equation (2.47),

$$p\left(u \to u'\right) = \frac{1}{\left(1-\alpha\right)E} dE' \quad ; \quad \alpha E \le E' \le E$$

giving

$$p\left(u \to u'\right) = \frac{1}{\left(1-\alpha\right)E} E_0 \exp\left(-u'\right) = \frac{1}{1-\alpha} \times \frac{1}{E_o \exp\left(-u\right)} \times E_0 \exp\left(-u'\right)$$

$$\therefore p\left(u \to u'\right) = \frac{1}{1-\alpha} \exp\left(u - u'\right)$$

To find the integration bounds, note that when $E' = E$, we have $u' = u$. In turn, when $E' = \alpha E$, we have

$$u' = \ln\left(\frac{E_0}{\alpha E}\right) = \ln\left(\frac{E_0}{E}\right) - \ln(\alpha) = u + \ln\left(\frac{1}{\alpha}\right)$$

In summary, we have shown that

$$\boxed{p(u \to u') = \begin{cases} \frac{1}{1-\alpha}\exp(u-u') \ , \ u \le u' \le u + \ln(1/\alpha) \\ 0 \ , \ \text{otherwise} \end{cases}}$$

Part (b): First note that

$$\varphi(E) = -\varphi(u)\,du/dE = \varphi(u)\frac{1}{E_0}\exp(u)$$

Then, equation (3.22) may be written as

$$\Sigma_s(u)\varphi(u)\frac{1}{E_0}\exp(u) = \frac{1}{1-\alpha}\int_E^{E/\alpha}\frac{1}{E'}\Sigma_s(E')\varphi(E')\,dE'$$

$$\therefore \Sigma_s(u)\varphi(u)\frac{1}{E_0}\exp(u) = -\frac{1}{1-\alpha}\int_u^{u-\ln\left(\frac{1}{\alpha}\right)}\frac{\exp(u')}{E_0}\Sigma_s(u')\varphi(u')\,du'$$

$$\therefore \Sigma_s(u)\varphi(u)\frac{1}{E_0}\exp(u) = \boxed{\frac{1}{1-\alpha}\int_{u-\ln\left(\frac{1}{\alpha}\right)}^{u}\exp(u'-u)\Sigma_s(u')\varphi(u')\,du'}$$

■ P3.10

Parts (a) and (c): All the needed data is in Table 3.1, provided we take $\Sigma_{si} = (\xi_i\Sigma_{si})/\xi_i$, yielding

$$\Sigma_s^{\mathrm{D_2O}} = \frac{0.18}{0.51} = 0.353$$

$$\Sigma_s^{\mathrm{H_2O}} = \frac{1.28}{0.93} = 1.376$$

Also,

$$\Sigma_{ai} = \frac{\xi_i\Sigma_{si}}{\left(\xi_i\Sigma_{si}/\Sigma_{ai}\right)}$$

$$\therefore \Sigma_a^{D_2O} = \frac{0.18}{21{,}000} = 8.57\times10^{-6}$$

$$\therefore \Sigma_a^{H_2O} = \frac{1.28}{58} = 0.0221$$

From equation (2.61), the averaged slowing down decrement is given by

$$\bar{\xi} = \frac{1}{\Sigma_s}\sum_i \xi_i\Sigma_{si}$$

For 1% contamination, the number densities and thus the macroscopic cross-sections of heavy water and water are replaced by 0.99 and 0.01 of their nominal values, respectively. Accordingly,

$$\bar{\xi} = \frac{\xi_{D_2O}\times0.99\Sigma_s^{D_2O} + \xi_{H_2O}\times0.01\Sigma_s^{H_2O}}{0.99\Sigma_s^{D_2O} + 0.01\Sigma_s^{H_2O}}$$

$$\therefore \bar{\xi} = \frac{0.51\times0.99\times0.353 + 0.91\times0.01\times1.376}{0.99\times0.353 + 0.01\times1.376} = \boxed{0.525}$$

The variation in slowing down ratio, in turn, is given by

$$\frac{\bar{\xi}\Sigma_s}{\Sigma_a} = \frac{\sum_i \xi_i\Sigma_{si}}{0.99\Sigma_a^{\mathrm{D_2O}} + 0.01\Sigma_a^{\mathrm{H_2O}}} = \frac{0.51\times0.99\times0.353 + 0.91\times0.01\times1.376}{0.99\times\left(8.57\times10^{-6}\right) + 0.01\times0.0221} = \boxed{832}$$

Thus, while the contamination has only small effects on the slowing down decrement, it decreases the slowing down ratio substantially as a result of the much larger absorption cross-section of water.

■ P4.3

The volume of the reactor is given by the sum of fuel, coolant (sodium), and structural material (iron):

$$V = V_f + V_{Na} + V_{Fe}$$

We were given $V_f/V = 0.30$, $V_{Na}/V = 0.50$, and $V_{Fe}/V = 0.20$. We also know that

$$V_f = V_{\mathrm{PuO_2}} + V_{UO_2}$$

and we were given $V_f = V_{PuO2}/V_f = 0.15$ and $V_{UO2}//V_f = 0.85$. Using volume weighting, we may write

$$k_\infty = \frac{\overline{v}\Sigma_f}{\Sigma_a} = \frac{\left(V_f/V\right)\overline{v}\Sigma_f^f}{\left(V_f/V\right)\Sigma_a^f + \left(V_{Na}/V\right)\Sigma_a^{\mathrm{Na}} + \left(V_{Fe}/V\right)\Sigma_a^{\mathrm{Fe}}}$$

$$\therefore k_\infty = \frac{0.30\overline{v}\Sigma_f^f}{0.30\Sigma_a^f + 0.50\Sigma_a^{\mathrm{Na}} + 0.20\Sigma_a^{\mathrm{Fe}}} \quad \text{(I)}$$

Using volume weighting for the fissile and fertile material in the fuel:

$$v\Sigma_f^f = \left(\frac{V_{\mathrm{PuO_2}}}{V_f}\right)v^{49}\Sigma_f^{\mathrm{PuO2}} + \left(\frac{V_{\mathrm{UO_2}}}{V_f}\right)v^{28}\Sigma_f^{\mathrm{UO2}} = 0.15v^{49}\Sigma_f^{\mathrm{PuO2}} + 0.85v^{28}\Sigma_f^{\mathrm{UO2}}$$

Also,

$$\Sigma_a^f = \left(\frac{V_{\mathrm{PuO_2}}}{V_f}\right)\Sigma_a^{\mathrm{PuO2}} + \left(\frac{V_{\mathrm{UO_2}}}{V_f}\right)\Sigma_a^{\mathrm{UO2}} = 0.15\Sigma_a^{\mathrm{PuO2}} + 0.85\Sigma_a^{\mathrm{UO2}}$$

Substituting the two previous results into (I) brings to

$$\therefore k_\infty = \frac{0.045v^{49}\Sigma_f^{\mathrm{PuO2}} + 0.255v^{28}\Sigma_f^{\mathrm{UO2}}}{0.045\Sigma_a^{\mathrm{PuO2}} + 0.255\Sigma_a^{\mathrm{UO2}} + 0.50\Sigma_a^{\mathrm{Na}} + 0.20\Sigma_a^{\mathrm{Fe}}}$$

Using $\Sigma = N\sigma = (\rho N_o/A)\sigma$ and ignoring the cross-sections of oxygen, we have

$$\therefore k_\infty = \frac{0.045\dfrac{\rho_{\mathrm{PuO_2}}}{A_{\mathrm{PuO_2}}}N_o v^{49}\sigma_f^{49} + 0.255\dfrac{\rho_{\mathrm{UO_2}}}{A_{\mathrm{UO_2}}}N_o v^{28}\sigma_f^{28}}{0.045\dfrac{\rho_{\mathrm{PuO_2}}}{A_{\mathrm{PuO_2}}}N_o\sigma_a^{49} + 0.255\dfrac{\rho_{\mathrm{UO_2}}}{A_{\mathrm{UO_2}}}N_o\sigma_a^{28} + 0.50\dfrac{\rho_{\mathrm{Na}}}{A_{\mathrm{Na}}}N_o\sigma_a^{\mathrm{Na}} + 0.20\dfrac{\rho_{\mathrm{Fe}}}{A_{\mathrm{Fe}}}N_o\sigma_a^{\mathrm{Fe}}}$$

Cancelling N_o,

$$\therefore k_\infty = \frac{0.045\dfrac{\rho_{\mathrm{PuO_2}}}{A_{\mathrm{PuO_2}}}v^{49}\sigma_f^{49} + 0.255\dfrac{\rho_{\mathrm{UO_2}}}{A_{\mathrm{UO_2}}}v^{28}\sigma_f^{28}}{0.045\dfrac{\rho_{\mathrm{PuO_2}}}{A_{\mathrm{PuO_2}}}\sigma_a^{49} + 0.255\dfrac{\rho_{\mathrm{UO_2}}}{A_{\mathrm{UO_2}}}\sigma_a^{28} + 0.50\dfrac{\rho_{\mathrm{Na}}}{A_{\mathrm{Na}}}\sigma_a^{\mathrm{Na}} + 0.20\dfrac{\rho_{\mathrm{Fe}}}{A_{\mathrm{Fe}}}\sigma_a^{\mathrm{Fe}}}$$

The pertaining mass numbers are $A_{Na} = 23$, $A_{Fe} = 55.85$, $A_{PuO2} = 239 + 2 \times 16 = 271$, and $A_{UO2} = 238 + 2 \times 16 = 270$; all other variables can be read from the given table, so that

$$k_\infty = \frac{0.045\times\dfrac{11}{271}\times 2.98\times 1.95 + 0.255\times\dfrac{11}{270}\times 2.47\times 0.05}{0.045\times\dfrac{11}{271}\times 2.40 + 0.255\times\dfrac{11}{270}\times 0.404 + 0.50\times\dfrac{0.97}{23}\times 0.0018 + 0.20\times\dfrac{7.87}{55.85}\times 0.0087}$$

$$\therefore \boxed{k_\infty = 1.343}$$

■ P4.4

The first step is to update the value of k_∞ accounting for the given nonleakage probability:

$$k_\infty = \frac{k}{P_{NL}} = \frac{1.0}{0.9} = 1.111$$

As before, the sum of volumes is such that $V = V_f + V_{Na} + V_{Fe}$ and we are given $V_f/V = 0.30$, $V_{Na}/V = 0.50$, and $V_{Fe}/V = 0.20$. Also, the total volume of fuel is given by the sum of the volume of plutonium dioxide, V_{PuO_2}, and uranium dioxide, V_{UO_2}:

$$V_f = V_{\mathrm{PuO}_2} + V_{\mathrm{UO}_2}$$

Let $x = V_{\mathrm{PuO}_2}/V$ and $1 - x = V_{\mathrm{UO}_2}/V$. Using volume weighting, we may write

$$k_\infty = \frac{\overline{\nu}\Sigma_f}{\Sigma_a} = \frac{\left(V_f/V\right)\overline{\nu}\Sigma_f^f}{\left(V_f/V\right)\Sigma_a^f + \left(V_{\mathrm{Na}}/V\right)\Sigma_a^{\mathrm{Na}} + \left(V_{\mathrm{Fe}}/V\right)\Sigma_a^{\mathrm{Fe}}}$$

$$\therefore k_\infty = \frac{\overline{\nu}\Sigma_f}{\Sigma_a} = \frac{0.30\overline{\nu}\Sigma_f^f}{0.30\Sigma_a^f + 0.50\Sigma_a^{\mathrm{Na}} + 0.20\Sigma_a^{\mathrm{Fe}}} \quad \text{(I)}$$

Using volume weighting for the fissile and fertile material in the fuel, we have

$$\nu\Sigma_f^f = x\nu^{49}\Sigma_f^{\mathrm{PuO2}} + (1-x)\nu^{28}\Sigma_f^{\mathrm{UO2}}$$

and

$$\Sigma_a^f = x\Sigma_a^{\mathrm{PuO2}} + (1-x)\Sigma_a^{\mathrm{UO2}}$$

Substituting in (I),

$$k_\infty = \frac{\overline{\nu}\Sigma_f}{\Sigma_a} = \frac{0.30\left[x\nu^{49}\Sigma_f^{\mathrm{PuO2}} + (1-x)\nu^{28}\Sigma_f^{\mathrm{UO2}}\right]}{0.30\left[x\Sigma_a^{\mathrm{PuO2}} + (1-x)\Sigma_a^{\mathrm{UO2}}\right] + 0.50\Sigma_a^{\mathrm{Na}} + 0.20\Sigma_a^{\mathrm{Fe}}}$$

Using $\Sigma = N\sigma = (\rho N_o/A)\sigma$ and ignoring the cross-sections of oxygen, we have

$$k_\infty = \frac{0.30\left[x\frac{\rho_{\mathrm{PuO}_2}}{A_{\mathrm{PuO}_2}}N_o\nu^{49}\sigma_f^{49} + (1-x)\frac{\rho_{\mathrm{UO}_2}}{A_{\mathrm{UO}_2}}N_o\nu^{28}\sigma_f^{28}\right]}{0.30\left[x\frac{\rho_{\mathrm{PuO}_2}}{A_{\mathrm{PuO}_2}}N_o\sigma_a^{49} + (1-x)\frac{\rho_{\mathrm{UO}_2}}{A_{\mathrm{UO}_2}}N_o\sigma_a^{28}\right] + 0.50\frac{\rho_{\mathrm{Na}}}{A_{\mathrm{Na}}}N_o\sigma_a^{\mathrm{Na}} + 0.20\frac{\rho_{\mathrm{Fe}}}{A_{\mathrm{Fe}}}N_o\sigma_a^{\mathrm{Fe}}}$$

Cancelling N_o,

$$k_\infty = \frac{0.30\left[x\frac{\rho_{\mathrm{PuO}_2}}{A_{\mathrm{PuO}_2}}\nu^{49}\sigma_f^{49} + (1-x)\frac{\rho_{\mathrm{UO}_2}}{A_{\mathrm{UO}_2}}\nu^{28}\sigma_f^{28}\right]}{0.30\left[x\frac{\rho_{\mathrm{PuO}_2}}{A_{\mathrm{PuO}_2}}\sigma_a^{49} + (1-x)\frac{\rho_{\mathrm{UO}_2}}{A_{\mathrm{UO}_2}}\sigma_a^{28}\right] + 0.50\frac{\rho_{\mathrm{Na}}}{A_{\mathrm{Na}}}\sigma_a^{\mathrm{Na}} + 0.20\frac{\rho_{\mathrm{Fe}}}{A_{\mathrm{Fe}}}\sigma_a^{\mathrm{Fe}}} = 1.111$$

The mass numbers A have already been calculated in the previous problem. Substituting these and other data, we obtain

$$k_\infty = \frac{0.30\left[x\times\frac{11}{271}\times 2.98\times 1.95 + (1-x)\times\frac{11}{270}\times 2.47\times 0.05\right]}{0.30\left[x\times\frac{11}{271}\times 2.40 + (1-x)\times\frac{11}{270}\times 0.404\right] + 0.50\times\frac{0.97}{23}\times 0.0018 + 0.20\times\frac{7.87}{55.85}\times 0.0087} = 1.111$$

$$\therefore k_\infty = \frac{0.30\left[0.236x + 0.00503(1-x)\right]}{0.30\left[0.0974x + 0.0165(1-x)\right] + 0.000283} = 1.111$$

This first-degree equation in x can be easily solved by expanding the terms in brackets, cross-multiplying, and solving for x. We can speed things up with Mathematica:

```
In[110]:= Solve[(0.30 * (0.236 * x + 0.00503 * (1 - x)))/(0.30 * (0.0974 * x + 0.0165 * (1 - x)) + 0.000283) == 1.111, x]

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[110]= {{x → 0.101705}}
```

That is, $x \approx 0.102$. Therefore, the fuel should be 10.2% PuO_2 and 89.8% UO_2 by volume.

