GATE Engineering Mathematics

◆35 Two-Mark Questions

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Here's a set of 35 fully solved problems for applicants to the GATE Engineering Mathematics sub-exam. As usual, all problems are solved step by step. The problems discussed herein are relatively advanced and hence qualify as two-mark problems. Easier problems can be found in the companion PDF posted in the Montogue Quiz website. ■

PROBLEMS

Problem 1. If A is a 5×1 matrix and B is a 5×5 matrix, then the matrix $C = AA^TB$ will have dimensions:
(A) 1×1 (a scalar)
(B) 1×5
(C) 5×1
(D) 5×5

Problem 2. A real 4×4 matrix A satisfies the equation $A^2 = I$, where I is the 4×4 identity matrix. One of the eigenvalues of A is:
(A) 0
(B) 1
(C) 2
(D) 4

Problem 3. The maximum value of the determinant among all 2×2 real-symmetric matrices with trace equal to 10 is:
(A) 20
(B) 25
(C) 30
(D) 100

Problem 4. Let A be a $n \times n$ invertible square matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Further, let A^{-1} be the inverse of A. Which of the following is **true**?
(A) The eigenvalues and eigenvectors of A^{-1} are the same as those of A.
(B) The eigenvalues of A^{-1} are the same as those of A, but the eigenvectors are different.
(C) The eigenvectors of A^{-1} are the same as those of A, but the eigenvalues are different.
(D) The eigenvalues and eigenvectors of A^{-1} are different from those of A.

Problem 5. For what values of a, if any, will the following system of equations in x, y, and z have a solution?

$$\begin{cases} 2x+3y=4 \\ x+y+z=4 \\ x+2y-z=a \end{cases}$$

(A) Any real number.
(B) 0
(C) 1
(D) The system will not have a solution regardless of the value of a.

Problem 6. If function $g(x)$ defined below is differentiable for all real values, what is the value of q?

$$g(x)=\begin{cases} px^3+2qx^2+6, & x\le 1 \\ 2px^2+qx+2, & x>1 \end{cases}$$

(A) 2
(B) 4
(C) 6
(D) 8

Problem 7. Let $f(x+y)=f(x)f(y)$ and $f(x)=1+xg(x)\gamma(x)$, where $\lim_{x\to 0} g(x)=a$ and $\lim_{x\to 0} \gamma(x)=b$. Then, $f'(x)=k\times f(x)$, where k is equal to:
(A) a/b
(B) ab
(C) $2ab$
(D) $ab+1$

Problem 8. The function $f(x)=x^2e^{-x}$ has a maximum for x equal to:
(A) -1
(B) 0
(C) 1
(D) 2

Problem 9. If at every point of a certain curve on the xy-plane the slope of the tangent equals $-4x/y$, then the curve is:
(A) a circle.
(B) an ellipse.
(C) a parabola.
(D) a hyperbola.

Problem 10. The length of the curve $y=\ln(\cos x)$ between $x=0$ and $x=\pi/3$ is most nearly:
(A) 0.88
(B) 1.32
(C) 1.74
(D) 1.91

Problem 11. Find the approximate area of the surface obtained by rotating the following curve about the x-axis.

$$y=\cos\left(\frac{x}{2}\right) \quad (0\le x\le \pi)$$

(A) 11.4
(B) 13.1
(C) 15.6
(D) 18.4

Problem 12. Regarding infinite series σ and τ, which of the following is true?

$$\sigma = \sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3} \quad ; \quad \tau = \sum_{n=1}^{\infty} \frac{n \cos(n\pi)}{2^n}$$

(A) Both series are convergent.
(B) Series σ converges and series τ diverges.
(C) Series τ converges and series σ diverges.
(D) Both series are divergent.

Problem 13. In order to replace *sin* x by $x - x^3/6$ with an error of magnitude no greater than 2×10^{-3}, we must have $|x| <$ ☐
(A) 0.45
(B) 0.61
(C) 0.75
(D) 0.83

Problem 14. Find the equation for the plane tangent to surface f at the point $P(3, 5, -4)$.

$$f: \quad x^2 + y^2 - z^2 = 18$$

(A) $3x - 4y + 5z = 16$
(B) $2x + 4y - 5z = 18$
(C) $3x + 5y + 4z = 18$
(D) $3x + 5y + 4z = 20$

