Montogue

GATE Engineering Mathematics

35 One-Mark Questions

Lucas Monteiro Nogueira

Here's a set of 35 fully solved problems for applicants to the GATE Engineering Mathematics sub-exam. As usual, all problems are solved step by step. The problems discussed herein are straightforward to solve and hence qualify as one-mark problems. More advanced problems can be found in the companion PDF posted in the Montogue Quiz website. ■

PROBLEMS

Problem 1. If A is a 2 × 1 matrix and B is a 2 × 2 matrix, then the matrix $C = A^T BA$ will have dimensions:

(A) 1 × 1 (a scalar)
(B) 1 × 2
(C) 2 × 1
(D) 2 × 2

Problem 2. Which of the following matrices has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 5$?

(A) $\begin{pmatrix} 1 & 1 \\ -1 & 8 \end{pmatrix}$ **(B)** $\begin{pmatrix} 4 & 1 \\ 1 & 5 \end{pmatrix}$ **(C)** $\begin{pmatrix} 2 & 2 \\ -3 & 8 \end{pmatrix}$ **(D)** $\begin{pmatrix} 3 & 2 \\ -1 & 6 \end{pmatrix}$

Problem 3. What is the value of determinant A?

$$A = \begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix}$$

(A) −2
(B) 0
(C) 2
(D) 4

Problem 4. Consider the following system of linear equations. This system has:

$$\begin{cases} 2x + y + z = 0 \\ y - z = 0 \\ x + y = 0 \end{cases}$$

(A) A unique solution.
(B) No solutions.
(C) An infinite number of solutions.
(D) Four real solutions.

Problem 5. If A is a 7×9 matrix with three linearly independent rows, what is the rank of A?
(A) 3
(B) 4
(C) 6
(D) 7

Problem 6. What is the value of the following limit?

$$\lim_{x \to 0} \frac{(1+x)^{1/3} - (1-x)^{1/3}}{x}$$

(A) 1/3
(B) 2/3
(C) 4/3
(D) ∞

Problem 7. If $y = e^{\tan x}$, then

$$\cos^2 x \frac{d^2 y}{dx^2} = ?$$

(A) $(-1 - \sin 2x)\frac{dy}{dx}$

(B) $(1 - \sin 2x)\frac{dy}{dx}$

(C) $\left(1 + \frac{1}{2}\sin 2x\right)\frac{dy}{dx}$

(D) $(1 + \sin 2x)\frac{dy}{dx}$

Problem 8. Let a continuous function f be such that $f(0) \neq 0$ and $f(x+y) = f(x)f(y)$ for all x and y. If $f(5) = 2$ and $f'(0) = 3$, then the derivative $f'(5)$ is equal to:
(A) 3
(B) 4
(C) 5
(D) 6

Problem 9. Function f satisfies the conclusion of the mean value theorem for derivatives in exactly one value $x = c$ in the interval $x \in [0,2]$. What is the value of c?

$$f(x) = 2x^2 - 3x + 1 \ ; \ x \in [0,2]$$

(A) 3/4
(B) 1
(C) 4/3
(D) 5/3

Problem 10. The parabolic arc $y = x^2$, $2 \le x \le 3$ is revolved around the x-axis. The volume of the solid of revolution thus obtained is:
(A) 30π
(B) $65\pi/2$
(C) 33π
(D) $69\pi/2$

Problem 11. The length of the curve $y = (2/3)x^{3/2}$ between $x = 0$ and $x = 1$ is most nearly:
(A) 0.67
(B) 1
(C) 1.22
(D) 1.33

Problem 12. Evaluate the following improper integral.

$$I = \int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx$$

(A) 1
(B) 2
(C) 3
(D) The integral diverges.

Problem 13. Regarding infinite series σ and τ, which of the following is true?

$$\sigma = \sum_{n=1}^{\infty} \frac{9^n}{3+10^n} \quad ; \quad \tau = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$$

(A) Both series are convergent.
(B) Series σ converges and series τ diverges.
(C) Series τ converges and series σ diverges.
(D) Both series are divergent.