■ P4.6

Let N be the number density of UO_2. Given the enrichment $\tilde{e} = 0.03$, we appeal to equation (4.49) (pp. 107) to compute factor η_T:

$$\eta_T = \frac{\nu \Sigma_f^f}{\Sigma_a^f} = \frac{\tilde{e} N \nu^{25} \sigma_f^{25}}{\tilde{e} N \sigma_a^{25} + (1-\tilde{e}) N \sigma_a^{28} + 2N\sigma_a^O}$$

Cancelling *N*,

$$\eta_T = \frac{\tilde{e} \nu^{25} \sigma_f^{25}}{\tilde{e} \sigma_a^{25} + (1-\tilde{e}) \sigma_a^{28} + 2\sigma_a^O}$$

Gleaning data from Table 3.2 and oxygen data from Table E-3, we get

$$\eta_T = \frac{0.03 \times 2.43 \times 505}{0.03 \times 591 + (1-0.03) \times 2.42 + 2 \times 0.0002} = \boxed{1.834}$$

Now, to find the resonance escape probability *p*, we appeal to equation (4.40) in page 104:

$$p = \exp\left[-\left(\frac{V_f}{V_m}\right)\frac{N_f}{\xi\Sigma_s} I\right] \text{ (I)}$$

Given *A* = 238 + 2 × 16 = 270, the number density of fuel is

$$N_f = \frac{\rho N_0}{A} = \frac{11.0 \times (6.02 \times 10^{23})}{270} = 2.45 \times 10^{22} \text{ g/cm}^3$$

The resonance integral I for UO_2 can be read from Table 4.3 (pp. 105),

$$I = 4.45 + 26.6\sqrt{4/\rho D} = 4.45 + 26.6\sqrt{4/(11 \times 1.1)} = 19.74 \text{ b}$$

Note that we have used 1.1 cm as the pin diameter instead of 1.0 cm because we were told that the Dancoff correction increases the fuel diameter for the resonance integral calculation by 10%. Taking $\xi\Sigma_s$ = 1.28 for water from Table 3.1, we substitute into (I) to obtain

$$p = \exp\left[-\left(\frac{V_f}{V_m}\right)\frac{N_f}{\xi\Sigma_s} I\right] = \exp\left[-\left(\frac{1}{2}\right) \times \frac{(2.45 \times 10^{22})}{1.28} \times (19.74 \times 10^{-24})\right]$$

$$\therefore \boxed{p = 0.828}$$

Next, the thermal utilization is given by equation (4.55) (pp. 108):

$$f = \frac{1}{1 + \varsigma (V_m N_m / V_f N_f)(\bar{\sigma}_{aT}^m / \bar{\sigma}_{aT}^f)} \text{ (II)}$$

Here, the number density ratio N_m/N_f is determined as

$$\frac{N_m}{N_f} = \frac{\rho_m N_0 / A_m}{\rho_f N_0 / A_f} = \frac{\rho_m A_f}{\rho_f A_m} = \frac{1.0 \times 270}{11 \times 18} = 1.364$$

The absorption cross-sections $\bar{\sigma}_{aT}^m$ = 0.5896 b and $\bar{\sigma}_{aT}^f$ = 6.540 b can be read from Table E-3; substituting in (II),

$$f = \frac{1}{1 + \varsigma (V_m/V_f)(N_m/N_f)(\bar{\sigma}_{aT}^m / \bar{\sigma}_{aT}^f)} = \frac{1}{1 + 1.16 \times 2 \times 1.364 \times (0.5896/6.540)} = \boxed{0.778}$$

Finally, k_∞ is calculated as

$$k_\infty = \varepsilon p f \eta_T = 1.24 \times 0.828 \times 0.778 \times 1.834 = \boxed{1.465}$$

■ P4.8

Part (a): The resonance escape probability is given by equation (4.40), namely

$$p = \exp\left[-\left(\frac{V_f}{V_m}\right)\frac{N_f}{\xi\Sigma_s} I\right]$$

As heavy water is replaced by light water, the only quantity changing is the slowing down power, $\xi\Sigma_s$. Referring to Table 3.1 on page 62, we see that the SDP of H_2O (= 1.28) is greater than that of D_2O (= 0.18). Accordingly, we surmise that the resonance escape probability will increase.

Part (b): The thermal utilization is given by equation (4.55), namely

$$f = \frac{1}{1+\varsigma\left(V_m/V_f\right)\left(N_m/N_f\right)\left(\bar{\sigma}_{aT}^m/\bar{\sigma}_{aT}^f\right)}$$

The major change will occur in the moderator thermal absorption cross-section, $\bar{\sigma}_{aT}^m$. The thermal disadvantage factor will change less. With reference to Table E-3, we see that $\bar{\sigma}_{aT}^m$ is much smaller for heavy water than for light water. Accordingly, the numerator in (4.55) will increase and the thermal utilization will decrease.
Part (c): Because of its very small thermal absorption cross-section, heavy water is considered to be the best moderator. Reactors can be built using natural uranium if heavy water is the moderator, but not with light water. Therefore, the net effect of replacing heavy water with ordinary water would be to decrease the value of k_∞.

■ P4.10

Part (a): The pertaining calculations have been performed in the solution to Problem 4.6. The resonance escape probability is such that

$$p = \exp\left[-\left(\frac{V_f}{V_m}\right)\frac{N_f}{\xi\Sigma_s}I\right] = \exp\left[-\left(\frac{V_m}{V_f}\right)^{-1}\times\frac{\left(2.45\times10^{22}\right)}{1.28}\times\left(19.74\times10^{-24}\right)\right]$$

$$\therefore p = \exp\left[-0.378\left(\frac{V_m}{V_f}\right)^{-1}\right] \quad \text{(I)}$$

This is the equation we need to plot in part (a).
Part (b): The thermal utilization *f* is such that

$$f = \frac{1}{1+\varsigma\left(V_m/V_f\right)\left(N_m/N_f\right)\left(\bar{\sigma}_{aT}^m/\bar{\sigma}_{aT}^f\right)} = \frac{1}{1+1.16\times\left(V_m/V_f\right)\times1.364\times\left(0.5896/6.540\right)}$$

$$\therefore f = \frac{1}{1+0.1426\left(V_m/V_f\right)} \quad \text{(II)}$$

This is the equation we need to plot in part (b).
Part (c): With ε and η_T unchanged at 1.24 and 1.834, respectively, we write the four-factor formula to obtain

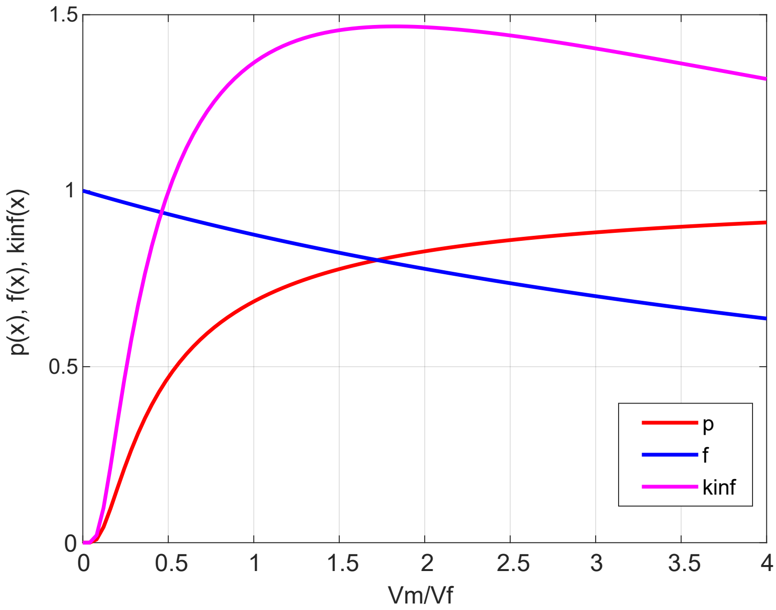
$$k_\infty = \varepsilon p f \eta_T = 1.24\times\exp\left[-0.378\left(\frac{V_m}{V_f}\right)^{-1}\right]\times\frac{1}{1+0.1426\left(V_m/V_f\right)}\times1.834$$

$$\therefore k_\infty = 2.274\exp\left[-0.378\left(\frac{V_m}{V_f}\right)^{-1}\right]\times\frac{1}{1+0.1426\left(V_m/V_f\right)} \quad \text{(III)}$$

This is the equation we need to plot in part (c).
Parts (d) and (e): The equations obtained in parts (a), (b), and (c) are plotted with the following MATLAB code (note that we've used a wider range of V_m/V_f values than the problem statement suggests):

```
m_f = linspace(0,4,100);
p = exp(-0.378.*m_f.^(-1));
f = 1./(1+0.1426.*m_f);
k_inf = 2.274.*p.*f;
figure
plot(m_f, p, 'LineWidth', 2, 'Color', 'red')
hold on
plot(m_f, f, 'LineWidth', 2, 'Color', 'blue')
plot(m_f, k_inf, 'LineWidth', 2, 'Color', 'magenta')
grid on
hold off
```

(See next page.)



Note that the resonance escape probability p increases steadily with increasing V_m/V_f. In turn, the thermal utilization f decreases steadily with increasing V_m/V_f. Finally, k_∞ at first rapidly increases with V_m/V_f but then begins to drop.

To find the maximum value of k_∞, we type

```
>> max(k_inf)

ans =

     1.4668
```

That is, $k_{\infty,\max}$ = 1.4668. This is entry number 46 in *k_inf*; the corresponding V_m/V_f is

```
>> m_f(46)

ans =

     1.8182
```

That is, the maximum k_∞ corresponds to a volume ratio $V_m/V_f \approx 1.818$.

■ P4.11

Part (a): As indicated by Table 4.3, the resonance integral increases when replacing U with UO_2. This effect is magnified by the lower density of UO_2.

Part (b): The thermal utilization may decrease slightly as a result of the lower density of UO_2. The thermal absorption cross-section of oxygen is so small that it has little effect.

Part (c): Since the thermal absorption cross-section of oxygen is so small, it causes only a very slight decrease in η_T. The decrease in density cancels out of the definition of η_T.

■ P4.12

With reference to the table given in Problem 3.4, calculating η_T is straightforward:

$$\eta_T = \frac{\nu\Sigma_f^f}{\Sigma_a^f} = \frac{0.02\cancel{N}\nu^{25}\sigma_f^{25} + 0.01\cancel{N}\nu^{49}\sigma_f^{49}}{0.02\cancel{N}\sigma_a^{25} + 0.01\cancel{N}\sigma_f^{49} + 0.97\cancel{N}\sigma_a^{28}} = \frac{0.02\nu^{25}\sigma_f^{25} + 0.01\nu^{49}\sigma_f^{49}}{0.02\sigma_a^{25} + 0.01\sigma_f^{49} + 0.97\sigma_a^{28}}$$

$$\therefore \eta_T = \frac{0.02\times 2.43\times 505 + 0.01\times 2.90\times 698}{0.02\times 591 + 0.01\times 973 + 0.97\times 2.42} = \boxed{1.874}$$

■ P5.6

Part (a): The period T we aim for is

$$\exp\left(\frac{2}{T}\right) = 10^3 \;\rightarrow\; T = \frac{2}{\ln\left(10^3\right)}$$

$$\therefore T = \frac{2}{\ln\left(10^3\right)} = 0290 \text{ h} = \boxed{17.4 \text{ min}}$$

Part (b): Since the period is very long, the positive reactivity must be quite small relative to β, and we may use equation (5.57):

$$T = \frac{\beta}{\rho\lambda} \;\rightarrow\; \frac{\rho}{\beta} = \frac{1}{\lambda T} \;\text{(I)}$$

The only missing quantity is λ, which can be established from equation (5.34); the delayed neutron properties can be gleaned from Table 5.1.

I	Approximate half-life (sec)	lambda	beta	beta/lambda
1	56	0.0124	0.00021	0.01697
2	23	0.0301	0.00142	0.04713
3	6.2	0.1118	0.00128	0.01145
4	2.3	0.3013	0.00257	0.00853
5	0.61	1.1361	0.00075	0.00066
6	0.23	3.0130	0.00027	0.00009
Total			0.0065	0.08483
1/lambda			13.05064	
lambda			0.07662	

As shown in the highlighted cell, we have $\lambda = 0.07662\ \text{s}^{-1}$. Substituting in (I),

$$\frac{\rho}{\beta} = \frac{1}{\lambda T} = \frac{1}{0.07662 \times (17.4 \times 60)} = 0.0125 \text{ dollars} = \boxed{1.25 \text{ cents}}$$

■ P5.9

Since only one cent of reactivity is involved, equation (5.57) is a reasonable approximation for both positive and negative periods of parts (a) and (b) respectively. Table 5.1 gives the delayed neutron fractions. However, we must use equation (5.34) to calculate λ. The calculations are summarized below.

Half-life (s)	beta(I)			beta(I)/lambda(I)		
	U-233	U-235	Pu-239	U-233	U-235	Pu-239
56	0.00023	0.00021	0.00007	0.018586	0.01697	0.005657
23	0.00078	0.00142	0.00063	0.025887	0.047128	0.020909
6.2	0.00064	0.00128	0.00044	0.005726	0.011452	0.003937
2.3	0.00074	0.00257	0.00069	0.002456	0.00853	0.00229
0.61	0.00014	0.00075	0.00018	0.000123	0.00066	0.000158
0.23	0.00008	0.00027	0.00009	2.66E-05	8.96E-05	2.99E-05
Sum	0.00261	0.0065	0.0021	0.052805	0.084829	0.032981
1/lambda				20.23176	13.05064	15.70501
lambda				0.049427	0.076625	0.063674

Then, the reactor periods are expressed as

$$T = \frac{\beta}{\rho\lambda} = \frac{\cancel{\beta}}{0.01\cancel{\beta}\lambda} = \frac{100}{\lambda}$$

Part (a): Using the λ values highlighted in blue, we have, for uranium-233,

$$T = \frac{100}{0.049427} = 2023 \text{ s} = \boxed{33.7 \text{ min}}$$

For uranium-235,

$$T = \frac{100}{0.076625} = 1305 \text{ s} = \boxed{21.8 \text{ min}}$$

For plutonium-239,

$$T = \frac{100}{0.063674} = 1571 \text{ s} = \boxed{26.2 \text{ min}}$$

Part (b): The periods for part (b) are simply $T = -33.7$ min for uranium-233, $T = -21.8$ min for uranium-235, and $T = -26.2$ min for plutonium-239.

■ P6.6

Part (a): In source-free spherical geometry, the neutron flux is given by equation (6.51), namely

$$\phi(r) = \frac{C_1}{r}\exp\left(\frac{r}{L}\right) + \frac{C_2}{r}\exp\left(-\frac{r}{L}\right)$$

Since in this problem there is no source at the origin, the flux $\phi(0)$ must be finite. However, with reference to the equation above, this can only hold if $C_2 = -C_1$, so that

$$\phi(r)=\frac{C_1}{r}\exp\left(\frac{r}{L}\right)-\frac{C_1}{r}\exp\left(-\frac{r}{L}\right)=C\frac{\sinh(r/L)}{r} \quad \text{(I)}$$

where $C = 2C_1$. Suppose that there are s'' neutrons/s/cm^2 impinging on the surface of the sphere. Then, the incoming partial current will be

$$J^-(R)=\frac{1}{4}\phi(R)+\frac{1}{2}D\frac{d}{dr}\phi(r)\Big|_{r=R}=s''$$

Inserting the flux distribution,

$$\frac{1}{4}C\frac{\sinh(R/L)}{R}+\frac{1}{2}DC\frac{\cosh(R/L)}{LR}-\frac{1}{2}DC\frac{\sinh(R/L)}{R^2}=s''$$

Solving for C:

$$C=\left[\left(1-2DR^{-1}\right)\sinh(R/L)+2DL^{-1}\cosh(R/L)\right]^{-1}4Rs''$$

Substituting C in (I) brings to

$$\phi(r)=\frac{\sinh(r/L)}{\left(1-2DR^{-1}\right)\sinh(R/L)+2DL^{-1}\cosh(R/L)}\left(\frac{R}{r}\right)4s''$$

Part (b): The fraction absorbed will be simply one minus the fraction reflected. The fraction reflected is just the albedo, namely

$$\alpha=\frac{J^+(R)}{J^-(R)}$$

But $J^+(R)$ is given by (6.32) and $J^-(R) = s''$, so that

$$\alpha=\frac{1}{s''}\left[\frac{1}{4}\phi(R)-\frac{1}{2}D\frac{d}{dr}\phi(r)\Big|_{r=R}\right] \quad \text{(II)}$$

We first take the flux derivative,

$$\frac{d}{dr}\phi(r)=\frac{L^{-1}\cosh(r/L)-r^{-1}\sinh(r/L)}{\left(1-2DR^{-1}\right)\sinh(R/L)+2DL^{-1}\cosh(R/L)}\left(\frac{R}{r}\right)4s''$$

Substituting $\phi(R)$ and the derivative $d\phi(R)/dr$ into (II),

$$\alpha=\left[\begin{array}{l}\dfrac{\sinh(R/L)}{\left(1-2DR^{-1}\right)\sinh(R/L)+2DL^{-1}\cosh(R/L)}\\ -2D\dfrac{L^{-1}\cosh(R/L)-R^{-1}\sinh(R/L)}{\left(1-2DR^{-1}\right)\sinh(R/L)+2DL^{-1}\cosh(R/L)}\end{array}\right]$$

We can simplify this lengthy expression with Mathematica's *Simplify* command:

```
In[139]:= Simplify[Sinh[R / L] / ((1 - 2 * d * R^-1) * Sinh[R / L] + 2 * d * L^-1 * Cosh[R / L]) -
   2 * d * (L^-1 * Cosh[R / L] - R^-1 * Sinh[R / L]) / ((1 - 2 * d * R^-1) * Sinh[R / L] + 2 * d * L^-1 * Cosh[R / L])]

Out[139]= (-2 d R Cosh[R/L] + L (2 d + R) Sinh[R/L]) / (2 d R Cosh[R/L] + L (-2 d + R) Sinh[R/L])
```

That is,

$$\alpha=\frac{-2DR\times\cosh(R/L)+L(2D+R)\times\sinh(R/L)}{2DR\times\cosh(R/L)+L(-2D+R)\times\sinh(R/L)} \quad \text{(III)}$$

For graphite, $D = 0.84$ cm and $\Sigma_a = 2.1\times10^{-4}$ cm^{-1}, so that

$$L=\sqrt{D/\Sigma_a}=\sqrt{0.84/\left(2.1\times10^{-4}\right)}=63.3\text{ cm}$$

Also, $R = 100$ cm; inserting these quantities into (III),

$$\alpha = \frac{-2\times 0.84\times 100\times \cosh\left(100/63.3\right)+63.3\times\left(2\times 0.84+100\right)\times\sinh\left(100/63.3\right)}{2\times 0.84\times 100\times \cosh\left(100/63.3\right)+63.3\times\left(-2\times 0.84+100\right)\times\sinh\left(100/63.3\right)}$$

$$\therefore \alpha = 0.976$$

Thus, because graphite has a very small absorption cross-section, 97.6% of the neutrons are reflected from the sphere, while only 100 – 97.6 = 2.4% are absorbed in the slab. Note that if the sphere had a very large radius ($R \to \infty$), equation (III) would further simplify to:

In[1]:= $\text{Limit}\left[\dfrac{-2\,d\,R\,\text{Cosh}\left[\frac{R}{L}\right]+L\,(2\,d+R)\,\text{Sinh}\left[\frac{R}{L}\right]}{2\,d\,R\,\text{Cosh}\left[\frac{R}{L}\right]+L\,(-2\,d+R)\,\text{Sinh}\left[\frac{R}{L}\right]},\ R\to\infty\right]$

Out[1]= $\dfrac{-2\,d+L}{2\,d+L}$ if $d\in\mathbb{R}$ && $L>0$

That is,

$$\lim_{R\to\infty}\alpha = \frac{-2D+L}{2D+L} = \frac{-2\times 0.84+63.3}{2\times 0.84+63.3} = 0.948$$

That is, in the upper limit of an infinite-size sphere, only about 5.2% of the incident neutrons would be absorbed in the graphite.