Problem 15. Find the directional derivative of $f(x,y) = \frac{xy}{\sqrt{2}}(x + y)$ at (1,1) in the direction of the unit vector $\vec{a} = \vec{i}/\sqrt{2} + \vec{j}/\sqrt{2}$.
(A) 1
(B) 2
(C) 3
(D) 4

Problem 16. Evaluate double integral I, where D is the region bounded by $y = 0$, $y = x^2$, and $x = 1$.

$$I = \iint_D x \cos y \, dA$$

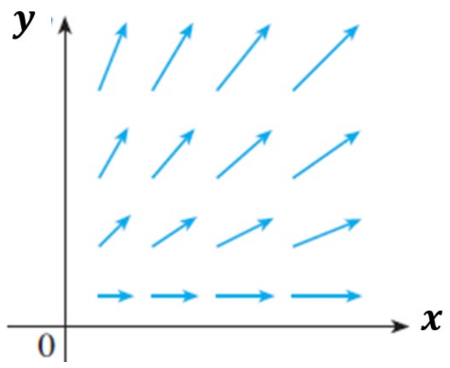
(A) $I \approx 0.23$
(B) $I \approx 0.36$
(C) $I \approx 0.46$
(D) $I \approx 0.72$

Problem 17. What is the curl of vector field **F** at point (0,1,1)?

$$\mathbf{F}(x,y,z) = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$$

(A) $\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$
(B) $\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$
(C) $\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$
(D) $\vec{0}$

Problem 18. For the two-dimensional vector field **F** sketched below, which of the following is **true**?



(A) div(**F**) < 0; |curl(**F**)| > 0
(B) div(**F**) = 0; |curl(**F**)| > 0
(C) div(**F**) > 0; |curl(**F**)| > 0
(D) div(**F**) > 0; |curl(**F**)| = 0

Problem 19. A table of values of a function $f(x,y)$ with continuous gradient is given. Find the integral I, where C is a curve described parametrically by the vector function $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (t^3 + t)\mathbf{j}$, $0 \leq t \leq 1$.

$$I = \int_C \nabla f \cdot d\mathbf{r}$$

x \ y	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

(A) 2
(B) 4
(C) 6
(D) 8

Problem 20. Find the work done by force **F** over the curve **r** for $0 \leq t \leq 1$ in the direction of increasing t.

$$\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$$

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k}$$

(A) 1/2
(B) 1
(C) 3/2
(D) 2

Problem 21. Find the optimum values of x_1 and x_2 for the following constrained maximization problem.

$$\max \ \ y = 2x_1 + 3x_2 \ \ ; \ \text{subject to } 2x_1^2 + 5x_2^2 = 10$$

(A) $x_1 = \frac{3}{5}\sqrt{\frac{17}{19}}$; $x_2 = \sqrt{\frac{18}{19}}$

(B) $x_1 = \frac{5}{3}\sqrt{\frac{16}{19}}$; $x_2 = \sqrt{\frac{18}{19}}$

(C) $x_1 = \frac{5}{3}\sqrt{\frac{17}{19}}$; $x_2 = \sqrt{\frac{17}{19}}$

(D) $x_1 = \frac{5}{3}\sqrt{\frac{18}{19}}$; $x_2 = \sqrt{\frac{18}{19}}$

Problem 22. Expression Z shown next can be simplified to give (note that $i = \sqrt{-1}$):

$$Z = \left(\frac{1+i}{\sqrt{2}}\right)^8 + \left(\frac{1-i}{\sqrt{2}}\right)^8$$

(A) $\sqrt{2}/2$
(B) 1
(C) $2i$
(D) 2

Problem 23. What is the value of the following limit?

$$\lim_{z \to i} \frac{z^2 + z + 1 - i}{z^2 - 3iz - 2}$$

(A) 0
(B) i
(C) $-2 + i$
(D) The limit does not exist.

Problem 24. The function $f(z) = xy + i(xy + x)$ has a derivative at exactly one point. What is the complex number z defined by this point? Is the function analytic at this point?
(A) $z = 1/2 + i/2$; the function is not analytic at this point.
(B) $z = -1/2 + i/2$; the function is analytic at this point.
(C) $z = -1/2 - i/2$; the function is analytic at this point.
(D) $z = -1/2 - i/2$; the function is not analytic at this point.