Problem 14. What is the radius of convergence of the following power series?

$$\rho(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(A) 0
(B) 1
(C) 2
(D) ∞

Problem 15. In the Taylor series expansion of e^x about $x = 2$, the coefficient of $(x-2)^4$ is:
(A) 1/4!
(B) $2^4/4!$
(C) $e^2/4!$
(D) $e^4/4!$

Problem 16. For a scalar function $f(x,y,z) = x^2 + 3y^2 + 2z^2$, the directional derivative at point $P(1, 2, -1)$ in the direction of the vector $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ is:
(A) –18
(B) $-3\sqrt{6}$
(C) $3\sqrt{6}$
(D) 18

Problem 17. Choose the alternative that represents integral I with the order of integration reversed.

$$I = \int_0^2 \int_{y-2}^0 dxdy$$

(A) $I = \int_0^2 \int_0^{x+2} dydx$

(B) $I = \int_{-2}^0 \int_0^{x+2} dydx$

(C) $I = \int_{-2}^0 \int_0^{x+1} dydx$

(D) $I = \int_{-1}^0 \int_0^{x+2} dydx$

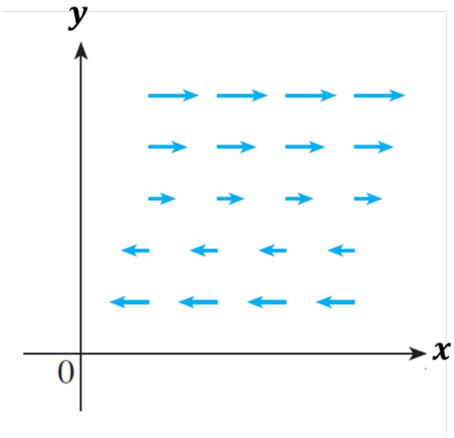
Problem 18. Evaluate double integral I, where D is the region bounded by $y = x$, $y = x^3$, and $x \geq 0$.

$$I = \iint_D \left(x^2 + 2y\right) dA$$

(A) $I \approx 0.15$
(B) $I \approx 0.27$
(C) $I \approx 0.38$
(D) $I \approx 0.47$

Problem 19. Find the divergence of the vector field $x^2z\mathbf{i} + xy^3\mathbf{j} - xz^2\mathbf{k}$ at point (1, –1, 1).
(A) –3
(B) –1
(C) 1
(D) 3

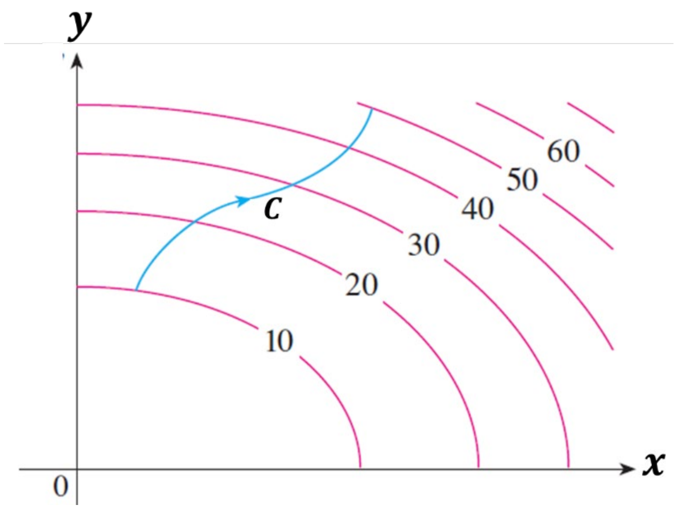
Problem 20. For the two-dimensional vector field **F** sketched below, which of the following is **true**?



(A) div(**F**) < 0 and curl(**F**) is a vector pointing in the negative *z*-direction.
(B) div(**F**) < 0 and curl(**F**) is a vector pointing in the positive *z*-direction.
(C) div(**F**) $= 0$ and curl(**F**) is a vector pointing in the negative *z*-direction.
(D) div(**F**) $= 0$ and curl(**F**) is a vector pointing in the positive *z*-direction.

Problem 21. The figure shows a curve *C* and a contour map of a function f whose gradient is continuous. Curve *C* is represented by a vector function **r**(*t*). Find the integral I:

$$I = \int_C \nabla f \cdot d\mathbf{r}$$



(A) 20
(B) 30
(C) 40
(D) 50

Problem 22. Find the optimum values of x_1 and x_2 for the following constrained maximization problem. What is the maximum value of y?

$$\max \ \ y = (x_1 + 2)(x_2 + 1) \ ; \ \text{subject to } x_1 + x_2 = 21$$

(A) 121
(B) 144
(C) 145
(D) 169

Problem 23. The complex number *z* shown next can be restated as:

$$z = \frac{2 - 3i}{-5 + i}$$

(A) $z = 0.5 + 0.5i$
(B) $z = 0.5 - 0.5i$
(C) $z = -0.5 + 0.5i$
(B) $z = -0.5 - 0.5i$

Problem 24. What is the value of the following limit?

$$\lim_{z \to i} \frac{\left(z^3 + i\right)}{\left(z^2 + 1\right)z}$$

(A) –3/2
(B) 3/2
(C) ∞
(D) The limit does not exist.