■ P6.10

Part (a): Equation (6.51) applies on both sides of the spherical shell; the equation is

$$\phi(r) = \frac{C_1}{r}\exp\left(r/L\right)+\frac{C_2}{r}\exp\left(-r/L\right)$$

Inside the shell we must have $C_1 = -C_2$ and therefore the solution may be restated as

$$\phi(r) = \frac{C_1}{r}\exp\left(r/L\right)-\frac{C_1}{r}\exp\left(-r/L\right) = \frac{\sinh\left(r/L\right)}{r}C \quad ; \quad 0\le r<R$$

Outside the shell we must have $C_1 = 0$, so that the solution vanishes at infinity. Thus, we may write

$$\phi(r) = \frac{\exp\left(-r/L\right)}{r}C' \quad ; \quad R<r\le\infty$$

The next step is to employ the interface conditions, (6.42) and (6.44), to determine the remaining two constants. From equation (6.42),

$$\phi\left(x_{0,-}\right)=\phi\left(x_{0,+}\right) \quad\to\quad \sinh\left(\frac{R}{L}\right)C = \exp\left(-\frac{R}{L}\right)C'$$

$$\therefore C' = \exp\left(\frac{R}{L}\right)\sinh\left(\frac{R}{L}\right)C \quad \text{(I)}$$

To employ equation (6.44), we need the currents at each side of the interface:

$$J\left(r_-\right) = -D\frac{d}{dr}\phi(r)\Big|_{r_0^-} = -D\left[\frac{\cosh\left(r/L\right)}{Lr}-\frac{\sinh\left(r/L\right)}{r^2}\right]C \quad 0\le r<R$$

$$J\left(r_+\right) = -D\frac{d}{dr}\phi(r)\Big|_{r_0^+} = D\left[\frac{\exp\left(-r/L\right)}{Lr}-\frac{\exp\left(-r/L\right)}{r^2}\right]C' \quad R<r\le\infty$$

For this spherical geometry problem, equation (6.44) becomes

$$-D\frac{d}{dr}\phi(r)\Big|_{R^-}+s'' = -D\frac{d}{dr}\phi(r)\Big|_{R^+}$$

$$\therefore -D\left[\frac{\cosh\left(R/L\right)}{LR}-\frac{\sinh\left(R/L\right)}{R^2}\right]C+s'' = D\left[\frac{\exp\left(-R/L\right)}{LR}+\frac{\exp\left(-R/L\right)}{R^2}\right]C'$$

Replacing C' with (I),

$$-D\left[RL^{-1}\cosh\left(R/L\right)-\sinh\left(R/L\right)\right]C+R^2s'' = D\left(RL^{-1}+1\right)\sinh\left(R/L\right)C$$

$$\therefore C = \underbrace{\left[\sinh(R/L) + \cosh(R/L)\right]^{-1}}_{=\exp(-R/L)} \frac{RLs''}{D} = \exp(-R/L)\frac{RLs''}{D}$$

Finally, we have

$$\phi(r) = \begin{cases} \left(\dfrac{R}{r}\right)\sinh(r/L)\exp(-R/L)\dfrac{Ls''}{D} & ; \quad 0 \le r < R \\ \left(\dfrac{R}{r}\right)\exp(-r/L)\sinh(R/L)\dfrac{Ls''}{D} & ; \quad R \le r < \infty \end{cases}$$

Part (b): Since *sinh(r/L)* ≈ *r/L* for small *r*, flux $\phi(0)$ is such that

$$\phi(0) = \left(\frac{R}{r}\right)\sinh(r/L)\exp(-R/L)\frac{Ls''}{D} \approx \exp\left(-\frac{R}{L}\right)\frac{Rs''}{D}$$

Also,

$$\phi(R) = \sinh\left(\frac{R}{L}\right)\exp\left(-\frac{R}{L}\right)\frac{Ls''}{D}$$

Dividing $\phi(0)$ by $\phi(R)$,

$$\frac{\phi(0)}{\phi(R)} = \frac{\cancel{\exp(-R/L)}\; R\cancel{s''}/\cancel{D}}{\sinh(R/L)\;\cancel{\exp(-R/L)}\; L\cancel{s''}/\cancel{D}} = \boxed{\frac{(R/L)}{\sinh(R/L)}}$$

■ P6.14

In equation (6.95) let $\sqrt{1-k_\infty} = m$. The first two terms of a Taylor series expansion of *sinh(x)* centered at *x* = 0 are

$$\sinh(x) = x + \frac{x^3}{6} + \mathbf{O}(x)$$

so that, substituting in (6.95),

$$\phi(r) = \frac{s_o'''}{m^2\Sigma_a}\left[1 - \frac{\tilde{R}}{r}\frac{\sinh\left(L^{-1}\sqrt{1-k_\infty}\,r\right)}{\sinh\left(L^{-1}\sqrt{1-k_\infty}\,\tilde{R}\right)}\right] = \frac{s_o'''}{m^2\Sigma_a}\left[1 - \frac{\tilde{R}}{r}\frac{\sinh\left(L^{-1}mr\right)}{\sinh\left(L^{-1}m\tilde{R}\right)}\right]$$

$$\therefore \phi(r) = \frac{s_o'''}{m^2\Sigma_a}\left[1 - \frac{\tilde{R}}{r}\frac{L^{-1}mr + \frac{1}{6}\left(L^{-1}mr\right)^3}{L^{-1}m\tilde{R} + \frac{1}{6}\left(L^{-1}m\tilde{R}\right)^3}\right]$$

$$\therefore \phi(r) = \frac{s_o'''}{m^2\Sigma_a}\left\{1 - \cancel{\frac{\tilde{R}}{r}}\cancel{\frac{r}{\tilde{R}}}\left[\frac{1 + \frac{1}{6}\left(L^{-1}mr\right)^2}{1 + \frac{1}{6}\left(L^{-1}m\tilde{R}\right)^2}\right]\right\}$$

$$\therefore \phi(r) = \frac{s_o'''}{m^2\Sigma_a}\left\{1 - \left[\frac{1 + \frac{1}{6}\left(L^{-1}mr\right)^2}{1 + \frac{1}{6}\left(L^{-1}m\tilde{R}\right)^2}\right]\right\} \quad \text{(I)}$$

For small x and y, we may use the approximation

$$x + y = \frac{(1+x)}{(1+y)} \approx (1+x)(1-y) \approx 1 + x - y$$

so that

$$\left[\frac{1 + \frac{1}{6}\left(L^{-1}mr\right)^2}{1 + \frac{1}{6}\left(L^{-1}m\tilde{R}\right)^2}\right] \approx 1 + \frac{1}{6}\left(L^{-1}mr\right)^2 - \frac{1}{6}\left(L^{-1}m\tilde{R}\right)^2$$

Substituting in (I),

$$\phi(r) = \frac{s_o'''}{m^2\Sigma_a}\left\{1 - \left[\frac{1 + \frac{1}{6}\left(L^{-1}mr\right)^2}{1 + \frac{1}{6}\left(L^{-1}m\tilde{R}\right)^2}\right]\right\} = \frac{s_o'''}{m^2\Sigma_a}\left\{1 - 1 - \frac{1}{6}\left(L^{-1}mr\right)^2 + \frac{1}{6}\left(L^{-1}m\tilde{R}\right)^2\right\}$$

$$\therefore \phi(r) = \frac{s_o'''}{6\Sigma_a L^2}\left(\tilde{R}^2 - r^2\right)$$

Alternatively,

$$\phi(r)=\frac{s_o'''}{6D}\left(\tilde{R}^2-r^2\right)$$

Turning to equation (6.103), let $\sqrt{1-k_\infty}=n$. Note that for small x a Taylor series expansion of *sin(x)* around $x = 0$ reads

$$\sin x = x-\frac{1}{6}x^3+\mathbf{O}(x)$$

so that

$$\phi(r)=\frac{s_o'''}{n^2\Sigma_a}\left[\frac{\tilde{R}\sin\left(L^{-1}\sqrt{k_\infty-1}r\right)}{r\sin\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)}-1\right]=\frac{s_o'''}{n^2\Sigma_a}\left[\frac{\tilde{R}\sin\left(L^{-1}nr\right)}{r\sin\left(L^{-1}n\tilde{R}\right)}-1\right]$$

$$\therefore \phi(r)=\frac{s_o'''}{n^2\Sigma_a}\left\{\frac{\tilde{R}}{r}\left[\frac{L^{-1}nr-{}^1\!/_6\left(L^{-1}nr\right)^3}{L^{-1}n\tilde{R}-{}^1\!/_6\left(L^{-1}n\tilde{R}\right)^3}\right]-1\right\}$$

$$\therefore \phi(r)=\frac{s_o'''}{n^2\Sigma_a}\left\{\cancel{\frac{\tilde{R}}{r}\frac{r}{\tilde{R}}}\left[\frac{L^{-1}nr-{}^1\!/_6\left(L^{-1}nr\right)^2}{L^{-1}n\tilde{R}-{}^1\!/_6\left(L^{-1}n\tilde{R}\right)^2}\right]-1\right\}\ \text{(II)}$$

As before, for small x and y, we may use the approximation $x + y \approx (1 - x)(1 + y) \approx 1 - x + y$, giving

$$\left[\frac{L^{-1}nr-{}^1\!/_6\left(L^{-1}nr\right)^3}{L^{-1}n\tilde{R}-{}^1\!/_6\left(L^{-1}n\tilde{R}\right)^3}\right]\approx 1-\frac{1}{6}\left(L^{-1}nr\right)^2+\frac{1}{6}\left(L^{-1}n\tilde{R}\right)^2$$

Substituting in (II),

$$\phi(r)=\frac{s_o'''}{n^2\Sigma_a}\left[1-\frac{1}{6}\left(L^{-1}nr\right)^2+\frac{1}{6}\left(L^{-1}n\tilde{R}\right)^2-1\right]$$

$$\therefore \phi(r)=\frac{s_o'''}{6\Sigma_a L^2}\left(\tilde{R}^2-r^2\right)$$

Clearly, equations (6.95) and (6.103) tend to the same result in the limit of $k_\infty \to 1$.

■ P6.15

Within the sphere (except at the origin) the flux is given by equation (6.92), but with the distributed source equal to zero:

$$\phi(r)=\frac{C_1}{r}\exp\left(L^{-1}\sqrt{1-k_\infty}r\right)+\frac{C_2}{r}\exp\left(-L^{-1}\sqrt{1-k_\infty}r\right)\ \text{(I)}$$

To obtain the two arbitrary constants, we first set the flux to zero at the extrapolated boundary, that is, $\phi(\tilde{R}) = 0$, giving

$$\phi(r=\tilde{R})=\frac{C_1}{\tilde{R}}\exp\left(L^{-1}\sqrt{1-k_\infty}\tilde{R}\right)+\frac{C_2}{\tilde{R}}\exp\left(-L^{-1}\sqrt{1-k_\infty}\tilde{R}\right)=0$$

$$\therefore C_1=-\exp\left(-2L^{-1}\sqrt{1-k_\infty}\tilde{R}\right)C_2$$

Substituting in (I),

$$\phi(r)=\left\{\exp\left[-L^{-1}\sqrt{1-k_\infty}r\right]-\exp\left[L^{-1}\sqrt{1-k_\infty}\left(r-2\tilde{R}\right)\right]\right\}\frac{C_2}{r}$$

or, in terms of hyperbolic functions,

$$\phi(r)=\sinh\left[L^{-1}\sqrt{1-k_\infty}\left(\tilde{R}-r\right)\right]\frac{C}{r}\ \text{(II)}$$

where *C* is an arbitrary constant. To determine *C*, we appeal to equation (6.53); the current *J(r)* is given by (6.52),

$$J(r)-D\frac{d}{dr}\phi(r)=\left\{\begin{aligned}&\frac{D}{Lr}\sqrt{1-k_\infty}\cosh\left[L^{-1}\sqrt{1-k_\infty}\left(\tilde{R}-r\right)\right]\\&+\frac{D}{r^2}\sinh\left[L^{-1}\sqrt{1-k_\infty}\left(\tilde{R}-r\right)\right]\end{aligned}\right\}C$$

so that

$$s_p=\lim_{r\to 0}4\pi r^2 J(r)=\lim_{r\to 0}4\pi\left\{\begin{matrix}\frac{Dr}{L}\sqrt{1-k_\infty}\cosh\left[L^{-1}\sqrt{1-k_\infty}\left(\tilde{R}-r\right)\right]\\ +D\sinh\left[L^{-1}\sqrt{1-k_\infty}\left(\tilde{R}-r\right)\right]\end{matrix}\right\}C$$

$$\therefore s_p=4\pi\left\{\begin{matrix}\frac{D\times 0}{L}\sqrt{1-k_\infty}\cosh\left[L^{-1}\sqrt{1-k_\infty}\left(\tilde{R}-0\right)\right]\\ +D\sinh\left[L^{-1}\sqrt{1-k_\infty}\left(\tilde{R}-0\right)\right]\end{matrix}\right\}C$$

$$\therefore s_p=4\pi D\sinh\left(L^{-1}\sqrt{1-k_\infty}\tilde{R}\right)C$$

$$\therefore C=\frac{s_p}{4\pi D\sinh\left(L^{-1}\sqrt{1-k_\infty}\tilde{R}\right)}$$

Substituting *C* in (II) gives the flux $\phi(r)$:

$$\boxed{\phi(r)=\frac{\sinh\left[L^{-1}\sqrt{1-k_\infty}\left(\tilde{R}-r\right)\right]}{\sinh\left[L^{-1}\sqrt{1-k_\infty}\tilde{R}\right]}\frac{s_p}{4\pi Dr}}$$

■ P6.16

Part (a): Within the sphere (except at the origin) the flux is given by Eq. (6.101), but with the distributed source set equal to zero:

$$\phi(r)=\frac{C_1}{r}\sin\left(L^{-1}\sqrt{k_\infty-1}r\right)+\frac{C_2}{r}\cos\left(L^{-1}\sqrt{k_\infty-1}r\right)\ \text{(I)}$$

To obtain the two arbitrary constants, we first set the flux to zero at the extrapolated boundary, that is, $\phi(\tilde{R})=0$; thus,

$$0=\frac{C_1}{\tilde{R}}\sin\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)+\frac{C_2}{\tilde{R}}\cos\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)$$

Solving for C_2,

$$C_2=-C_1\tan\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)$$

Substituting in (I),

$$\phi(r)=\frac{C_1}{r}\left[\sin\left(L^{-1}\sqrt{k_\infty-1}r\right)-\tan\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)\cos\left(L^{-1}\sqrt{k_\infty-1}r\right)\right]\ \text{(II)}$$

We can determine C_1 with equation (6.53); we first need the current $J(r)$:

$$J(r)=-D\frac{d}{dr}\phi(r)=D\frac{C_1}{r^2}\left[\sin\left(L^{-1}\sqrt{k_\infty-1}r\right)-\tan\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)\cos\left(L^{-1}\sqrt{k_\infty-1}r\right)\right]$$
$$-\frac{DC_1L^{-1}\sqrt{k_\infty-1}}{r}\left[\cos\left(L^{-1}\sqrt{k_\infty-1}r\right)+\tan\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)\sin\left(L^{-1}\sqrt{k_\infty-1}r\right)\right]$$