Problem 25. Letting $\tilde{z} = x - iy$, find the integral I along the complex contour C defined by the parabola $y = (1 - x)^2$.

$$I = \int_i^1 \tilde{z} dz$$

(A) $-2i/3$
(B) $-i/3$
(C) $-i/6$
(D) $i/3$

Problem 26. The residue of complex function $f(z)$ at $z = 3$ is:

$$f(z) = \frac{1}{(z+3)^2(z-3)^2}$$

(A) –2/81
(B) –1/108
(C) –1/216
(D) 1/108

Problem 27. A fair die is rolled twice. The probability that an even number will follow an odd number is:
(A) 1/6
(B) 1/4
(C) 1/3
(D) 2/3

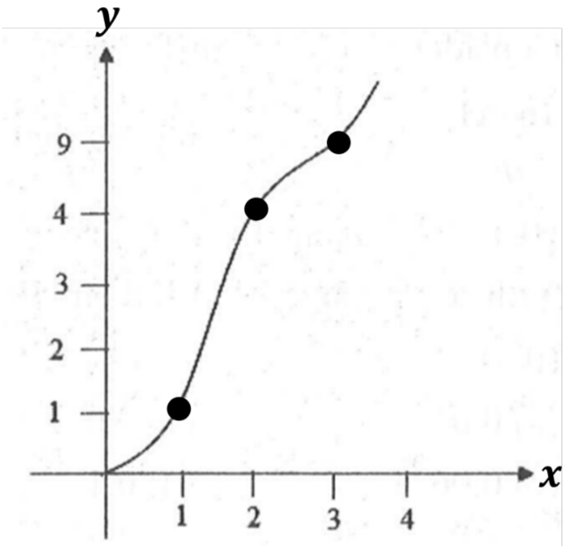
Problem 28. The mean and standard deviation of a normally distributed dataset are 25 and 6, respectively. Suppose that 25 is subtracted from every term in the dataset, and then the result is divided by 6. Which of the following best describes the resulting distribution?
(A) It has a mean of 0, a standard deviation of 6, and its shape is unknown.
(B) It has a mean of 0, a standard deviation of 6, and its shape is normal.
(C) It has a mean of 1, a standard deviation of 0, and its shape is unknown.
(D) It has a mean of 0, a standard deviation of 1, and its shape is normal.

Problem 29. A manufacturer produces ping-pong balls that are packed into boxes of 200 units. If quality control studies indicate that 1% of the ping-pong balls produced are defective, what percentage of the boxes will contain 2 or more defectives? Assume Poisson statistics.
(A) 47.1%
(B) 59.4%
(C) 66.2%
(D) 70.0%

Problem 30. An integer between 1 (inclusive) and 300 (inclusive) is chosen at random. What is the probability that the number chosen is a multiple of 3 **or** 5?
(A) 1/3
(B) 2/5
(C) 7/15
(D) 8/15

Problem 31. During the month of July, the probability that it will rain in a given day is 3/5. In a day without rain, the probability that soccer team Engineering Mathematics United will win a game is 7/10. If there is rain, the probability that EMU will win a game is 3/10. Knowing that EMU won a game in July, the probability that there was rain in that day is:
(A) 7/23
(B) 9/22
(C) 9/23
(D) 11/23

Problem 32. The following curve on the *xy*-plane passes through three known points (1,1), (2,4), and (3,9). Using the trapezoidal rule with an interval spacing $h = 1$, the area under the curve in the interval $x \in [1; 3]$ is:

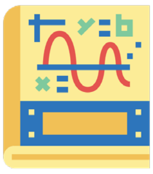


(A) 8
(B) 26/3
(C) 9
(D) 28/3

Problem 33. A nonzero polynomial $f(x)$ of degree 3 has roots at abscissae $x = 1$, $x = 2$, and $x = 3$. In view of this information, which of the following must be **true**?
(A) $f(0) \times f(4) < 0$
(B) $f(0) \times f(4) > 0$
(C) $f(0) + f(4) < 0$
(D) $f(0) + f(4) > 0$

Problem 34. If, for some continuous function f, we have $f(1) = 2$, $f(2) = 4$, and $f(4) = 14$, what is the value of $f(3)$ obtained with Lagrange interpolation?
(A) 7
(B) 8
(C) 9
(D) 11

Problem 35. If the equation $sin(x) - x^2 = 0$ is solved by the Newton-Raphson method with an initial guess $x = 1$, the value of x after 2 iterations will be, most nearly:
(A) 0.8770
(B) 0.8826
(C) 0.8914
(D) 0.9032



Answer key is on the next page!
Solutions also begin on the next page.