Problem 25. An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x,y) + iv(x,y)$. If $u(x,y) = 2xy$, then $v(x,y)$ must have the form
(A) $x^2 + y^2$ + constant
(B) $x^2 - y^2$ + constant
(C) $-x^2 + y^2$ + constant
(D) $-x^2 - y^2$ + constant

Problem 26. A box contains 3 blue balls and 5 red balls. Three balls are selected randomly from the box, one after another, without replacement. The probability that the selected set contains one blue ball and two red balls is:
(A) 13/28
(B) 1/2
(C) 15/28
(D) 4/7

Problem 27. A fair dice is tossed two times. The probability that the second toss results in a value that is higher than the first toss is:
(A) 1/18
(B) 4/9
(C) 5/12
(D) 1/2

Problem 28. A discrete random variable *X* takes values from 1 to 5 with the probabilities shown in the table. Find the mean μ and variance σ^2 of the random variable in question.

X	1	2	3	4	5
P(X)	0.3	0.3	0.1	0.2	0.1

(A) $\mu = 2.4$; $\sigma^2 = 8.0$
(B) $\mu = 2.4$; $\sigma^2 = 8.1$
(C) $\mu = 2.5$; $\sigma^2 = 8.0$
(D) $\mu = 2.5$; $\sigma^2 = 8.1$

Problem 29. A continuous random variable is uniformly distributed over the interval 3 to 8. The variance of this random variable is:
(A) 4/3
(B) 2
(C) 25/12
(D) 16/3

Problem 30. When all entries in a finite dataset are the same and different from zero, all but one of the following statistical measurements equals zero. Which one is it?
(A) Mean
(B) Standard deviation
(C) Variance
(D) Interquartile range

Problem 31. Four players *A, B, C*, and *D* compete in a sports championship. The probability that *A* will win the championship is three times greater than the probability that *B* will win. The probability that *B* will win the championship is two times greater than the probability that *C* will win. The probability that *C* will win the championship is three times greater than the probability that *D* will win. What is the probability that player *C* will win the championship?
(A) 1/28
(B) 3/28
(C) 1/7
(D) 5/28

Problem 32. Two factories A and B manufacture 5000 mechanical parts per day. Factory A manufactures 2000 parts, of which 4% are defective. Factory B manufactures the remaining 3000 parts, of which 1% are defective. A part is sampled from the daily output of the two factories, and the part is found to be defective. What is the probability that the sampled part was manufactured by factory A?
(A) 4/7
(B) 2/3
(C) 8/11
(D) 4/5

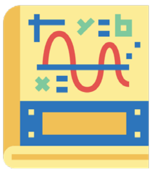
Problem 33. The following partial differential equation is classified as:

$$5\frac{\partial^2 z}{\partial x^2}+9\frac{\partial^2 z}{\partial x \partial y}+6\frac{\partial^2 z}{\partial y^2}=xy$$

(A) Elliptic.
(B) Parabolic.
(C) Hyperbolic.
(D) More information is needed.

Problem 34. When used to compute the area under a polynomial function, Simpson's rule for integration gives an exact (i.e., zero-error) result when the degree of the polynomial is less than or equal to:
(A) 1
(B) 2
(C) 3
(D) 4

Problem 35. The only real root of function $f(x) = x + \sqrt{x} - 3$ can be obtained using the Newton-Raphson method. If the starting value is $x = 2$, then the value of x to be used in the next iteration is:
(A) 0.739
(B) 1.331
(C) 1.694
(D) 2.306



Answer key is on the next page!
Solutions also begin on the next page.

ANSWER KEY

Problem	Answer	Problem	Answer
1	A	19	D
2	D	20	C
3	A	21	C
4	C	22	B
5	A	23	C
6	B	24	B
7	D	25	C
8	D	26	C
9	B	27	C
10	B	28	D
11	C	29	C
12	B	30	A
13	A	31	B
14	D	32	C
15	C	33	A
16	B	34	C
17	B	35	C
18	B		

SOLUTIONS

1 ➡ A

Firstly, if A is 2×1, then A^T is 1×2. The matrix product of A^T (1×2) and B (2×2) will be 1×2. Finally, the product of this 1×2 matrix and matrix A (2×1) will be 1×1 (i.e., a scalar).