But for small *x*, *sin(x)* ≈ *x* and *cos(x)* ≈ 1, giving

$$J(r)=D\frac{C_1}{r^2}\left[L^{-1}\sqrt{k_\infty-1}r-\tan\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)\right]$$
$$-\frac{DC_1L^{-1}\sqrt{k_\infty-1}}{r}\left[1+\tan\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)L^{-1}\sqrt{k_\infty-1}r\right]$$

Substituting into (6.53),

$$s_p=\lim_{r\to 0}4\pi r^2 J(r)=\lim_{r\to 0}4\pi r^2\left\{\begin{matrix}D\frac{C_1}{r^2}\left[L^{-1}\sqrt{k_\infty-1}r-\tan\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)\right]\\ -\frac{DC_1L^{-1}\sqrt{k_\infty-1}}{r}\left[1+\tan\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)L^{-1}\sqrt{k_\infty-1}r\right]\end{matrix}\right\}$$

$$\therefore s_p=\lim_{r\to 0}4\pi\left\{\begin{matrix}DC_1\left[L^{-1}\sqrt{k_\infty-1}r-\tan\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)\right]\\ -DC_1L^{-1}\sqrt{k_\infty-1}\left[r+\tan\left(L^{-1}\sqrt{k_\infty-1}\tilde{R}\right)L^{-1}\sqrt{k_\infty-1}r^2\right]\end{matrix}\right\}$$

$$\therefore s_p = 4\pi \left\{ \begin{array}{c} DC_1\left[L^{-1}\sqrt{k_\infty - 1}\times 0 - \tan\left(L^{-1}\sqrt{k_\infty - 1}\tilde{R}\right)\right] \\ -DC_1L^{-1}\sqrt{k_\infty - 1}\left[0 + \tan\left(L^{-1}\sqrt{k_\infty - 1}\tilde{R}\right)L^{-1}\sqrt{k_\infty - 1}\times 0^2\right] \end{array} \right\}$$

$$\therefore s_p = -4\pi D C_1 \tan\left(L^{-1}\sqrt{k_\infty - 1}\tilde{R}\right)$$

$$\therefore C_1 = -\frac{s_p}{4\pi D \tan\left(L^{-1}\sqrt{k_\infty - 1}\tilde{R}\right)}$$

I've highlighted the tangent term to make the final calculations easier to follow. Substituting C_1 into (II) brings to

$$\phi(r) = \frac{1}{r}\times\left(-\frac{s_p}{4\pi D\tan\left(L^{-1}\sqrt{k_\infty - 1}r\right)}\right)\times\left[\sin\left(L^{-1}\sqrt{k_\infty - 1}r\right) - \tan\left(L^{-1}\sqrt{k_\infty - 1}\tilde{R}\right)\cos\left(L^{-1}\sqrt{k_\infty - 1}r\right)\right]$$

$$\therefore \boxed{\phi(r) = \frac{s_p}{4\pi D r}\left[\cos\left(L^{-1}\sqrt{k_\infty - 1}r\right) - \frac{\sin\left(L^{-1}\sqrt{k_\infty - 1}r\right)}{\tan\left(L^{-1}\sqrt{k_\infty - 1}\tilde{R}\right)}\right]}$$

The equation above gives the flux distribution we were asked to determine.
Part (b): The equation determined in part (a) becomes singular when the denominator of the second term vanishes:

$$\tan\left(L^{-1}\sqrt{k_\infty - 1}\tilde{R}\right) = 0 \quad \rightarrow \quad \frac{\sqrt{k_\infty - 1}\tilde{R}}{L} = \pi$$

$$\therefore \frac{k_\infty}{1 + \left(\pi L/\tilde{R}\right)^2} = 1$$

This result is identical to the criticality condition (6.105).

■ P6.17

We begin by setting $k_\infty = 1$ in equation (6.82), which can be restated as

$$-\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\phi(r)\right] = \frac{s_o'''}{D}$$

This ODE can be integrated directly. Multiplying through by $-r^2 dr$ and carrying out the integration, we have

$$\int d\left[r^2\frac{d}{dr}\phi(r)\right] = -\int\frac{s_o'''}{D}r^2 dr$$

$$\therefore r^2\frac{d}{dr}\phi(r) = -\frac{s_o'''}{D}\frac{r^3}{3} + C_1$$

Multiplying through by dr/r^2 and integrating a second time,

$$\int d\phi(r) = -\frac{s_o'''}{D}\int\frac{r}{3}dr + C_1\int\frac{dr}{r^2}$$

$$\therefore \phi(r) = -\frac{s_o'''}{D}\frac{r^2}{6} - C_1\frac{1}{r} + C_2 \quad \text{(I)}$$

It remains to determine constants C_1 and C_2. Since this is a distributed source, the flux must be finite at the origin, which in turn prohibits us from dividing by zero in the right-hand side of (I); thus, $C_1 = 0$, giving

$$\phi(r) = -\frac{s_o'''}{D}\frac{r^2}{6} + C_2 \quad \text{(II)}$$

Also, the flux must vanish at the extrapolated boundary $\tilde{R}$, so (II) becomes

$$\phi(r = \tilde{R}) = -\frac{s_o'''}{D}\frac{\tilde{R}^2}{6} + C_2 = 0$$

$$\therefore C_2 = \frac{s_o'''}{D}\frac{\tilde{R}^2}{6}$$

Substituting in (I),

$$\phi(r) = -\frac{s_o'''}{D}\frac{r^2}{6} + \frac{s_o'''}{D}\frac{\tilde{R}^2}{6}$$

$$\therefore \boxed{\phi(r) = \frac{s_o'''}{6D}\left(\tilde{R}^2 - r^2\right)}$$

Notice that this happens to be the same flux profile to which equations (6.95) and (6.103) tend to in the limit of $k_\infty \to 1$, as we've established in Problem 6.14.

■ P6.18

Parts (a) to (d): For small x, we have $\sinh(x) \approx x$ and $\sin(x) \approx x$. Accordingly, (6.95) simplifies to

$$\phi(0) = \lim_{r\to 0}\frac{s'''}{(1-k_\infty)\Sigma_a}\left[1 - \frac{\tilde{R}}{r}\frac{\sinh\left(L^{-1}\sqrt{1-k_\infty}\,r\right)}{\sinh\left(L^{-1}\sqrt{1-k_\infty}\,\tilde{R}\right)}\right]$$

$$\therefore \phi(0) \approx \lim_{r\to 0}\frac{s'''}{(1-k_\infty)\Sigma_a}\left[1 - \frac{\tilde{R}}{r}\frac{L^{-1}\sqrt{1-k_\infty}\,r}{\sinh\left(L^{-1}\sqrt{1-k_\infty}\,\tilde{R}\right)}\right]$$

$$\therefore \phi(0) \approx \frac{s'''}{(1-k_\infty)\Sigma_a}\left[1 - \frac{\left(\tilde{R}/L\right)\sqrt{1-k_\infty}}{\sinh\left(\sqrt{1-k_\infty}\,\tilde{R}/L\right)}\right] \quad \text{(I)}$$

In turn, (6.103) simplifies to

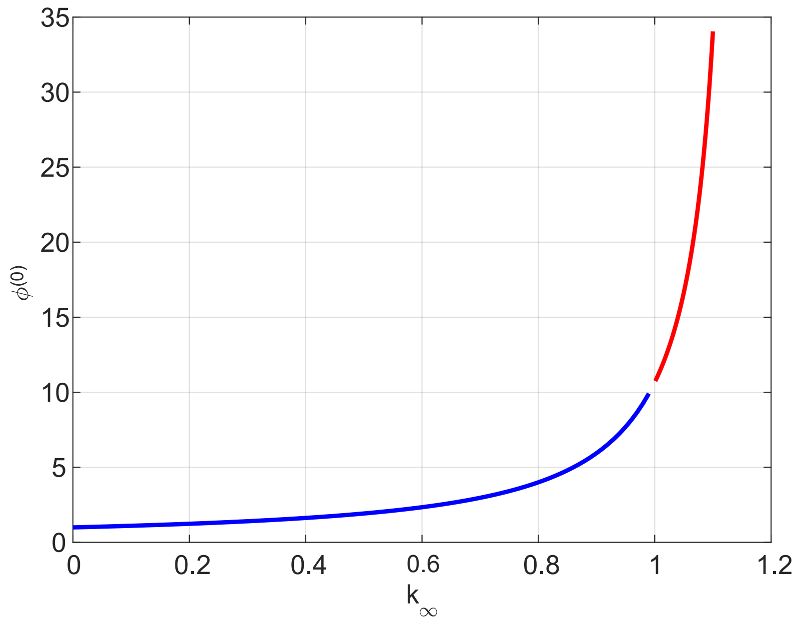
$$\phi(0) = \lim_{r\to 0}\frac{s'''}{(k_\infty-1)\Sigma_a}\left[\frac{\tilde{R}}{r}\frac{\sin\left(L^{-1}\sqrt{1-k_\infty}\,r\right)}{\sin\left(L^{-1}\sqrt{1-k_\infty}\,\tilde{R}\right)} - 1\right]$$

$$\therefore \phi(0) \approx \frac{s'''}{(k_\infty-1)\Sigma_a}\left[\frac{\left(\tilde{R}/L\right)\sqrt{k_\infty-1}}{\sin\left(\sqrt{1-k_\infty}\,\tilde{R}/L\right)} - 1\right] \quad \text{(II)}$$

Equations (I) and (II) can be normalized with respect to $\phi(0) \propto s_0'''/\Sigma_a$. The remaining step is to plot $\phi(0)$ with respect to the specified ranges of values; the pertaining MATLAB code for part (b) is shown next:

```
kinf_low = linspace(0,1,100);
kinf_high = linspace(1,1.1,100);
RL = 8;
phi_low = 1./(1-kinf_low).*(1 - RL.*sqrt(1-kinf_low)./sinh(sqrt(1-kinf_low)*RL));
phi_high = 1./(kinf_high-1).*(RL.*sqrt(kinf_high - 1)./sin(sqrt(kinf_high - 1).*RL) - 1);
plot(kinf_low, phi_low, 'LineWidth', 2, 'Color', 'blue')
hold on
plot(kinf_high, phi_high, 'LineWidth', 2, 'Color', 'red')
hold off
grid on
```

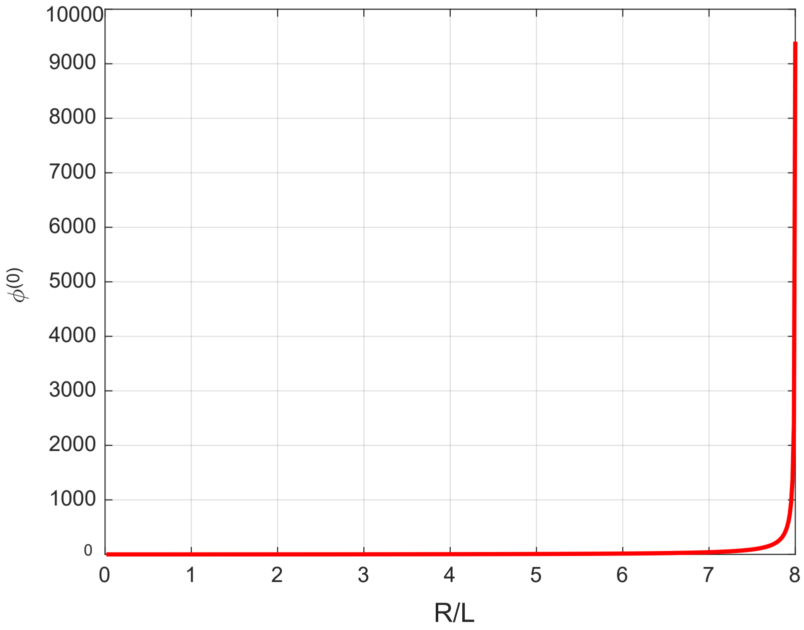
As can be seen, with a fixed $\tilde{R}/L = 8$, the reactor approaches criticality by increasing k_∞, and $\phi(0)$ rises slowly at first but very rapidly for $k_\infty > 1$. As $k_\infty \to 1.154$, the flux tends to infinity. Note that we've truncated the plotting range as (0, 1.1) instead of (0, 1.154) for visualization purposes.



The code for part (c) is shown in continuation:

```
kinf = 1.154;
RL = linspace(0,8,500);
phi = 1./(kinf-1).*(RL.*sqrt(kinf - 1)./sin(sqrt(kinf - 1).*RL) -
1);
plot(RL, phi, 'LineWidth', 2, 'Color', 'red')
grid on
```

As can be seen, increasing the core radius causes the flux to rise correspondingly, especially at $\tilde{R}/L$ greater than 7. The flux tends to infinity as $\tilde{R}/L \to 8$, that is, as criticality is reached.



■ P7.2

Since the core is uniform, minimizing the critical mass is equivalent to minimizing the core volume. Conversely, we can ask: for a given volume, what value of the height-to-diameter ratio yields the minimum buckling, and therefore the minimum leakage? Let χ denote the height-to-diameter ratio, that is,

$$\chi = \frac{H}{D}$$

The cylinder volume can be stated as

$$V = \frac{\pi D^2}{4} H = \frac{\pi D^2}{4} \times D\chi = \frac{1}{4}\chi\pi D^3 \quad \text{(I)}$$

The buckling is given by equation (7.20),

$$B^2 = \left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2 = \left(\frac{4.810}{D}\right)^2 + \left(\frac{\pi}{\chi D}\right)^2$$

$$\therefore B^2 = \frac{1}{D^2}\left[4.810^2 + \left(\frac{\pi}{\chi}\right)^2\right] \quad \text{(II)}$$

Solving (I) for diameter,

$$V = \frac{1}{4}\chi\pi D^3 \quad \to \quad D = \left(\frac{4V}{\pi\chi}\right)^{\frac{1}{3}}$$

Substituting in (II),

$$B^2 = \frac{1}{D^2}\left[4.810^2 + \left(\frac{\pi}{\chi}\right)^2\right] = \left(\frac{\pi\chi}{4V}\right)^{\frac{2}{3}}\left[4.810^2 + \left(\frac{\pi}{\chi}\right)^2\right]$$

$$\therefore B^2 = \left(\frac{\pi}{4V}\right)^{\frac{2}{3}}\left(4.810^2\chi^{2/3} + \pi^2\chi^{-4/3}\right)$$

To minimize the buckling B, we first differentiate the equation above,

In[9]:= $\text{Simplify}\left[\text{D}\left[\left(\frac{\text{Pi}}{4 * \text{V}}\right)^{\frac{2}{3}} * \left(4.810^2 * \chi^{2/3} + \text{Pi}^2 * \chi^{-4/3}\right), \chi\right]\right]$

Out[9]= $\dfrac{13.1298 \left(\frac{1}{\text{V}}\right)^{2/3} \boxed{\left(-0.853178 + 1.\,\chi^2\right)}}{\chi^{7/3}}$

Then, we set the result above to zero, which means that the term highlighted in red must be such that

$$-0.853+\chi^2=0$$

$$\therefore \chi=\sqrt{0.853}=\boxed{0.924}$$

That is, the smallest critical mass for a bare cylindrical reactor is attained when the height-to-diameter ratio equals approximately 0.924.

■ P7.3

Part (a): For a cylinder with a height-to-diameter ratio of one, $2R = D = H$. Referring to equation (7.20), we have

$$B^2=\left(\frac{\pi}{H}\right)^2+\left(\frac{2.405}{R}\right)^2=\left(\frac{\pi}{D}\right)^2+\left(\frac{4.810}{D}\right)^2=\frac{33.01}{D^2}$$

Setting (7.6) to one and replacing B^2 with the result above,

$$k=\frac{k_\infty}{1+M^2B^2}\;\rightarrow\;1.0=\frac{1.36}{1+18^2\times\dfrac{33.01}{D^2}}$$

$$\therefore D=172.4\text{ cm}$$

The critical assembly must have a height and diameter of approximately 1.72 meters.