ANSWER KEY

Problem	Answer	Problem	Answer
1	D	19	C
2	B	20	A
3	B	21	D
4	C	22	D
5	B	23	C
6	A	24	D
7	B	25	A
8	D	26	B
9	B	27	B
10	B	28	D
11	B	29	B
12	A	30	C
13	C	31	C
14	C	32	C
15	C	33	A
16	A	34	B
17	D	35	A
18	D		

SOLUTIONS

1 ➡ D

Firstly, if A is 5 × 1, then A^T is 1 × 5. The matrix product of A (5 × 1) and A^T (1 × 5) will be 5 × 5. Finally, the product of this 5 × 5 matrix and matrix B (5 × 5) will be 5 × 5.

2 ➡ B

Let λ denote an eigenvalue of A, so that λ^2 is an eigenvalue of A^2. Since $A^2 = I$, and 1 is the only eigenvalue of the identity matrix I, we may write

$$\lambda^2 = 1 \rightarrow \boxed{\lambda = \pm 1}$$

3 ➡ B

Let matrix A be denoted as

$$A = \begin{pmatrix} m & 0 \\ 0 & 10-m \end{pmatrix}$$

This matrix has determinant $f(m) = m \times (10 - m) = 10m - m^2$. Clearly, $f(m)$ is a concave-down parabolic curve of the form $f(x) = ax^2 + bx$, $a < 0$, and as such attains a maximum value at $m = -b/2a = -10/2(-1) = 5$. It follows that

$$\left[\text{Det}(A)\right]_{\max} = f(m=5) = 10 \times 5 - 5^2 = \boxed{25}$$

4 ➡ C

In general, the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A, but the eigenvectors of both matrices are the same.

5 ➡ B

Let R denote a certain row and C denote a certain column. The matrix for the system under consideration is shown next.

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 3 & 0 & 4 \\ 1 & 2 & -1 & a \end{array}\right]$$

Making $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - R_1$, we obtain:

$$A=\left[\begin{array}{ccc|c}1 & 1 & 1 & 4\\ 0 & 1 & -2 & -4\\ 0 & 1 & -2 & a-4\end{array}\right]$$

Making $R_3 \to R_3 - R_2$, we obtain:

$$A=\left[\begin{array}{ccc|c}1 & 1 & 1 & 4\\ 0 & 1 & -2 & -4\\ 0 & 0 & 0 & a\end{array}\right]$$

The system described by the matrix above will have a solution if $a = 0$.

6 ➡ A

The key to solve this problem is to observe that, if $g(x)$ is to be differentiable for all values of x, it must be so at $x = 1$, which is the abscissa that separates the two expressions of piecewise function g. For this to be the case, both pieces of g must be equal at $x = 1$, and both pieces of the first derivative $g'(x)$ must be equal at $x = 1$. Applying the first requirement,

$$px^3+2qx^2+6=2px^2+qx+2$$
$$\therefore p\times1^3+2q\times1^2+6=2p\times1^2+q\times1+2$$
$$\therefore p+2q+6=2p+q+2$$
$$\therefore p-q=4\ \ \text{(I)}$$

To apply the second requirement, we differentiate $g(x)$,

$$g'(x)=\begin{cases}3px^2+4qx, & x\le 1\\ 4px+q, & x>1\end{cases}$$

Equating the two pieces of $g'(x)$ at $x = 1$,

$$3px^2+4qx=4px+q$$
$$\therefore 3p\times1^2+4q\times1=4p\times1+q$$
$$\therefore 3p+4q=4p+q$$
$$\therefore p=3q$$

Substituting in (I) and solving for q,

$$p-q=4\ \ \to\ \ 3q-q=4$$
$$\therefore 2q=4$$
$$\therefore \boxed{q=2}$$

7 ➡ B

Appealing to the definition of derivative and using $f(x + y) = f(x)f(y)$, we have

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{f(x)f(h)-f(x)}{h}$$
$$\therefore f'(x)=f(x)\lim_{h\to0}\frac{f(h)-1}{h}$$
$$\therefore f'(x)=f(x)\lim_{h\to0}\frac{\left[1+hg(h)\gamma(h)\right]-1}{h}$$
$$\therefore f'(x)=f(x)\lim_{h\to0}\frac{\cancel{h}g(h)\gamma(h)}{\cancel{h}}$$
$$\therefore f'(x)=f(x)\lim_{h\to0}g(h)\gamma(h)$$
$$\therefore f'(x)=ab\times f(x)$$
$$\therefore \boxed{k=ab}$$