2 ➡ D

In order for a matrix to have $\lambda_1 = 4$ and $\lambda_2 = 5$ as eigenvalues, its trace must equal $\lambda_1 + \lambda_2 = 9$ and its determinant must equal $\lambda_1\lambda_2 = 4 \times 5 = 20$. Matrix (D) obeys both conditions.

3 ➡ A

Let R denote a certain row and C denote a certain column. We first apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$ to obtain

$$A = \begin{vmatrix} x+1 & 1 & 2 \\ x+3 & 2 & 3 \\ x+7 & 3 & 4 \end{vmatrix}$$

Then, we apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$ to obtain

$$A = \begin{vmatrix} x+1 & 1 & 2 \\ 2 & 1 & 1 \\ 4 & 1 & 1 \end{vmatrix}$$

Lastly, we apply $R_3 \to R_3 - R_2$, giving

$$A = \begin{vmatrix} x+1 & 1 & 2 \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{vmatrix}$$

Expanding along the third row, we get

$$A = 2\times(-1)^{3+1}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 0\times(-1)^{3+2}\begin{vmatrix} x+1 & 2 \\ 2 & 1 \end{vmatrix} + 0\times(-1)^{3+3}\begin{vmatrix} x+1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$\therefore A = -2+0+0 = \boxed{-2}$$

4 ➡ C

Let *R* denote a certain row and *C* denote a certain column. The matrix for the system under consideration is shown next.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

Making $R_1 \to R_1 + R_2$ yields

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

Making $R_1 \to R_1 - 2R_2$ yields

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

This matrix has rank 2, which is less than the order of the matrix (= 3). We conclude that an infinite number of solutions exist for the system in question. The reduced system has the form

$$\begin{cases} x + y = 0 \\ y - z = 0 \end{cases}$$

If $z = k$, then $y = k$ and $x = -k$; accordingly, the solution vector is

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad (k \in \Re)$$

5 ➡ A

The rank ρ of a matrix is the largest number of linearly independent row (or equivalently, column) vectors it contains. In the present case, $\rho(A) = 3$.

6 ➡ B

Applying L'Hospital's rule, we obtain

$$\lim_{x\to 0} \frac{(1+x)^{1/3} - (1-x)^{1/3}}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{3}(1+x)^{-2/3} + \frac{1}{3}(1-x)^{-2/3}}{1}$$

$$\therefore \lim_{x\to\infty} \frac{\frac{1}{3}(1+x)^{-2/3} + \frac{1}{3}(1-x)^{-2/3}}{1} = \frac{\frac{1}{3} + \frac{1}{3}}{1} = \boxed{\frac{2}{3}}$$

Alternatively, we could use the algebraic identity

$$a - b = \frac{a^3 - b^3}{a^2 + ab + b^2}$$

so that

$$\lim_{x\to 0} \frac{(1+x)^{1/3} - (1-x)^{1/3}}{x} = \lim_{x\to 0} \left\{ \left(\frac{1}{x}\right) \times \left[\frac{(1+x) - (1-x)}{(1-x)^{2/3} + (1-x)^{1/3}(1+x)^{1/3} + (1+x)^{2/3}} \right] \right\}$$

$$\therefore \lim_{x\to 0} \frac{(1+x)^{1/3} - (1-x)^{1/3}}{x} = \lim_{x\to 0} \left[\frac{2}{(1-x)^{2/3} + (1-x)^{1/3}(1+x)^{1/3} + (1+x)^{2/3}} \right]$$

$$\therefore \lim_{x\to 0} \frac{(1+x)^{1/3} - (1-x)^{1/3}}{x} = \frac{2}{1+1+1} = \boxed{\frac{2}{3}}$$

7 ➡ D

Firstly, we take the given expression for y and apply logarithms to both sides,

$$y = e^{\tan x} \quad \to \quad \ln y = \tan x$$

Then, we differentiate on both sides,

$$\ln y = \tan x \quad \to \quad \frac{1}{y}\frac{dy}{dx} = \sec^2 x$$

$$\therefore \frac{1}{\sec^2 x}\frac{dy}{dx} = y$$

$$\therefore \cos^2 x \frac{dy}{dx} = y$$

Differentiating a second time,

$$\cos^2 x \frac{d^2 y}{dx^2} - 2\sin x \cos x \frac{dy}{dx} = \frac{dy}{dx}$$

$$\therefore \cos^2 x \frac{d^2 y}{dx^2} = \left(1 + \underbrace{2\sin x \cos x}_{=\sin(2x)}\right)\frac{dy}{dx}$$