Part (b): With $H' = H/2 = 172/2 = 86$ cm, the value of k when the two halves are isolated from each other becomes

$$k=\frac{k_\infty}{1+M^2B^2}=\frac{1.36}{1+18^2\times\left[\left(\dfrac{4.810}{172}\right)^2+\left(\dfrac{\pi}{86}\right)^2\right]}=\boxed{0.807}$$

■ P7.6

Part (a): We begin by writing out equation (7.7) in Cartesian coordinates:

$$\nabla^2\phi+B^2\phi=0$$

$$\therefore\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi(x,y,z)+B^2\phi(x,y,z)=0$$

Substituting the given $\phi(x,y,z)$,

$$C\cos\left(\frac{\pi y}{a}\right)\cos\left(\frac{\pi z}{a}\right)\frac{d^2}{dx^2}\left[\cos\left(\frac{\pi x}{a}\right)\right]+C\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{a}\right)\frac{d^2}{dx^2}\left[\cos\left(\frac{\pi y}{a}\right)\right]$$
$$+C\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\frac{d^2}{dz^2}\left[\cos\left(\frac{\pi z}{a}\right)\right]+B^2C\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\cos\left(\frac{\pi z}{a}\right)=0$$

Dividing through by $\phi(x,y,z)$,

$$\frac{\dfrac{d^2}{dx^2}\cos(\pi x/a)}{\cos(\pi x/a)}+\frac{\dfrac{d^2}{dy^2}\cos(\pi y/a)}{\cos(\pi y/a)}+\frac{\dfrac{d^2}{dz^2}\cos(\pi z/a)}{\cos(\pi z/a)}+B^2=0\;\;\text{(I)}$$

Computing the derivatives on the left-hand side,

$$\frac{d^2}{dx^2}\cos(\pi x/a)=-\left(\frac{\pi}{a}\right)\frac{d}{dx}\sin\left(\frac{\pi x}{a}\right)=-\left(\frac{\pi}{a}\right)^2\cos\left(\frac{\pi x}{a}\right)$$

$$\frac{d^2}{dy^2}\cos(\pi y/a)=-\left(\frac{\pi}{a}\right)^2\cos\left(\frac{\pi x}{a}\right)$$

$$\frac{d^2}{dz^2}\cos(\pi z/a)=-\left(\frac{\pi}{a}\right)^2\cos\left(\frac{\pi z}{a}\right)$$

Substituting in (I),

$$\frac{-\left(\frac{\pi}{a}\right)^2\cos(\pi x/a)}{\cos(\pi x/a)}+\frac{-\left(\frac{\pi}{a}\right)^2\cos(\pi y/a)}{\cos(\pi y/a)}+\frac{-\left(\frac{\pi}{a}\right)^2\cos(\pi z/a)}{\cos(\pi z/a)}+B^2=0 \;\; \text{(I)}$$

$$\therefore -\left(\frac{\pi}{a}\right)^2-\left(\frac{\pi}{a}\right)^2-\left(\frac{\pi}{a}\right)^2+B^2=0$$

$$\therefore B^2=3\left(\frac{\pi}{a}\right)^2$$

Accordingly, the flux distribution is correct, since it satisfies the diffusion equation provided the buckling is set to $B^2 = 3(\pi/a)^2$; the flux is positive within the reactor, and meets the boundary conditions because

$$\cos\left(\frac{\pi x}{a}\right)\Bigg|_{x=\pm a/2}=\cos\left(\pm\frac{\pi}{2}\right)=0$$

and similarly for y and z.

Part (b): The reactor power is given by equation (7.22),

$$P=\gamma\int\Sigma_f\phi dV$$

Substituting the given $\phi(x,y,z)$,

$$P=\gamma\Sigma_f\int\phi(x,y,z)dV=\gamma\Sigma_f C\int_{-a/2}^{a/2}\cos\left(\frac{\pi x}{a}\right)dx\int_{-a/2}^{a/2}\cos\left(\frac{\pi y}{a}\right)dy\int_{-a/2}^{a/2}\cos\left(\frac{\pi z}{a}\right)dz$$

The integrals on the right-hand side are such that

$$\int_{-a/2}^{a/2}\cos\left(\frac{\pi x}{a}\right)dx=\left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right)\Bigg|_{-a/2}^{a/2}=\frac{a}{\pi}\times\left[1-(-1)\right]=\frac{2a}{\pi}$$

and similarly for y and z. It follows that

$$P=\gamma\Sigma_f C\times\left(\frac{2a}{\pi}\right)\times\left(\frac{2a}{\pi}\right)\times\left(\frac{2a}{\pi}\right)=\gamma\Sigma_f C\left(\frac{2a}{\pi}\right)^3$$

Since the volume $V = a^3$, we can solve the result above for constant C:

$$P=\gamma\Sigma_f C\left(\frac{2a}{\pi}\right)^3 \;\rightarrow\; C=\frac{P}{\gamma\Sigma_f}\left(\frac{\pi}{2a}\right)^3$$

$$\therefore \boxed{C=\frac{\pi^3}{8}\frac{P}{\gamma\Sigma_f V}}$$

Part (c): We've already determined the buckling in part (a):

$$\boxed{B^2=3\left(\frac{\pi}{a}\right)^2}$$

Part (d): As usual in criticality calculations, we appeal to equation (7.6) with $L = M$:

$$k=\frac{k_\infty}{1+M^2B^2} \;\rightarrow\; k_\infty=k\left(1+M^2B^2\right)$$

$$\therefore k_\infty=1.0\times\left[1+M^2\times3\left(\frac{\pi}{a}\right)^2\right]$$

$$\therefore k_\infty=1.0\times\left[1+20^2\times3\left(\frac{\pi}{200}\right)^2\right]=\boxed{1.296}$$

■ P7.12

The flux distribution is given by (6.107), which when combined with equation (6.104) becomes

$$\phi(r)=\frac{C_1}{r}\sin\left(\frac{\pi r}{R}\right)$$

The maximum flux is at the center of the sphere. For small *r*, *sin*(πr/*R*) ≈ πr/*R*, giving

$$\phi(0)\approx\frac{C_1}{r}\times\frac{\pi r}{R}=\frac{\pi C_1}{R}\quad\text{(I)}$$

The volume-averaged flux is given by

$$\bar{\phi}=\frac{1}{V}\int\phi(r)dV=\frac{1}{4/3\,\pi R^3}\int_0^R\phi(r)4\pi r^2dr$$

$$\therefore\bar{\phi}=\frac{3}{4\pi R^3}\int_0^R\frac{C_1}{r}\sin\left(\frac{\pi r}{R}\right)4\pi r^2dr$$

$$\therefore\bar{\phi}=\frac{3}{4\pi R^3}\int_0^R C_1\sin\left(\frac{\pi r}{R}\right)4\pi rdr$$

$$\therefore\bar{\phi}=\frac{3C_1}{R^3}\int_0^R\sin\left(\frac{\pi r}{R}\right)rdr$$

The integral on the right-hand side can be evaluated with Mathematica:

```
In[18]:= Integrate[Sin[Pi*r/R]*r, {r, 0, R}, Assumptions → R > 0]

Out[18]= R^2/π
```

Therefore,

$$\bar{\phi}=\frac{3C_1}{R^3}\times\frac{R^2}{\pi}=\frac{3C_1}{\pi R}$$

Combining the result above and the result in (I), the maximum-to-average ratio becomes

$$\frac{\phi(0)}{\bar{\phi}}=\frac{\pi C_1/R}{3C_1/\pi R}=\frac{\pi^2}{3}=\boxed{3.29}$$

■ P7.13

The flux distribution within the core is given by equation (6.107) (pp. 161):

$$\phi(r)=\frac{C_1}{r}\sin\left(L^{-1}\sqrt{k_\infty-1}\times r\right)\ \ ;\ \ 0\le r\le R$$

In the reflector, the flux distribution is given by (6.51) (pp. 150):

$$\phi(r)=\frac{C_1}{r}\exp\left(\frac{r}{L}\right)+\frac{C_2}{r}\exp\left(-\frac{r}{L}\right)$$

For the boundary condition $\phi(r\to\infty)=0$ to hold, we must have $C_1=0$, so the relation above simplifies to

$$\phi(r)=\frac{C_2}{r}\exp\left(-\frac{r}{L}\right)\ \ ;\ \ R\le r\le\infty$$

We proceed to impose the interfacial condition (6.42):

$$\phi(r_{0-})=\phi(r_{0+})\ \to\ \left[\frac{C_1}{r}\sin\left(L^{-1}\sqrt{k_\infty-1}\times r\right)\right]\Bigg|_{r=R}=\left[\frac{C_2}{r}\exp\left(-\frac{r}{L}\right)\right]\Bigg|_{r=R}$$

$$\therefore\frac{C_1}{R}\sin\left(L^{-1}\sqrt{k_\infty-1}R\right)=\frac{C_2}{R}\exp\left(-\frac{R}{L}\right)\quad\text{(I)}$$

Similarly, using interface condition (6.43),

$$D(r_{0-})\frac{d}{dr}\phi(r_{0-})=D(r_{0+})\frac{d}{dr}\phi(r_{0+})$$

$$\therefore -D\frac{C_1}{R^2}\sin\left(L^{-1}\sqrt{k_\infty-1}R\right)+D\frac{C_1\sqrt{k_\infty-1}}{RL}\cos\left(L^{-1}\sqrt{k_\infty-1}R\right)$$
$$=-D\frac{C_2}{R^2}\exp\left(-\frac{R}{L}\right)-D\frac{C_2}{RL}\exp\left(-\frac{R}{L}\right)$$

Multiplying this latter result by R^2/D,

$$-C_1\sin\left(L^{-1}\sqrt{k_\infty-1}R\right)+\frac{RC_1\sqrt{k_\infty-1}}{L}\cos\left(L^{-1}\sqrt{k_\infty-1}R\right)$$
$$=-C_2\exp\left(-\frac{R}{L}\right)-\frac{RC_2}{L}\exp\left(-\frac{R}{L}\right)\quad \text{(II)}$$

Then, we divide (II) by (I) and manipulate, giving

$$-\frac{C_1\sin\left(L^{-1}\sqrt{k_\infty-1}R\right)}{\frac{C_1}{R}\sin\left(L^{-1}\sqrt{k_\infty-1}R\right)}+\frac{\frac{RC_1\sqrt{k_\infty-1}}{L}\cos\left(L^{-1}\sqrt{k_\infty-1}R\right)}{\frac{C_1}{R}\sin\left(L^{-1}\sqrt{k_\infty-1}R\right)}$$
$$=\frac{-C_2\exp\left(-\frac{R}{L}\right)}{\frac{C_2}{R}\exp\left(-\frac{R}{L}\right)}-\frac{\frac{RC_2}{L}\exp\left(-\frac{R}{L}\right)}{\frac{C_2}{R}\exp\left(-\frac{R}{L}\right)}$$

$$\therefore -R+\frac{\frac{RC_1\sqrt{k_\infty-1}}{L}\cos\left(L^{-1}\sqrt{k_\infty-1}R\right)}{\frac{C_1}{R}\sin\left(L^{-1}\sqrt{k_\infty-1}R\right)}=-R-\frac{\frac{RC_2}{L}\exp\left(-\frac{R}{L}\right)}{\frac{C_2}{R}\exp\left(-\frac{R}{L}\right)}$$

$$\therefore \frac{\frac{RC_1\sqrt{k_\infty-1}}{L}\cos\left(L^{-1}\sqrt{k_\infty-1}R\right)}{\frac{C_1}{R}\sin\left(L^{-1}\sqrt{k_\infty-1}R\right)}=-\frac{\frac{RC_2}{L}\exp\left(-\frac{R}{L}\right)}{\frac{C_2}{R}\exp\left(-\frac{R}{L}\right)}$$

$$\therefore \frac{\frac{RC_1\sqrt{k_\infty-1}}{L}\cot\left(L^{-1}\sqrt{k_\infty-1}R\right)}{\frac{C_1}{R}}=-\frac{\frac{RC_2}{L}}{\frac{C_2}{R}}$$

$$\therefore \frac{R^2\sqrt{k_\infty-1}}{L}\cot\left(L^{-1}\sqrt{k_\infty-1}R\right)=-\frac{R^2}{L}$$

$$\therefore \sqrt{k_\infty-1}\cot\left(\sqrt{k_\infty-1}\frac{R}{L}\right)=-1$$

$$\therefore \boxed{\sqrt{k_\infty-1}\cot\left(\sqrt{k_\infty-1}\frac{R}{L}\right)+1=0}$$

The result above is the transcendental criticality condition we were asked to obtain.

■ P7.14

Within the internal reflector the flux is given by equation (6.51) but with $C_2 = -C_1$, so that the flux is finite at the origin. Thus, with $C = C_1/2$, we may write

$$\phi(r)=\frac{C_1}{r}\left[\exp\left(\frac{r}{L}\right)-\exp\left(-\frac{r}{L}\right)\right]=\frac{C}{r}\sinh\left(\frac{r}{L}\right)\quad ;\quad 0\le r\le R$$

With the source term set equal to zero, equation (6.102) specifies the flux in the core,

$$\phi(r)=\frac{C_1}{r}\sin\left(\sqrt{k_\infty-1}\frac{r}{L}\right)+\frac{C_2}{r}\cos\left(\sqrt{k_\infty-1}\frac{r}{L}\right)\quad ;\quad R\le r\le 2R$$

To meet the boundary condition $\phi(2R) = 0$, the latter equation yields

$$C_2 = -C_1 \tan\left(\sqrt{k_\infty - 1} \times \frac{2R}{L}\right)$$

so that

$$\phi(r) = \frac{C_1}{r}\left[\sin\left(\sqrt{k_\infty - 1}\frac{r}{L}\right) - \tan\left(\sqrt{k_\infty - 1} \times \frac{2R}{L}\right)\cos\left(\sqrt{k_\infty - 1}\frac{r}{L}\right)\right]$$

or, using trigonometric identities,

$$\phi(r) = \frac{C'}{r}\sin\left[\sqrt{k_\infty - 1}\frac{(2R-r)}{L}\right] \quad ; \quad R \le r \le 2R$$

Next, we apply interface condition (6.42), namely

$$\phi(r_{0-}) = \phi(r_{0+}) \;\rightarrow\; \left[\frac{C}{r}\sinh\left(\frac{r}{L}\right)\right]\Bigg|_{r=R} = \left\{\frac{C'}{r}\sin\left[\sqrt{k_\infty - 1}\frac{(2R-r)}{L}\right]\right\}\Bigg|_{r=R}$$

$$\therefore \frac{C}{R}\sinh\left(\frac{R}{L}\right) = \frac{C'}{R}\sin\left[\sqrt{k_\infty - 1}\frac{(2R-R)}{L}\right]$$

$$\therefore \frac{C}{R}\sinh\left(\frac{R}{L}\right) = \frac{C'}{R}\sin\left(\sqrt{k_\infty - 1}\frac{R}{L}\right) \quad \text{(I)}$$

Similarly, using interface condition (6.43),

$$D(r_{0-})\frac{d}{dr}\phi(r_{0-}) = D(r_{0+})\frac{d}{dr}\phi(r_{0+})$$

$$\therefore -D\frac{C}{R^2}\sinh\left(\frac{R}{L}\right) + D\frac{C}{RL}\cosh\left(\frac{R}{L}\right) = -D\frac{C'}{R^2}\sin\left(\sqrt{k_\infty - 1}\frac{R}{L}\right) - D\frac{C'}{RL}\sqrt{k_\infty - 1}\cos\left(\sqrt{k_\infty - 1}\frac{R}{L}\right) \quad \text{(II)}$$

Dividing (II) by (I) brings to

$$\frac{-D\frac{C}{R^2}\sinh\left(\frac{R}{L}\right) + D\frac{C}{RL}\cosh\left(\frac{R}{L}\right)}{\frac{C}{R}\sinh\left(\frac{R}{L}\right)} = \frac{-D\frac{C'}{R^2}\sin\left(\sqrt{k_\infty - 1}\frac{R}{L}\right) - D\frac{C'}{RL}\sqrt{k_\infty - 1}\cos\left(\sqrt{k_\infty - 1}\frac{R}{L}\right)}{\frac{C'}{R}\sin\left(\sqrt{k_\infty - 1}\frac{R}{L}\right)}$$

$$\therefore \frac{-D\frac{1}{R}\sinh\left(\frac{R}{L}\right) + D\frac{1}{L}\cosh\left(\frac{R}{L}\right)}{\sinh\left(\frac{R}{L}\right)} = \frac{-D\frac{1}{R}\sin\left(\sqrt{k_\infty - 1}\frac{R}{L}\right) - D\frac{1}{L}\sqrt{k_\infty - 1}\cos\left(\sqrt{k_\infty - 1}\frac{R}{L}\right)}{\sin\left(\sqrt{k_\infty - 1}\frac{R}{L}\right)}$$

$$\therefore -\frac{D}{R} + \frac{D}{L}\coth\left(\frac{R}{L}\right) = -\frac{D}{R} - \frac{D}{L}\sqrt{k_\infty - 1}\cot\left(\sqrt{k_\infty - 1}\frac{R}{L}\right)$$

$$\therefore \boxed{\coth\left(\frac{R}{L}\right) = -\sqrt{k_\infty - 1}\cot\left(\sqrt{k_\infty - 1}\frac{R}{L}\right)}$$

This is the criticality condition.