8 ➡ D

Differentiating f once, we obtain

$$f(x) = x^2 e^{-x} \quad \rightarrow \quad f'(x) = 2xe^{-x} - x^2 e^{-x}$$

$$\therefore f'(x) = xe^{-x}(2 - x)$$

Setting the result above to zero, the resulting expression is satisfied if x = 0 or x = 2. To find whether these values correspond to maxima or minima, we differentiate f a second time:

$$f''(x) = -xe^{-x} + \left(e^{-x} - xe^{-x}\right)(-x + 2)$$

Substituting x = 0,

$$f''(x) = -0 \times e^{-0} + \left(e^{-0} - 0 \times e^{-0}\right)(-0 + 2) = 2 > 0$$

Since $f''(x = 0) > 0$, we conclude that x = 0 corresponds to a local minimum. Substituting x = 2,

$$f''(x) = -2 \times e^{-2} + \left(e^{-2} - 2 \times e^{-2}\right)(-2 + 2) = -2e^{-2} < 0$$

Since $f''(x = 2) < 0$, we conclude that x = 2 corresponds to a local maximum.

9 ➡ B

Separating variables and integrating,

$$\frac{dy}{dx} = \frac{-4x}{y} \quad \rightarrow \quad ydy = -4xdx$$

$$\therefore \int ydy = \int -4xdx$$

$$\therefore \frac{y^2}{2} = -2x^2 + C$$

$$\therefore \frac{y^2}{2} + 2x^2 = C$$

$$\therefore \frac{y^2}{4} + x^2 = C$$

$$\therefore \frac{y^2}{4C} + \frac{x^2}{C} = 1$$

This equation represents an ellipse.

10 ➡ B

Firstly, the derivative of y is $y' = -\tan x$. The differential arc element then becomes

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + (-\tan x)^2} = \sqrt{1 + \tan^2 x} = \sec x$$

so that

$$L = \int_0^{\pi/3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^{\pi/3} \sec x \, dx$$

$$\therefore L = \ln\left|\sec x + \tan x\right|\Big|_{x=0}^{x=\pi/3}$$

$$\therefore L = \ln\left|2 + \sqrt{3}\right| - \ln\left|1 + 0\right|$$

$$\therefore L = \ln\left|2 + \sqrt{3}\right| \approx \boxed{1.32}$$

11 ➡ B

In general, the surface area obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

In the present case, with $y = \cos(x/2)$ and $y' = -(1/2)\sin(x/2)$, we set up the integral

$$S=\int_0^{\pi}2\pi\cos\left(\frac{x}{2}\right)\sqrt{1+\left[-\frac{1}{2}\sin\left(\frac{x}{2}\right)\right]^2}\,dx$$

$$\therefore S=\int_0^{\pi}2\pi\cos\left(\frac{x}{2}\right)\sqrt{1+\frac{1}{4}\sin^2\left(\frac{x}{2}\right)}dx$$

Introducing the substitution $u = \sin(x/2)$, so that $du = (1/2)\cos(x/2)dx$, the integral above becomes

$$S=2\pi\int_0^{\pi}(2du)\times\sqrt{1+\frac{1}{4}u^2}$$

$$\therefore S=2\pi\int_0^{\pi}\sqrt{4+u^2}\,du$$

$$\therefore S=\left\{2\pi\left[\frac{1}{2}u\sqrt{4+u^2}+2\ln\left|u+\sqrt{u^2+4}\right|\right]\right\}_{u(0)=0}^{u(\pi)=1}$$

$$\therefore S=2\pi\left[\begin{array}{l}\frac{1}{2}\times1\times\sqrt{4+1^2}+2\ln\left|1+\sqrt{1^2+4}\right|\\-\frac{1}{2}\times0\times\sqrt{4+0^2}-2\ln\left|0+\sqrt{0^2+4}\right|\end{array}\right]$$

$$\therefore S=2\pi\left[\frac{\sqrt{5}}{2}+2\ln\left|1+\sqrt{5}\right|-2\ln|2|\right]$$

$$\therefore S=\pi\sqrt{5}+4\pi\ln\left(\frac{1+\sqrt{5}}{2}\right)\approx\boxed{13.1}$$

12 ➡ A

Consider first series σ. For increasing k, we can state that

$$\frac{k\sin^2 k}{1+k^3}\le\frac{k}{1+k^3}<\frac{k}{k^3}=\frac{1}{k^2}\quad(\forall k\ge1)$$