$$\therefore \cos^2 x \frac{d^2 y}{dx^2} = \boxed{(1+\sin 2x)\frac{dy}{dx}}$$

8 ➡ D

Appealing to the definition of derivative and using *f(x + y) = f(x)f(y)* as given in the problem statement, we have

$$f'(5) = \lim_{h\to 0}\frac{f(5+h)-f(5)}{h} = \lim_{h\to 0}\frac{f(5)f(h)-f(5+0)}{h}$$

$$\therefore f'(5) = \lim_{h\to 0}\frac{f(5)f(h)-f(5)f(0)}{h}$$

$$\therefore f'(5) = f(5)\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$\therefore f'(5) = f(5)f'(0)$$

$$\therefore f'(5) = 2\times 3 = \boxed{6}$$

9 ➡ B

The mean value theorem for derivatives reads

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

In the present case, a = 0, $f(0)$ = 2 × 0^2 – 3 × 0 + 1 = 1, b = 2, and $f(2)$ = 2 × 2^2 – 3 × 2 + 1 = 3, with the result that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \to \quad 4c - 3 = \frac{3-1}{2-0}$$

$$\therefore 4c - 3 = 1$$

$$\therefore \boxed{c = 1}$$

10 ➡ B

The volume V of the solid in question is given by

$$V = \int_2^3 2\pi x f(x)\,dy = \int_2^3 2\pi \times x \times x^2 dy$$

$$\therefore V = 2\pi\int_2^3 x^3 dy$$

$$\therefore V = 2\pi \times \left.\frac{x^4}{4}\right|_{x=2}^{x=3}$$

$$\therefore V = 2\pi \times \frac{1}{4} \times \left(3^4 - 2^4\right)$$

$$\therefore \boxed{V = \frac{65\pi}{2}}$$

11 ➡ C

Applying the arc length integral, we write

$$\text{Length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where

$$\frac{d}{dx}\left(\frac{2}{3}x^{3/2}\right) = \frac{2}{3} \times \frac{3}{2} \times x^{1/2} = x^{1/2}$$

so that

$$\text{Length} = \int_0^1 \sqrt{1 + \left(x^{1/2}\right)^2}\, dx = \int_0^1 \sqrt{1 + x}\,dx = \left.\frac{2}{3}(x+1)^{3/2}\right|_{x=0}^{x=1}$$

$$\therefore \text{Length} = \frac{2}{3}(1+1)^{3/2} - \frac{2}{3}(0+1)^{3/2}$$

$$\therefore \text{Length} = \frac{2}{3} \times 2^{3/2} - \frac{2}{3} \approx \boxed{1.22}$$

12 ➡ B

The first step is to restate the improper integral with a limit:

$$I = \int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to\infty} \int_3^t \frac{1}{(x-2)^{3/2}}\,dx$$

Then, using $u = x - 2$, $du = dx$ and carrying out the integration,

$$I = \lim_{t\to\infty} \int_3^t \frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to\infty} \left[-2(x-2)^{-1/2}\right]_{x=3}^{x=t}$$

$$\therefore I = \lim_{t\to\infty} \left[-2(x-2)^{-1/2}\right]_{x=3}^{x=t} = \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t-2}} + \frac{2}{\sqrt{1}}\right) = 0 + 2 = \boxed{2}$$

13 ➡ A

Regarding series σ, note that

$$\frac{9^n}{3 + 10^n} < \frac{9^n}{10^n} = (0.9)^n \;\; ; \; n \ge 1$$

The series

$$\sigma' = \sum_{n=1}^{\infty} 0.9^n$$

is a convergent geometric series; therefore, σ converges by the comparison test. Series τ can be restated as

$$\tau = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3 + 5n} = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

where $b_n = 1/(3 + 5n) > 0$. Further, we know that b_n is decreasing and $\lim_{n\to\infty} b_n = 0$, so the series converges by the alternating series test.

14 ➡ D

Using the ratio test, we may write

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right|$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right|$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|\lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|\times 0 = 0 < 1 \text{ for all real } x$$

Thus, by the ratio test, the radius of convergence $R = \infty$.