■ P7.16

In the core region we write equation (7.7) for a slab geometry with $B^2 = (k_\infty - 1)/L^2$ and $L^2 = D/\Sigma_a$:

$$\frac{d^2\phi(x)}{dx^2} + B^2\phi(x) = 0$$

The solution in this case is

$$\phi(x) = C_1'\sin(Bx) + C_2'\cos(Bx) \quad ; \quad 0 \le x \le a$$

In the reflector region the solution is given by (6.20):

$$\phi(x) = C_1\exp\left(\frac{x}{L}\right) + C_2\exp\left(-\frac{x}{L}\right) \quad ; \quad a \le x \le 2a$$

We next apply boundary and interface conditions. On the left we have $\phi(0) = 0$, giving

$$\phi(0) = \underbrace{C_1'\sin(B\times 0)}_{=0} + C_2'\cos(B\times 0) = 0$$

$$\therefore C_2' = 0$$

so that

$$\phi(0) = C_1'\sin(Bx) \;\; ; \;\; 0 \le x \le a$$

On the right, we impose $\phi(2a) = 0$, so that $C_2 = -C_1 exp(4a/L)$ and

$$\phi(x) = C_1\left[\exp\left(\frac{x}{L}\right) - \exp\left(\frac{4a}{L}\right)\exp\left(-\frac{x}{L}\right)\right]$$

This latter result can be expressed in terms of hyperbolic functions:

$$\phi(x) = C\sinh\left[\frac{(2a-x)}{L}\right] \;\; ; \;\; a \le x \le 2a$$

Next, we appeal to interface condition (6.42),

$$\phi(x_{0-} = a) = \phi(x_{0+} = a) \;\; \to \;\; C_1'\sin(B\times a) = C\sinh\left(\frac{2a-a}{L}\right)$$

$$\therefore C_1'\sin(Ba) = C\sinh\left(\frac{a}{L}\right) \quad \text{(I)}$$

Likewise, using interface condition (6.43),

$$D(x_{0-})\frac{d}{dr}\phi(x_{0-}) = D(x_{0+})\frac{d}{dr}\phi(x_{0+})$$

$$\therefore DC_1'B\cos(Ba) = -DCL^{-1}\cosh\left(\frac{a}{L}\right) \quad \text{(II)}$$

Dividing (I) by (II),

$$\frac{\cancel{C'}_1\sin(Ba)}{\cancel{D}\cancel{C'}_1B\cos(Ba)} = \frac{\cancel{C}\sinh(a/L)}{-\cancel{D}\cancel{C}L^{-1}\cosh(a/L)}$$

$$\therefore B^{-1}\tan(Ba) = -L\tanh\left(\frac{a}{L}\right)$$

Replacing B gives the final criticality condition:

$$\frac{L}{\sqrt{k_\infty - 1}}\tan\left(\frac{\sqrt{k_\infty - 1}}{L}a\right) = -L\tanh\left(\frac{a}{L}\right)$$

$$\therefore \boxed{\tan\left(\sqrt{k_\infty - 1}\frac{a}{L}\right) = -\sqrt{k_\infty - 1}\tanh\left(\frac{a}{L}\right)}$$

■ P8.1

Part (a): Let V and V' be the old and new volumes, respectively, and assume a height-to-diameter ratio $\chi = H/D$; we proceed to write

$$V = \frac{1}{4}\pi\chi D^3 \;\; ; \;\; V' = \frac{1}{4}\pi\chi D'^3$$

so that

$$\frac{D'}{D} = \left(\frac{V'}{V}\right)^{\frac{1}{3}} = (1.2)^{\frac{1}{3}} = 1.0627$$

Using the equation for buckling of a cylindrical core,

$$B^2 = \left(\frac{2.405}{D/2}\right)^2 + \left(\frac{\pi}{\chi D}\right)^2$$

$$\therefore B^2 = \frac{1}{D^2}\times\underbrace{\left[(2\times 2.405)^2 + \left(\frac{\pi}{\chi}\right)^2\right]}_{=C}$$

$$\therefore B^2 = \frac{C}{D^2}$$

The ratio of bucklings is

$$\left(\frac{B'}{B}\right)^2 = \frac{C/D'^2}{C/D^2} = \left(\frac{D}{D'}\right)^2 = \left(\frac{1}{1.0627}\right)^2 = 0.8855$$

The nonleakage probability of the initial reactor is $P_{NL} = 1 - P_L = 1 - 0.065 = 0.935$. Recalling that buckling and nonleakage probability are related by equation (7.8) (pp. 169), we may write

$$P_{NL} = \frac{1}{1+L^2B^2} \rightarrow B^2 = L^{-2}\left(\frac{1}{P_{NL}} - 1\right)$$

$$\therefore \left(\frac{B'}{B}\right)^2 = \frac{\cancel{L^{2}}\left(1/P'_{NL} - 1\right)}{\cancel{L^{2}}\left(1/P_{NL} - 1\right)} = \frac{\left(1/P'_{NL} - 1\right)}{\left(1/P_{NL} - 1\right)}$$

$$\therefore P'_{NL} = \left[1 + \left(\frac{1}{P_{NL}} - 1\right)\left(\frac{B'}{B}\right)^2\right]^{-1}$$

$$\therefore P'_{NL} = \left[1 + \left(\frac{1}{0.935} - 1\right) \times 0.8855\right]^{-1} = 0.94201$$

Finally, the leakage probability of the updated reactor is

$$P'_L = 1 - P'_{NL} = 1 - 0.9420 = \boxed{0.058}$$

Part (b): Noting that $k'_{\infty}P'_{NL} = k_{\infty}P_{NL} = 1$, the change in enrichment needed to accommodate the specified increase in power is calculated as

$$\frac{k'_{\infty} - k_{\infty}}{k_{\infty}} = \frac{k'_{\infty}}{k_{\infty}} - 1 = \frac{0.9350}{0.94201} - 1 = -0.00744 = \boxed{-0.744\%}$$

■ P8.2

Parts (a), (b) and (c): For a bare uniform core, it follows from (8.22) (pp. 204) that $P'''_{\text{max}} = 3.63\bar{P}''' = 3.63P/V$. Then, noting that 500 W/cm^3 = 500 MW/m^3,

$$P'''_{\text{max}} = \frac{3.63P}{V} \rightarrow V = \frac{3.63P}{P'''_{\text{max}}}$$

$$\therefore V = \frac{3.63 \times P}{500} = 0.00726P$$

But for a cylinder with a height-to-diameter ratio of one, volume $V = \pi H^3/4$, or

$$V = \frac{\pi H^3}{4} \rightarrow H = \left(\frac{4V}{\pi}\right)^{\frac{1}{3}}$$

$$\therefore H = \left(\frac{4 \times 0.00726P}{\pi}\right)^{\frac{1}{3}} = 0.2099P^{\frac{1}{3}} \quad \text{(I)}$$

This equation indicates that the core height is proportional to power$^{1/3}$. Next, the core buckling is given by (7.20) (pp. 172), which for a core with height-to-diameter ratio of one becomes

$$B^2 = \left(\frac{2 \times 2.405}{D}\right)^2 + \left(\frac{\pi}{H}\right)^2 = \left(\frac{2 \times 2.405}{H}\right)^2 + \left(\frac{\pi}{H}\right)^2$$

$$\therefore B^2 = 33.01H^{-2}$$

Replacing H with result (I),

$$\therefore B^2 = 33.01H^{-2} = 33.01 \times \left(0.2099P^{\frac{1}{3}}\right)^{-2}$$

$$\therefore B^2 = 749.2P^{-\frac{2}{3}} \quad \text{(II)}$$

This second result indicates that buckling is inversely proportional to power$^{2/3}$. To find the nonleakage probability, we appeal to equation (7.8) (pp. 169),

$$P_{NL} = \frac{1}{1+M^2B^2} = \frac{1}{1+0.2^2 \times \left(749.2P^{-\frac{2}{3}}\right)^2}$$

$$\therefore P_{NL} = \frac{1}{1+0.2^2 \times 749.2P^{-\frac{2}{3}}} = \frac{1}{1+29.97P^{-2/3}} \quad \text{(III)}$$

The calculations for all three reactor power ratings are summarized below.

Power (MW)	300	1000	3000
Height (eq. (I)) (m)	1.405	2.099	3.027
Buckling (eq. (II)) (m^-2)	16.718	7.492	3.602
PNL (eq. (III))	0.599	0.769	0.874

■ P8.6

Part (a): The peak-to-average power density in a uniform cylindrical reactor is 3.63; we can use this to establish the reactor volume *V*:

$$P'''_{\max} = 3.63\overline{P}''' = \frac{3.63P}{V} \rightarrow V = \frac{3.63P}{P'''_{\max}}$$

$$\therefore V = \frac{3.63 \times 3000}{250} = \boxed{43.6 \text{ m}^3}$$

With the height-to-diameter ratio set to one, the core size becomes

$$V = \pi\frac{D^2}{4}H = \pi\frac{H^3}{4}$$

$$\therefore H = \left(\frac{4V}{\pi}\right)^{\frac{1}{3}}$$

$$\therefore H = \left(\frac{4 \times 43.6}{\pi}\right)^{\frac{1}{3}} = \boxed{3.82 \text{ m}}$$

Part (b): For the cladding heat flux, $q''_{\max} = 3.63\overline{q}''$, so that

$$q''_{\max} = 3.63\overline{q}'' \rightarrow \overline{q}'' = \frac{q''_{\max}}{3.63}$$

$$\therefore \overline{q}'' = \frac{125}{3.63} = 34.4 \text{ W/cm}^2$$

Let *N* be the number of rods and *a* be their radius; the total heat transfer surface is $2\pi aHN$, and the reactor power can be expressed as

$$P = 2\pi aHN\overline{q}'' = 3000 \times 10^6 \text{ W}$$

Solving for *aN*,

$$P = 2\pi aHN\overline{q}'' = 3 \times 10^9 \rightarrow aN = \frac{3 \times 10^9}{2\pi H\overline{q}''}$$

$$\therefore aN = \frac{3 \times 10^9}{2\pi \times 382 \times 34.4} = 36{,}330 \text{ cm}^2$$

However, if *p* is the lattice pitch, then

$$Np^2 = \frac{\pi D^2}{4} = \frac{\pi \times 382^2}{4} = 114{,}600 \text{ cm}^2 \quad \text{(I)}$$

Dividing through by *aN*,

$$\frac{Np^2}{aN} = \frac{p^2}{a} = \frac{114{,}600}{36{,}330} = 3.154 \text{ cm}$$

Also, given that the moderator to fuel volume ratio is set at 2.0,

$$\frac{p^2-\pi a^2}{\pi a^2}=\frac{1}{\pi}\left(\frac{p}{a}\right)^2-1=2.0$$

$$\therefore \frac{1}{\pi}\left(\frac{p}{a}\right)^2=3.0$$

$$\therefore \left(\frac{p}{a}\right)^2=3.0\pi$$

$$\therefore \frac{p}{a}=\sqrt{3.0\pi}=3.070$$

The lattice pitch p then becomes

$$\frac{p^2}{a}=p\left(\frac{p}{a}\right) \rightarrow 3.154=p\times 3.070$$

$$\therefore p=\frac{3.154}{3.070}=\boxed{1.027\text{ cm}}$$

The fuel element diameter is

$$d=2a=2\left(\frac{a}{p}\right)p=2\times\frac{1}{3.070}\times 1.027=0.669\text{ cm}=\boxed{6.69\text{ mm}}$$

Part (c): To find the approximate number of fuel elements, we solve equation (I) in part (b) for N:

$$Np^2=114{,}600 \rightarrow N=\frac{114{,}600}{p^2}$$

$$\therefore N=\frac{114{,}600}{1.027^2}\approx\boxed{108{,}700}$$

Part (d): Note that $F_r = 2.32$ for a uniform cylindrical core. Solving (8.42) (pp. 208) for $\dot{W}$, we get

$$T_{0,\max}=\frac{PF_r}{\dot{W}c_p}+T_r \rightarrow \dot{W}=\frac{F_rP}{c_p\left(T_{0,\max}-T_i\right)}$$

The specific heat of water at the specified temperature can be taken as 6.4 kJ/kg·°C (see page 210), so that

$$\dot{W}=\frac{2.32\times\left(3.0\times 10^9\right)}{6400\times(330-290)}=\boxed{27{,}200\text{ kg/s}}$$

Taking $\rho = 0.676$ kg/m^3 as the density of pressurized water at the specified temperature (page 210), the average coolant velocity becomes

$$\bar{v}=\frac{W}{\rho N\left(p^2-\pi a^2\right)}=\frac{27{,}200}{\left(0.676\times 10^{-3}\right)\times 108{,}700\times\left[1.027^2-\pi\times(0.669/2)^2\right]}=526\text{ cm/s}$$

$$\therefore \boxed{\bar{v}=5.26\text{ m/s}}$$

■ P8.10

With the new peaking factors $F_r = 1.30$ and $F_z = 1.46$, all we have to do is follow the steps outlined in Section 8.3. For the core volume we evaluate Eq. (8.26) at the point of maximum linear heat rate to obtain

$$V=\frac{A_{\text{cell}}PF_rF_z}{q'_{\max}}=\frac{p^2PF_rF}{q'_{\max}}=\frac{1.536^2\times\left(3.0\times 10^9\right)\times 1.30\times 1.46}{400}=3.36\times 10^7\text{ cm}^3$$

$$\therefore V=33.6\text{ m}^3$$

For a height-to-diameter ratio of one, the core height H becomes

$$V=\pi\left(\frac{H}{2}\right)^2H \rightarrow H=\left(\frac{4V}{\pi}\right)^{\frac{1}{3}}$$

$$\therefore H = \left(\frac{4\times 33.6}{\pi}\right)^{\frac{1}{3}} = 3.50\text{ m}$$

The core-averaged power density is determined as

$$\bar{P}''' = \frac{P}{V} = \frac{3000}{33.6} = 89.3\text{ MW/m}^3$$

To find the number of fuel elements, we divide the core cross-sectional area by the area of a lattice cell:

$$N = \frac{\pi R^2}{A_{\text{cell}}} = \frac{\pi (H/2)^2}{p^2} = \frac{\pi\times(350/2)^2}{1.536^2} = 40,780$$

The mass flow rate is obtained by solving (8.42) for $\dot{W}$:

$$\dot{W} = \frac{1}{c_p}\frac{PF_r}{(T_{0,\max} - T_i)} = \frac{1}{6400}\times\frac{(3000\times 10^6)\times 1.30}{330-290} = 15,230\text{ kg/s}$$

$$\therefore W = 1.52\times 10^7\text{ g/s}$$

Taking 0.676 g/cm^3 as the density of pressurized water at 300ºC, the mean coolant velocity is

$$\bar{v} = \frac{W}{\rho N(p^2 - \pi a^2)} = \frac{1.52\times 10^7}{0.676\times 40,780\times(1.536^2 - \pi\times 0.509^2)} = 357\text{ cm/s}$$

$$\therefore \bar{v} = 3.57\text{ m/s}$$

Note that using the same lattice but with the lowered peaking factors, the core size dropped from 4.34 m to 3.50 m, the number of fuel pins dropped from 62,702 to 40,780, the mass flow rate dropped from 27,200 kg/s to 15,200 kg/s, and the average flow speed dropped slightly from 4.15 m/s to 3.57 m/s.

■ P8.11

The calculations are identical to those of Section 3 and Problem 8.10. Firstly, the reactor volume is

$$V = \frac{A_{\text{cell}} P F_r F_z}{q'_{\max}} = \frac{p^2 P F_r F}{q'_{\max}} = \frac{1.536^2\times(3.0\times 10^9)\times 1.00\times 1.00}{400} = 1.77\times 10^7\text{ cm}^3$$

$$\therefore V = 17.7\text{ m}^3$$

The core height H is

$$H = \left(\frac{4V}{\pi}\right)^{\frac{1}{3}} = \left[\frac{4\times(17.7\times 10^6)}{\pi}\right]^{\frac{1}{3}} = 283\text{ cm} = 2.83\text{ m}$$

The mass flow rate is obtained by solving (8.42) for $\dot{W}$:

$$\dot{W} = \frac{1}{c_p}\frac{PF_r}{(T_{0,\max} - T_i)} = \frac{1}{6400}\times\frac{(3000\times 10^6)\times 1.00}{330-290} = 11,720\text{ kg/s}$$

$$\therefore W = 1.52\times 10^7\text{ g/s}$$

The core-averaged power density is

$$\bar{P}''' = \frac{P}{V} = \frac{3000}{17.7} = 170\text{ MW/m}^3$$

The number of fuel elements is

$$N = \frac{\pi R^2}{A_{\text{cell}}} = \frac{\pi (H/2)^2}{p^2} = \frac{\pi\times(283/2)^2}{1.536^2} = 26,660$$

Lastly, the mean coolant velocity is

$$\bar{v} = \frac{W}{\rho N(p^2 - \pi a^2)} = \frac{1.17\times 10^7}{0.676\times 26,660\times(1.536^2 - \pi\times 0.509^2)} = 421\text{ cm/s}$$

$$\therefore \bar{v} = 4.21\text{ m/s}$$

Note that using the same lattice but with the peaking factors set to ideal values of 1.0, the core size dropped from 4.34 m to 2.83 m, the number of fuel pins dropped from 62,702 to 26,660, the mass flow rate dropped from 27,200 kg/s to 11,720 kg/s, and the average flow speed increased slightly from 4.15 m/s to 4.21 m/s.