15 ➡ C

The Taylor series expansion of e^x about $(x - 2)$ is

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \frac{e^2}{4!}(x-2)^4 + \ldots$$

16 ➡ B

To find the directional derivative in question, we first compute the gradient field of function *f*,

$$\nabla f = 2x\mathbf{i} + 6y\mathbf{j} + 4z\mathbf{k}$$

Then, we evaluate it at *P*(1, 2, −1),

$$\nabla f_{(1,2,-1)} = 2\times 1\mathbf{i} + 6\times 2\mathbf{j} + 4\times(-1)\mathbf{k}$$

$$\therefore \nabla f_{(1,2,-1)} = 2\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$$

Then, we proceed to evaluate the directional derivative

$$(D_{\mathbf{a}}f)_{(1,2,-1)} = (2\mathbf{i} + 12\mathbf{j} - 4\mathbf{k})\cdot\frac{\mathbf{i} - \mathbf{j} + 2\mathbf{k}}{\sqrt{1^2 + (-1)^2 + 2^2}}$$

$$\therefore (D_{\mathbf{a}}f)_{(1,2,-1)} = (2\mathbf{i} + 12\mathbf{j} - 4\mathbf{k})\cdot\frac{\mathbf{i} - \mathbf{j} + 2\mathbf{k}}{\sqrt{6}}$$

$$\therefore (D_{\mathbf{a}}f)_{(1,2,-1)} = \frac{2\times 1 + 12\times(-1) - 4\times 2}{\sqrt{6}}$$

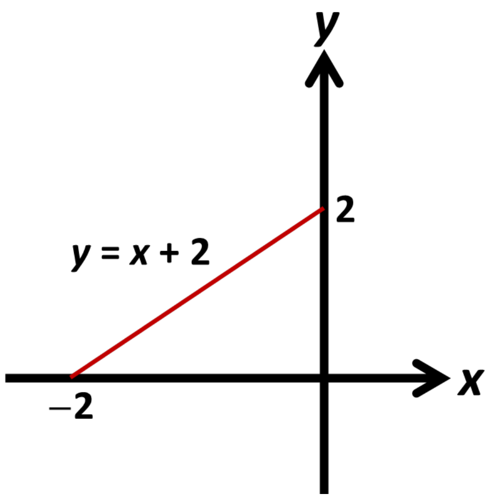
$$\therefore (D_{\mathbf{a}}f)_{(1,2,-1)} = \frac{2 - 12 - 8}{\sqrt{6}}$$

$$\therefore (D_{\mathbf{a}}f)_{(1,2,-1)} = -\frac{18}{\sqrt{6}} = -\frac{18\sqrt{6}}{6} = \boxed{-3\sqrt{6}}$$

17 ➡ B

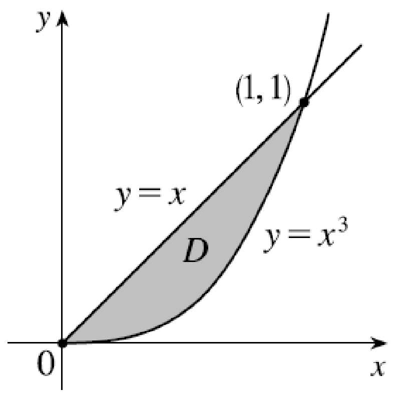
The integration area is sketched below. Clearly, integral I can be restated as

$$I = \int_0^2 \int_{y-2}^0 dxdy = \boxed{\int_{-2}^0 \int_0^{x+2} dydx}$$



18 ➡ B

The integration region *D* is sketched below.



The integral we're looking for is then

$$I=\iint_D\left(x^2+2y\right)dA=\int_0^1\int_{x^3}^{x}\left(x^2+2y\right)dydx$$

$$\therefore I=\int_0^1\left[\left(x^2y+y^2\right)\Big|_{x^3}^{x}\right]dx$$

$$\therefore I=\int_0^1\left[\left(x^2\times x+x^2\right)-\left(x^2\times x^3+x^6\right)\right]dx$$

$$\therefore I=\int_0^1\left(x^3+x^2-x^5-x^6\right)dx$$

$$\therefore I=\left(\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^6}{6}-\frac{x^7}{7}\right)\Bigg|_0^1=\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{7}=\frac{23}{84}\approx\boxed{0.274}$$

19 ➡ D

The divergence of the vector field in question is

$$\nabla\cdot\mathbf{F}=2xz+3xy^2-2xz=3xy^2$$

Substituting (x = 1, y = –1, z = 1), we obtain

$$\nabla\cdot\mathbf{F}=3\times1\times(-1)^2=\boxed{3}$$

20 ➡ C

The vector field has the form $\boldsymbol{F}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$, but we know that $R=0$ for the field in question. In addition, the y-component of each vector of **F** is 0, so $Q=0$, giving

$$\frac{\partial Q}{\partial x}=\frac{\partial Q}{\partial y}=\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial x}=\frac{\partial R}{\partial y}=\frac{\partial R}{\partial z}=0$$

Further, note that P increases as y increases, so $\partial P/\partial y>0$, but P doesn't change in the x- or z-directions, so $\partial P/\partial x=\partial P/\partial z=0$. Then, we can write, for the divergence of **F**,

$$\nabla\cdot\mathbf{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0+0+0=0$$

and for the curl,

$$\nabla\times\mathbf{F}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$\therefore\nabla\times\mathbf{F}=(0-0)\mathbf{i}+(0-0)\mathbf{j}+\left(0-\frac{\partial P}{\partial y}\right)\mathbf{k}=-\frac{\partial P}{\partial y}\mathbf{k}$$

Since $\partial P/\partial y>0$, we surmise that $-\partial P/\partial y\mathbf{k}$ is a vector pointing in the negative z-direction.

21 ➡ C

Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a\le t\le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then, by the fundamental theorem for line integrals,

$$\int_C\nabla f\cdot d\mathbf{r}=f\left(\mathbf{r}(b)\right)-f\left(\mathbf{r}(a)\right)$$

Accordingly, $I = \int_C \nabla f \cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C. From the graph, we find that $I = 50 - 10 = 40$.

22 ➡ B

The Lagrangean function that describes this optimization problem is

$$L = (x_1 + 2)(x_2 + 1) + \lambda(21 - x_1 - x_2)$$

The first-order conditions constitute the following system of equations:

$$\begin{cases} (x_2 + 1) - \lambda = 0 \ \text{(I)} \\ (x_1 + 2) - \lambda = 0 \ \text{(II)} \\ 21 - x_1 - x_2 = 0 \ \text{(III)} \end{cases}$$

Solving (II) for λ gives $\lambda = x_1 + 2$. Substituting into (I), we obtain

$$(x_2 + 1) - (x_1 + 2) = 0$$
$$\therefore x_2 = x_1 + 1 \ \text{(IV)}$$

Substituting into (III) and solving for x_1,

$$21 - x_1 - (x_1 + 1) = 0$$
$$\therefore 20 - 2x_1 = 0$$
$$\therefore x_1 = 10$$

Substituting into (IV),

$$x_2 = x_1 + 1 = 10 + 1 = 11$$

The maximum value of y then becomes

$$y_{\max} = (x_1 + 2)(x_2 + 1) = (10 + 2) \times (11 + 1) = \boxed{144}$$

23 ➡ C

All we have to do is rationalize the complex number by multiplying it by $(-5 - i)/(-5 - i)$, with the result that

$$z = \frac{2 - 3i}{-5 + i} \times \frac{-5 - i}{-5 - i} = \frac{-10 - 2i + 15i + 3i^2}{(-5)^2 - i^2}$$

$$\therefore z = \frac{-10 - 2i + 15i - 3}{26}$$

$$\therefore z = \frac{-13 + 13i}{26}$$

$$\therefore \boxed{z = -0.5 + 0.5i}$$

24 ➡ B

The function inside the limit is not defined at $z = i$ because the denominator becomes zero at this point. We can easily circumvent this problem by using L'Hospital's rule:

$$\lim_{z \to i} \frac{z^3 + i}{(z^2 + 1)z} \overset{\text{L'H}}{=} \lim_{z \to i} \frac{3z^2}{3z^2 + 1}$$

$$\therefore \lim_{z \to i} \frac{z^3 + i}{(z^2 + 1)z} = \frac{3 \times i^2}{3 \times i^2 + 1}$$

$$\therefore \lim_{z \to i} \frac{z^3 + i}{(z^2 + 1)z} = \frac{-3}{-3 + 1}$$

$$\therefore \lim_{z \to i} \frac{z^3 + i}{(z^2 + 1)z} = \boxed{\frac{3}{2}}$$

25 ➡ C

Firstly, note that the derivatives of the real component $u(x,y)$ are

$$\frac{\partial u}{\partial x}=2y$$

and

$$\frac{\partial u}{\partial y}=2x$$

Now, imaginary component v can be expressed by the differential form

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy \quad \text{(I)}$$

But, by the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \to \quad \frac{\partial v}{\partial y}=2y$$

and

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \quad \to \quad \frac{\partial v}{\partial x}=-2x$$

so that, substituting in (I) and integrating,

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=-2xdx+2ydy$$

$$\therefore v=-2\frac{x^2}{2}+2\frac{y^2}{2}+\text{Constant}$$

$$\therefore \boxed{v=-x^2+y^2+\text{Constant}}$$

26 ➡ C

The probability in question is

$$\Pr=\frac{C_1^3C_2^5}{C_3^8}=\frac{\frac{3!}{1!2!}\times\frac{5!}{2!3!}}{\frac{8!}{3!5!}}=\frac{3\times\frac{5\times4\times3!}{2\times3!}}{\frac{8\times7\times6\times5!}{3\times2\times5!}}$$

$$\therefore \Pr=\frac{3\times\frac{5\times4}{2}}{\frac{8\times7\times6}{3\times2}}=\frac{30}{56}=\boxed{\frac{15}{28}}$$

27 ➡ C

The total number of possible results is 6 × 6 = 36. The number of events of the type '2nd value is higher than 1st value' is n{(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)} = 15. Hence, the probability we aim for is 15/36 = 5/12.