■ P8.12

We first restate (8.42) as

$$F_r = \frac{Wc_p\left(T_{o,\max} - T_i\right)}{P} = \frac{\bar{v}\rho A_{\text{flow}} c_p\left(T_{o,\max} - T_i\right)}{P}$$

Denoting the modified values with primes, the ratio of peaking factors F_r'/F_r is calculated to be

$$\frac{F_r'}{F_r} = \frac{\bar{v}'\cancel{\rho A_{\text{flow}} c_p}\left(T_{o,\max}' - T_i\right)/\cancel{P}}{\bar{v}\cancel{\rho A_{\text{flow}} c_p}\left(T_{o,\max} - T_i\right)/\cancel{P}} = \frac{\bar{v}'}{\bar{v}}\frac{\left(T_{o,\max}' - T_i\right)}{\left(T_{o,\max} - T_i\right)}$$

$$\therefore \frac{F_r'}{F_r} = 0.9 \times \frac{(325-290)}{(330-290)} = 0.788$$

In order to accommodate the specified changes, the radial peaking factor must be reduced by 21.2%.

■ P8.13

Letting $\theta(t) = T_f(t) - T_c(0)$ since T_c remains constant, we first restate equation (8.49) as

$$\frac{d}{dt}\theta(t) = \frac{1}{M_f c_f}P_o \exp\left(\frac{t}{T}\right) - \frac{1}{\tau}\theta(t)$$

Rearranging and multiplying through by $\exp(t/\tau)$,

$$\frac{d}{dt}\left[\theta(t) + \frac{1}{\tau}\theta(t)\right]\exp\left(\frac{t}{\tau}\right) = \frac{1}{M_f c_f}P_o \exp\left(\frac{t}{T}\right)\exp\left(\frac{t}{\tau}\right)$$

or, equivalently,

$$\frac{d}{dt}\left[\theta(t)\exp\left(\frac{t}{\tau}\right)\right] = \frac{1}{M_f c_f}P_o \exp\left[\left(\frac{1}{T} + \frac{1}{\tau}\right)t\right]$$

Integrating between 0 and t:

$$\theta(t)\exp\left(\frac{t}{\tau}\right) - \theta(0) = \frac{1}{M_f c_f}P_o\int_0^t \exp\left[\left(\frac{1}{T} - \frac{1}{\tau}\right)t'\right]dt'$$

$$\therefore \theta(t)\exp\left(\frac{t}{\tau}\right) - \theta(0) = \frac{1}{M_f c_f}P_o\left\{\frac{\exp\left[\left(1/T + 1/\tau\right)t'\right] - 1}{1/T + 1/\tau}\right\}$$

Solving for $\theta(t)$,

$$\theta(t) = \theta(0)\exp\left(-\frac{t}{\tau}\right) + \frac{1}{M_f c_f}P_o\left[\frac{\exp(t/\tau) - \exp(-t/\tau)}{(1/T + 1/\tau)}\right]$$

To find the initial value $\theta(0)$, we refer to equation (8.32):

$$\theta(0) = T_f(0) - T_c(0) = R_f P_o = \frac{\tau}{M_f c_f}P_o$$

where we have used $\tau = M_f c_f R_f$ in the lattermost passage. Then, noting that $\theta(t) = T_f(0) - T_c(0)$, we obtain

$$T_f(t) = T_c(0) + \frac{\tau}{M_f c_f}P_o \exp\left(-\frac{t}{\tau}\right) + \frac{1}{M_f c_f}P_o\left[\frac{\exp(t/T) + \exp(-t/\tau)}{(1/T + 1/\tau)}\right]$$

Eliminating $M_f c_f$ yields

$$T_f(t) = T_c(0) + \frac{\tau}{\frac{\tau}{R_f}}P_o \exp\left(-\frac{t}{\tau}\right) + \frac{1}{\frac{\tau}{R_f}}P_o\left[\frac{\exp(t/T) + \exp(-t/\tau)}{(1/T + 1/\tau)}\right]$$

Performing some algebraic manipulation,

$$T_f(t) = T_c(0) + \frac{P_o R_f}{1+\tau/T}\left[\exp(t/T) + (t/T)\exp(-t/\tau)\right]$$

which is the relationship we were asked to demonstrate.

■ P9.4

Part (a): We first take the average coolant temperature

$$\overline{T}_c = \frac{1}{2}\left(T_i - \overline{T}_o\right) = \frac{1}{2}\times(350+500) = 425^{\circ}\mathrm{C}$$

The thermal resistance is then

$$R_f = \frac{\overline{T}_f - \overline{T}_c}{P} = \frac{1150-425}{1000} = \boxed{0.725^{\circ}\,\mathrm{C/MW}}$$

The mass flow rate is, in turn,

$$\dot{W} = \frac{P}{c_p\left(\overline{T}_o - T_i\right)} = \frac{1000\times10^6}{1250\times(500-350)} = \boxed{5333\ \mathrm{kg/s}}$$

Part (b): The temperature defect is given by the integral (9.34), which in this case simplifies to

$$D_T = \left(\alpha_f + \alpha_c\right)\left(T_i - T_r\right) = \left[\left(-1.8\times10^{-5}\right) + \left(0.45\times10^{-5}\right)\right]\times(350-180)$$

$$\therefore \boxed{D_T = -0.00230}$$

For the power defect, we write

$$D_P = \left[R_f\alpha_f + \left(2Wc_p\right)^{-1}\left(\alpha_f + \alpha_c\right)\right]P$$

$$\therefore D_P = \left\{0.725\times\left(-1.8\times10^{-5}\right) + \left[2\times5333\times\left(1250\times10^{-6}\right)\right]^{-1}\times\left[\left(-1.8\times10^{-5}\right)+\left(0.45\times10^{-5}\right)\right]\right\}\times1000$$

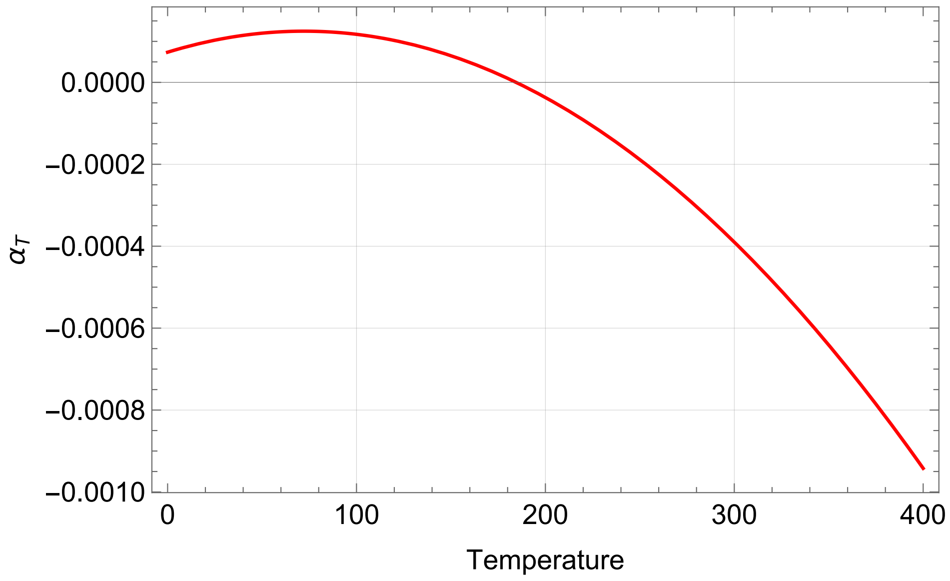
$$\therefore \boxed{D_P = -0.0141}$$

■ P9.5

Part (a): The core will be overmoderated inasmuch as the reactor's temperature coefficient is positive. For the problem at hand,

$$\alpha_T = \alpha_f + \alpha_m = -\frac{7.2\times10^{-4}}{\sqrt{273+T}} + \left(30 + 1.5T - 0.010T^2\right)\times10^{-6}$$

The easiest way to establish the range of temperatures for which α_T is positive is to plot the relationship above. As shown below, the temperature coefficient is positive (and hence, the core is overmoderated) in the range $0 \le T \le 170^{\circ}$C.



Part (b): To find the temperature defect, all we have to do is integrate α_T over the range $T \in (35, 290)$:

$$D_T = -\int_{35}^{290}\frac{7.2\times10^{-4}}{\sqrt{273+T}}dT + \int_{35}^{290}\left(30+1.5T-0.010T^2\right)\times10^{-6}\,dT$$

The integration can be carried out with the following MATLAB code:

```
alpha_f = @(T) -720./sqrt(273 + T);
alpha_m = @(T) (30 + 1.5.*T - 0.01.*T.^2);
DT_f = integral(alpha_f, 35, 290);
DT_m = integral(alpha_m, 35, 290);
DT = (DT_f + DT_m).*10.^-6

DT =

   -0.0202
```

That is, $D_T = -0.0202$.

Part (c): The power defect is given by equation (9.35), namely

$$D_P = \int_{T_i}^{\overline{T}_f(p)} \alpha_f\left(\overline{T}_f\right) d\overline{T}_f + \int_{T_i}^{\overline{T}_c(p)} \alpha_e\left(\overline{T}_c\right) d\overline{T}_c$$

Here, $T_i = 290$°C, $R_f = 0.45$°C/MW(t), $W = 68\times10^6$ kg/h = 18,890 kg/s, and $c_p = 6400$ J/kg°C, so that, using the formulas given in page 230,

$$\overline{T}_c = \frac{1}{2Wc_p}P + T_i = \frac{1}{2\times 18{,}890\times 6400}\times\left(3000\times10^6\right) + 290 = 302.4^\circ\mathrm{C}$$

$$\overline{T}_f = R_f P + \overline{T}_c = 0.45\times 3000 + 302.4 = 1652.4^\circ\mathrm{C}$$

Accordingly,

$$D_P = \left[-\int_{290}^{1652.4} \frac{720}{\sqrt{273+\overline{T}_f}} d\overline{T}_f + \int_{290}^{302.4}\left(30 + 1.5\overline{T}_c - 0.01\overline{T}_c^2\right) d\overline{T}_c\right]\times 10^{-6}$$

These calculations, including the two integrals on the right-hand side, can be performed with the following MATLAB code:

```
alpha_f = @(T) -720./sqrt(273 + T);
alpha_m = @(T) (30 + 1.5.*T - 0.01.*T.^2);
DP_f = integral(alpha_f, 290, 1652.4);
DP_m = integral(alpha_m, 290, 302.4);
DP = (DP_f + DP_m).*10.^-6

DP =

   -0.0340
```

That is, $D_T = -0.0340$.

■ P10.1

The xenon concentration following shutdown is given by equation (10.19):

$$X(t) = \Sigma_f \phi \left[\frac{\left(\gamma_I + \gamma_X\right)}{\lambda_X + \sigma_{aX}\phi} e^{-\lambda_X t} + \frac{\gamma_I}{\lambda_I - \lambda_X}\left(e^{-\lambda_X t} - e^{\lambda_I t}\right)\right]$$

To find the maximum, we differentiate the relation above and set the result to zero, giving

$$\frac{dX}{dt} = \Sigma_f \phi \left[\frac{\left(\gamma_I + \gamma_X\right)\lambda_X}{\lambda_X + \sigma_{aX}\phi} e^{-\lambda_X t} + \frac{\gamma_I}{\lambda_I - \lambda_X}\left(-\lambda_X e^{-\lambda_X t} + \lambda_I e^{\lambda_I t}\right)\right] = 0$$

For a very large flux ϕ, the first term in brackets tends to zero, and optimizing the relation above boils down to

$$\left(-\lambda_X e^{-\lambda_X t} + \lambda_I e^{-\lambda_I t}\right) = 0$$

$$\therefore \lambda_X e^{-\lambda_X t} = \lambda_I e^{-\lambda_I t}$$

$$\therefore \ln\lambda_X + \ln e^{-\lambda_X t} = \ln\lambda_I + \ln e^{-\lambda_I t}$$

$$\therefore \ln\lambda_X - \lambda_X t = \ln\lambda_I - \lambda_I t$$

$$\therefore t = \frac{1}{\lambda_I - \lambda_X}\ln\left(\lambda_I/\lambda_X\right) \quad \text{(I)}$$

Using the half-life values given in decay series (10.11), we have

$$\lambda_I = \frac{0.693}{t_{1/2,I}} = \frac{0.693}{6.7} = 0.1034\ \text{h}^{-1}$$

$$\lambda_X = \frac{0.693}{t_{1/2,X}} = \frac{0.693}{9.2} = 0.0753\ \text{h}^{-1}$$

Finally, we substitute into (I) to obtain

$$\therefore t = \frac{1}{0.1034 - 0.0753}\ln\left(0.1034/0.0753\right) = 11.286 \approx \boxed{11.3\ \text{h}}$$

■ P10.3

To find the ratio, we appeal to equation (10.15) (page 248),

$$X = \frac{\left(\gamma_I + \gamma_X\right)}{\lambda_X + \sigma_{aX}\phi}\Sigma_f\phi = \frac{\left(\gamma_I + \gamma_X\right)}{\lambda_X + \sigma_{aX}\phi}\sigma_f^{25}N^{25}\phi$$

$$\therefore \frac{X}{N^{25}} = \frac{\left(\gamma_I + \gamma_X\right)}{\lambda_X + \sigma_{aX}\phi}\sigma_f^{25}\phi \quad \text{(I)}$$

Using half-life data from (10.11), $\lambda_X = 0.693/t_{1/2,X} = 0.693/9.2 = 0.0753\ \text{h}^{-1} = 2.09\times10^{-5}\ \text{s}^{-1}$. Also, the absorption cross-section of xenon-135 is 2.65×10^{6} b, as given in page 247. Referring to Table 10.1, we read $\gamma_I = 0.0639$ and $\gamma_X = 0.00237$. Finally, $\sigma_f^{25} = 505$ b is taken from Table 3.2. Substituting in (I) brings to

$$\frac{X}{N^{25}} = \frac{\left(0.0639 + 0.00237\right)}{2.09\times10^{-5} + \left(2.65\times10^{-18}\right)\times\phi}\times\left(505\times10^{-24}\right)\phi$$

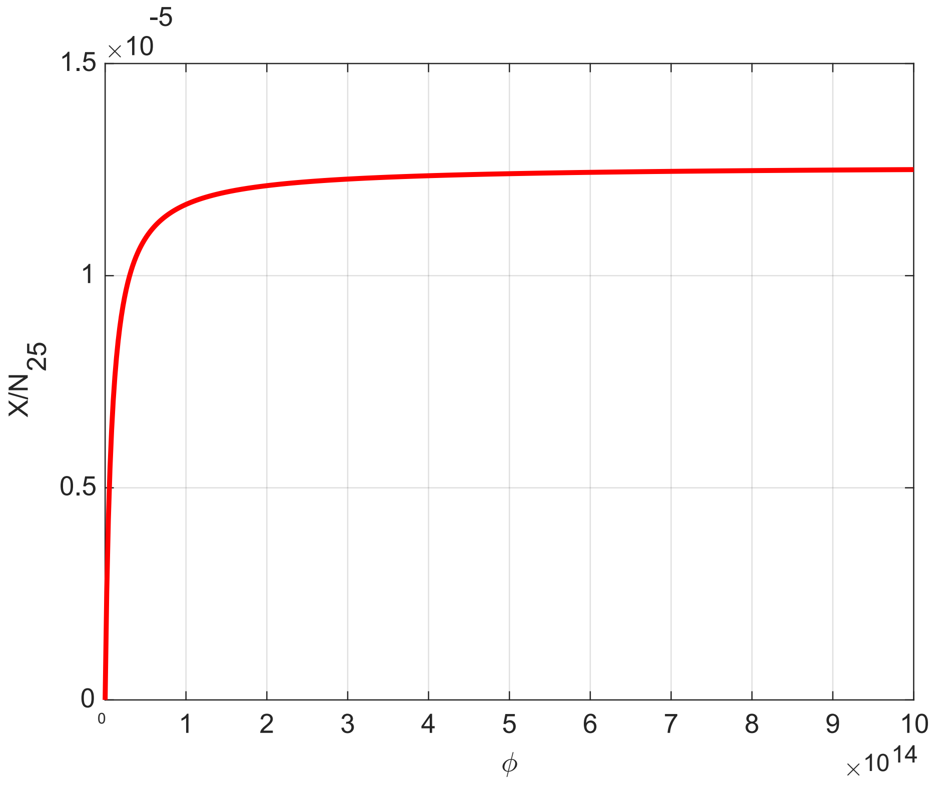
$$\therefore \frac{X}{N^{25}} = \frac{1}{2.09\times10^{-5} + \left(2.65\times10^{-18}\right)\times\phi}\times\left(3.35\times10^{-23}\right)\phi$$

$$\therefore \frac{X}{N^{25}} = \frac{1}{2.09\times10^{-5}\left[1 + \frac{\left(2.65\times10^{-18}\right)}{2.09\times10^{-5}}\times\phi\right]}\times\left(3.35\times10^{-23}\right)\phi$$

$$\therefore \frac{X}{N^{25}} = \frac{1}{1 + 1.27\times10^{-13}\phi}\times1.60\times10^{-18}\phi$$