28 ➡ D

As in the case of any discrete random variable, the mean μ is given by

$$\mu=\sum xP(x)=\begin{bmatrix}1\times0.3+2\times0.3+3\times0.1\\+4\times0.2+5\times0.1\end{bmatrix}=\boxed{2.5}$$

In turn, the variance σ^2 is

$$\sigma^2=\sum x^2P(x)=\begin{bmatrix}1^2\times0.3+2^2\times0.3+3^2\times0.1\\+4^2\times0.2+5^2\times0.1\end{bmatrix}=\boxed{8.1}$$

29 ➡ C

All we have to do is apply the formula

$$\text{Var}[X]=\frac{(b-a)^2}{12}=\frac{(8-3)^2}{12}=\boxed{\frac{25}{12}}$$

30 ➡ A

For a sample S with entries $\{k, k, k, \ldots, k\}$, $k \neq 0$, the standard deviation, variance, and interquartile range are all equal to 0, but the mean $\mu[S] = k$.

31 ➡ B

The sample space is $\Omega = \{w_1, w_2, w_3, w_4\}$, where

w_1 = A wins the championship;
w_2 = B wins the championship;
w_3 = C wins the championship;
w_4 = D wins the championship;

Let $P(w_4) = p$. Based on the given information, we may write

$$P(w_3) = 3P(w_4) = 3p$$
$$P(w_2) = 2P(w_3) = 2\times 3p = 6p$$
$$P(w_1) = 3P(w_2) = 3\times 6p = 18p$$

Since the probabilities must add up to 1, we have

$$P(w_1) + P(w_2) + P(w_3) + P(w_4) = 1$$
$$\therefore 18p + 6p + 3p + p = 1$$
$$\therefore p = \frac{1}{28}$$

Thus, the probabilities that each player will win become

$$p(w_1) = \frac{18}{28}\ ;\ p(w_2) = \frac{6}{28}\ ;\ \boxed{p(w_3) = \frac{3}{28}}\ ;\ p(w_4) = \frac{1}{28}$$

32 ➡ C

The probability we're looking for is

$$P(\text{A / Defective}) = \frac{P(\text{A})P(\text{Defective/A})}{P(\text{A})P(\text{Defective/A}) + P(\text{B})P(\text{Defective/B})}$$

$$\therefore P(\text{A / Defective}) = \frac{\frac{2}{5}\times\frac{4}{100}}{\frac{2}{5}\times\frac{4}{100} + \frac{3}{5}\times\frac{1}{100}}$$

$$\therefore P(\text{A / Defective}) = \frac{\frac{8}{500}}{\frac{8}{500} + \frac{3}{500}}$$

$$\therefore P(\text{A / Defective}) = \boxed{\frac{8}{11}}$$

33 ➡ A

A typical second-order partial differential equation with two independent variables has the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

The classification of such a 2nd order PDE hinges on the value of $B^2 - 4AC$:

$$\text{If}\begin{cases} B^2 - 4AC < 0 \rightarrow \text{Elliptic equation} \\ B^2 - 4AC = 0 \rightarrow \text{Parabolic equation} \\ B^2 - 4AC > 0 \rightarrow \text{Hyperbolic equation} \end{cases}$$

In the present case, $A = 5$, $B = 9$, and $C = 6$, giving

$$B^2 - 4AC = 9^2 - 4\times(5\times 6) = 81 - 120 = -39 < 0$$

Thus, the PDE in question is elliptic.

34 ➡ C

Simpson's rule gives an exact result if the degree of the polynomial is less than or equal to 3.

35 ➡ C

Per the Newton-Raphson method, the value of x to be used in iteration $n+1$ is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In the present case, $f(x) = x + \sqrt{x} - 3$ and

$$f'(x) = 1 + \frac{1}{2\sqrt{x}}$$

so that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2+\sqrt{2}-3}{1+\frac{1}{2\sqrt{2}}} = \boxed{1.69398}$$

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