The expression above is plotted below. As can be seen, when the flux is very large, X/N^{25} tends to a maximum value such that

$$\lim_{\phi\to\infty}\left(X/N^{25}\right) = \frac{1.60\times10^{-18}}{1.27\times10^{-13}} = \boxed{1.26\times10^{-5}}$$



■ P10.4

Parts (a) and (b): Taking γ = 3.1×10^{-11} W·s as the energy release per fission and noting that 1 MW/m^3 = 1 W/cm^3, the average flux becomes

$$\bar{\phi} = \frac{\bar{P}'''}{\gamma\Sigma_f} = \frac{80}{\left(3.1\times10^{-11}\right)\times0.203} = 1.27\times10^{13}\ \text{n/cm}^2/\text{s}$$

Multiplying this by peaking factor F_q gives the *maximum* flux:

$$\phi_{\max} = \frac{F_q\bar{P}'''}{\gamma\Sigma_f} = 2.0\times\left(1.27\times10^{13}\right) = 2.54\times10^{13}\ \text{n/cm}^2/\text{s}$$

Using data from (10.11), we compute the decay constants

$$\lambda_I = \frac{0.693}{t_{1/2,I}} = \frac{0.693}{6.7} = 0.103\ \text{hr}^{-1} = 2.86\times10^{-5}\ \text{s}^{-1}$$

$$\lambda_X = \frac{0.693}{t_{1/2,X}} = \frac{0.693}{9.2} = 0.0753\ \text{hr}^{-1} = 2.09\times10^{-5}\ \text{s}^{-1}$$

Also, σ_{aX} = 2.65×10^6 b, γ_I = 0.0639, and γ_X = 0.00237 (Table 10.1), so that, referring to equation (10.15),

$$X = \frac{(\gamma_I+\gamma_X)}{\lambda_X+\sigma_{aX}\phi}\Sigma_f\phi = \frac{(0.0639+0.00237)}{2.09\times10^{-5}+2.65\times10^{-18}\phi}\times0.203\phi$$

$$\therefore X = \frac{0.0135\phi}{2.09\times10^{-5}+2.65\times10^{-18}\phi}$$

To answer part (a), we substitute $\bar{\phi}$ = 1.27×10^{13} n/cm^2/s above to obtain

$$X = \frac{0.0135\times\left(1.27\times10^{13}\right)}{2.09\times10^{-5}+2.65\times10^{-18}\times\left(1.27\times10^{13}\right)} = \boxed{3.14\times10^{15}\ \text{cm}^{-3}}$$

To answer part (b), we substitute $\phi_{\max}$ = 2.54×10^{13} n/cm^2/s,

$$X = \frac{0.0135\times\left(2.54\times10^{13}\right)}{2.09\times10^{-5}+2.65\times10^{-18}\times\left(2.54\times10^{13}\right)} = \boxed{3.89\times10^{15}\ \text{cm}^{-3}}$$

Note that although the flux doubles from part (a) to part (b), the xenon concentration rises by only ≈24%.

Parts (c) and (d): To find the average samarium concentration, we take γ_P = 0.0107 from Table 10.1 and σ_{aS} = 41,000 b from page 250, so that

$$S = \frac{\gamma_P\Sigma_f}{\sigma_{aS}} = \frac{0.0107\times0.203}{41{,}000\times10^{-24}} = \boxed{5.30\times10^{16}\ \text{cm}^{-3}}$$

Since the samarium concentration is independent of the flux, it is the same throughout the reactor (so long as Σ_f is uniform).

■ P10.6

Part (a): On a time scale of hours, assume that the samarium decays instantaneously, so we may write

$$\frac{d}{dt}E = \gamma_s\Sigma_f\phi - \lambda E$$

The solution of this ODE is

$$E(t) = \frac{\gamma_s\Sigma_f\phi}{\lambda}\left[1-\exp(-\lambda t)\right]\ \text{(I)}$$

Similarly, for gadolinium,

$$\frac{d}{dt}G = \lambda E - \sigma_{aG}\phi G$$

Replacing *E* with (I),

$$\frac{d}{dt}G = \gamma_s\Sigma_f\phi\left[1-\exp(-\lambda t)\right] - \sigma_{aG}\phi G$$

Using an integrating factor *exp(*$\sigma_{aG}\phi t$*)*, we have

$$\frac{d}{dt}\left[G\exp\left(\sigma_{aG}\phi t\right)\right]=\gamma_s\Sigma_f\phi\left[1-\exp\left(-\lambda t\right)\right]\exp\left(\sigma_{aG}\phi t\right)$$

Integrating from -0 to t, with the initial condition $G(0) = 0$,

$$G\exp\left(\sigma_{aG}\phi t\right)=\gamma_s\Sigma_f\int_0^t\left\{\exp\left(\sigma_{aG}\phi t'\right)-\exp\left[\left(\sigma_{aG}\phi-\lambda\right)t'\right]\right\}dt'$$

$$\therefore G\exp\left(\sigma_{aG}\phi t\right)=\gamma_s\Sigma_f\phi\left\{\frac{\exp\left(\sigma_{aG}\phi t\right)-1}{\sigma_{aG}\phi}-\frac{\exp\left[\left(\sigma_{aG}\phi-\lambda\right)t\right]-1}{\sigma_{aG}\phi-\lambda}\right\}$$

Simplifying,

$$G\left(t\right)=\frac{\gamma_s\Sigma_f}{\sigma_{aG}}\left\{1-\frac{1}{\sigma_{aG}\phi-\lambda}\left[\sigma_{aG}\phi\exp\left(-\lambda t\right)-\lambda\exp\left(-\sigma_{aG}\phi t\right)\right]\right\}$$

The equation above describes the decay of gadolinium.
Part (b): As $t \to \infty$, the second term on the right-hand side of the expression obtained in part (a) tends to zero, and $G(\infty)$ becomes simply

$$G\left(\infty\right)=\frac{\gamma_s\Sigma_f}{\sigma_{aG}}=\frac{\left(7.0\times10^{-5}\right)\times\Sigma_f}{240{,}000\times10^{-24}}=2.92\times10^{14}\Sigma_f\quad\text{(II)}$$

To determine the fission cross-section, we use $P''' = \gamma\Sigma_f\phi$ and note that 1 MW/m^3 = 1 W/cm^3:

$$P'''=\gamma\Sigma_f\phi\quad\to\quad\Sigma_f=\frac{P'''}{\gamma\phi}$$

$$\therefore\Sigma_f=\frac{P'''}{\gamma\phi}=\frac{100}{\left(3.1\times10^{-11}\right)\times\left(8.0\times10^{12}\right)}=0.403\text{ cm}^{-1}$$

Substituting in (II) brings to

$$G\left(\infty\right)=\left(2.92\times10^{14}\right)\times0.403=\boxed{1.18\times10^{14}\text{ cm}^{-3}}$$

Part (c): Following shutdown the Eu decays at a rate

$$E\left(t\right)=\frac{\gamma_s\Sigma_f\phi}{\lambda}\exp\left(-\lambda t\right)$$

However, no additional Gd is produced from Eu, since $\phi = 0$. Moreover, since Gd is stable its concentration remains at $G(\infty) = 1.18\times10^{14}$ cm^{-3}.

■ P10.9

We first rearrange equation (10.12) (page 247) and multiply through by $\exp(\lambda_I t)$:

$$\frac{d}{dt}I\left(t\right)=\gamma_I\bar{\Sigma}_f\phi-\lambda_I I\left(t\right)$$

$$\therefore\frac{d}{dt}I\left(t\right)+\lambda_I I\left(t\right)=\gamma_I\bar{\Sigma}_f\phi$$

$$\therefore\left[\frac{d}{dt}I\left(t\right)+\lambda_I I\left(t\right)\right]\exp\left(\lambda_I t\right)=\gamma_I\bar{\Sigma}_f\phi\exp\left(\lambda_I t\right)$$

$$\therefore\left[\frac{d}{dt}I\left(t\right)\exp\left(\lambda_I t\right)\right]=\gamma_I\bar{\Sigma}_f\phi\exp\left(\lambda_I t\right)$$

Integrating between 0 and t,

$$I\left(t\right)\exp\left(\lambda_I t\right)-I\left(0\right)=\int_0^t\gamma_I\Sigma_f\phi\exp\left(\lambda_I t'\right)dt'$$

$$\therefore I\left(t\right)\exp\left(\lambda_I t\right)-I\left(0\right)=\frac{\gamma_I\Sigma_f\phi}{\lambda_I}\left[\exp\left(\lambda_I t\right)-1\right]$$

Solving for $I(t)$,

$$I\left(t\right)=I\left(0\right)e^{-\lambda_I t}+\frac{\gamma_I\Sigma_f\phi}{\lambda_I}\left(1-e^{-\lambda_I t}\right)\quad;\quad t\in\left(0;12\right)\text{h}\quad\text{(I)}$$

Let $t' = t - 12$ h. Then, while the reactor is shut down, no iodine is produced and thus

$$I(t) = I(12)e^{-\lambda_I t'} \quad ; \quad t \in (12;\, 24)\,\mathrm{h}$$

Using the periodic boundary condition,

$$I(0) = I(24) = \left[I(12)\exp(-\lambda_I \times t)\right]\Big|_{t=12}$$

$$\therefore I(0) = I(24) = \left[I(0)e^{-\lambda_I \times 12} + \frac{\gamma_I \Sigma_f \phi}{\lambda_I}\left(1 - e^{-\lambda_I \times 12}\right)\right] \times e^{-\lambda_I \times 12}$$

Solving for $I(0)$:

$$I(0) = \frac{\gamma_I \Sigma_f \phi}{\lambda_I} \frac{\left(1 - e^{-\lambda_I \times 12}\right)}{\left(1 - e^{-\lambda_I \times 24}\right)} e^{-\lambda_I \times 12} = \frac{\gamma_I \Sigma_f \phi}{\lambda_I} \frac{1}{\left(e^{\lambda_I \times 12} + 1\right)}$$

Substituting $I(0)$ into (I),

$$\boxed{I(t) = \frac{\gamma_I \Sigma_f \phi}{\lambda_I} \frac{1}{\left(e^{\lambda_I \times 12} + 1\right)} e^{-\lambda_I t} + \frac{\gamma_I \Sigma_f \phi}{\lambda_I}\left(1 - e^{-\lambda_I t}\right) \quad ; \quad t \in (0;\, 12)\,\mathrm{h}}$$

In turn,

$$I(t) = \frac{\gamma_I \Sigma_f \phi}{\lambda_I} \frac{e^{\lambda_I \times 12}}{\left(e^{\lambda_I \times 12} + 1\right)} e^{-\lambda_I t'} \quad ; \quad t' \in (0;\, 12)\,\mathrm{h}$$

Replacing t' with t ultimately gives

$$\boxed{I(t) = \frac{\gamma_I \Sigma_f \phi}{\lambda_I} \frac{e^{\lambda_I \times 24}}{\left(e^{\lambda_I \times 12} + 1\right)} e^{-\lambda_I t'} \quad ; \quad t \in (12;\, 24)\,\mathrm{h}}$$

■ P10.13

Part (a): We begin with equation (10.38) on page 255:

$$CR(t) = \frac{\sigma_\gamma^{28} N^{28}(0)}{\sigma_a^{25} N^{25}(t) + \sigma_a^{49} N^{49}(t)}$$

At the beginning of life, $N^{49}(0)$ and the equation simplifies to

$$CR(t) = \frac{\sigma_\gamma^{28} N^{28}(0)}{\sigma_a^{25} N^{25}(t) + \cancel{\sigma_a^{49} N^{49}(t)}} = \frac{\sigma_\gamma^{28} N^{28}(0)}{\sigma_a^{25} N^{25}(t)} \quad \text{(I)}$$

The enrichment is

$$\tilde{e} = \frac{N^{25}(0)}{N^{25}(0) + N^{28}(0)} = 0.04$$

Solving for $N^{28}(0)/N^{25}(0)$,

$$\frac{N^{28}(0)}{N^{25}(0)} = \frac{1}{\tilde{e}} - 1 = \frac{1}{0.04} - 1 = 24$$

Using thermal cross-section data from Table 3.2 and substituting the result above in (I), we obtain

$$CR(t) = \frac{2.42}{591} \times 24.0 = \boxed{0.0983}$$

Part (b): Firstly, note that, per equation (10.31),

$$N^{25}(t) = N^{25}(0)\exp\left[-\sigma_a^{25}\Phi(t)\right]$$

while equation (10.37) reads

$$N^{49}(t) = \frac{\sigma_\gamma^{28}}{\sigma_a^{49}} N^{28}(0)\left\{1 - \exp\left[-\sigma_a^{49}\Phi(t)\right]\right\}$$

Substituting these into the equation for conversion ratio,

$$CR(t)=\frac{\sigma_\gamma^{28}N^{28}(0)}{\sigma_a^{25}N^{25}(0)\exp\left[-\sigma_a^{25}\Phi(t)\right]+\sigma_\gamma^{28}N^{28}(0)\left\{1-\exp\left[-\sigma_a^{49}\Phi(t)\right]\right\}}$$

Simplifying,

$$CR(t)=\frac{1}{\exp\left[-\sigma_a^{25}\Phi(t)\right]+CR(0)\left\{1-\exp\left[-\sigma_a^{49}\Phi(t)\right]\right\}}CR(0)\ \text{(II)}$$

For 50% burnup, we restate (10.31) as

$$\frac{N^{25}(t)}{N^{25}(0)}=\exp\left[-\sigma_a^{25}\Phi(t)\right]=0.50$$

Solving for $\sigma_a^{25}\Phi(t)$,

$$\exp\left[-\sigma_a^{25}\Phi(t)\right]=0.50\ \ \rightarrow\ \ \sigma_a^{25}\Phi(t)=-\ln(0.50)$$

$$\therefore\ \sigma_a^{25}\Phi(t)=-\ln(0.50)=0.693$$

Similarly for the product $\sigma_a^{49}\Phi(t)$,

$$\sigma_a^{49}\Phi(t)=-\frac{\sigma_a^{49}}{\sigma_a^{25}}\ln(0.50)=\frac{973}{591}\times 0.693=1.14$$

Substituting the pertaining data into (II), we have

$$CR(t)=\frac{1}{\exp\left[-\sigma_a^{25}\Phi(t)\right]+CR(0)\left\{1-\exp\left[-\sigma_a^{49}\Phi(t)\right]\right\}}CR(0)\ \text{(II)}$$

$$\therefore\ CR(t)=\frac{1}{\exp(-0.693)+0.0983\times\left[1-\exp(-1.14)\right]}\times 0.0983=\boxed{0.173}$$

Part (c): The fraction in question is

$$\frac{P'''^{49}}{P'''}=\frac{P'''^{49}}{P'''^{25}+P'''^{49}}=\frac{\gamma\sigma_f^{49}N^{49}(t)\phi}{\gamma\sigma_f^{25}N^{25}(t)\phi+\gamma\sigma_f^{49}N^{49}(t)\phi}$$

Using equations (10.31) and (10.37),

$$\frac{P'''^{49}}{P'''}=\frac{\sigma_f^{49}\left(\sigma_\gamma^{28}/\sigma_a^{49}\right)N^{28}(0)\left\{1-\exp\left[-\sigma_a^{49}\Phi(t)\right]\right\}}{\sigma_f^{25}N^{25}(0)\exp\left[-\sigma_a^{25}\Phi(t)\right]+\sigma_f^{49}\left(\sigma_\gamma^{28}/\sigma_a^{49}\right)N^{28}(0)\left\{1-\exp\left[\sigma_a^{49}\Phi(t)\right]\right\}}$$

This can be restated as

$$\frac{P'''^{49}}{P'''}=\frac{\left(\sigma_f^{49}/\sigma_f^{25}\right)\left(\sigma_\gamma^{28}/\sigma_a^{49}\right)\left[N^{28}(0)/N^{25}(0)\right]\left\{1-\exp\left[-\sigma_a^{49}\Phi(t)\right]\right\}}{\exp\left[-\sigma_a^{25}\Phi(t)\right]+\left(\sigma_f^{49}/\sigma_f^{25}\right)\left(\sigma_\gamma^{28}/\sigma_a^{49}\right)\left[N^{28}(0)/N^{25}(0)\right]\left\{1-\exp\left[\sigma_a^{49}\Phi(t)\right]\right\}}\ \text{(III)}$$

Using the pertaining cross-sections from Table 3.2 on page 77, we may write

$$\left(\frac{\sigma_f^{49}}{\sigma_f^{25}}\right)\left(\frac{\sigma_\gamma^{28}}{\sigma_a^{49}}\right)=\left(\frac{698}{505}\right)\times\left(\frac{2.42}{973}\right)=0.00344$$

Substituting into (III),

$$\frac{P'''^{49}}{P'''}=\frac{0.00344\times 24\times\left[1-\exp(-1.14)\right]}{\exp(-0.693)+0.00344\times 24\times\left[1-\exp(-1.14)\right]}=0.10096\approx\boxed{10.1\%}$$