Since $\sum_{k=1}^{\infty}(1/k^2)$ is a convergent p-series, we conclude that σ is a convergent series by the comparison test.
Next, note that due to the periodicity of $\cos(n\pi)$, series τ can be restated as

$$\tau=\sum_{n=1}^{\infty}\frac{n\cos(n\pi)}{2^n}=\sum_{n=1}^{\infty}(-1)^n\frac{n}{2^n}$$

It is easy to see that $n/2^n$ is decreasing for $n \ge 2$ because the denominator 2^n increases much more rapidly than the numerator n with increasing n. This can be ascertained by noting that

$$\frac{d}{dx}\left(\frac{x}{2^x}\right)=2^{-x}-x\times\ln2\times2^{-x}=2^{-x}(1-x\ln2)<0\quad\left(\text{for }x>\frac{1}{\ln2}\right)$$

Also, from L'Hospital's rule,

$$\lim_{x\to\infty}\frac{x}{2^x}\overset{\text{L'H}}{=}\lim_{x\to\infty}\frac{1}{\ln2\times2^x}=0$$

It follows that τ converges by the alternating series test.

13 ➡ C

Firstly, recall that the third term in the series expansion of $\sin x$ is $|x|^5/5!$:

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}+O(x)$$

Thus, by the alternating series theorem,

$$\text{Error}=\frac{|x|^5}{5!}<2\times10^{-3}$$

$$\therefore |x|^5 < 5! \times 2\times 10^{-3}$$

$$\therefore |x| < \left(5! \times 2\times 10^{-3}\right)^{\frac{1}{5}}$$

$$\therefore \boxed{|x| < 0.752}$$

14 ➡ C

Firstly, the gradient of f is

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$$

Substituting the coordinates of P brings to

$$\nabla f = 2\times 3\mathbf{i} + 2\times 5\mathbf{j} - 2\times(-4)\mathbf{k}$$
$$\therefore \nabla f = 6\mathbf{i} + 10\mathbf{j} + 8\mathbf{k}$$

Thus, the tangent plane we aim for is

$$6(x-3) + 10(y-5) + 8(z+4) = 0$$

$$\therefore 6x - 18 + 10y - 50 + 8z + 32 = 0$$

$$\therefore 6x + 10y + 8z - 36 = 0$$

$$\therefore \boxed{3x + 5y + 4z = 18}$$

15 ➡ C

We first compute the gradient of f:

$$f(x,y) = \frac{1}{\sqrt{2}}\left(x^2y + xy^2\right)$$

$$\therefore \nabla f(x,y) = \frac{1}{\sqrt{2}}\left(2xy + y^2\right)\mathbf{i} + \frac{1}{\sqrt{2}}\left(x^2 + 2xy\right)\mathbf{j}$$

Substituting (x = 1, y = 1):

$$\nabla f(1,1) = \frac{1}{\sqrt{2}}\left(2\times 1\times 1 + 1^2\right)\mathbf{i} + \frac{1}{\sqrt{2}}\left(1^2 + 2\times 1\times 1\right)\mathbf{j}$$

$$\therefore \nabla f(1,1) = \frac{3}{\sqrt{2}}\mathbf{i} + \frac{3}{\sqrt{2}}\mathbf{j}$$

The unit vector given is

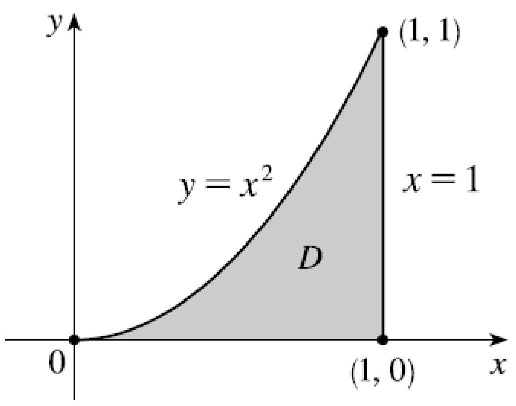
$$\mathbf{a} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

so that

$$\nabla f(1,1)\cdot\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{\sqrt{2}}\times\frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}}\times\frac{1}{\sqrt{2}} = \boxed{3}$$

16 ➡ A

The integration region D is sketched below.



The integral we're looking for is then

$$I = \iint_D x\cos y\, dA = \int_0^1\int_0^{x^2} x\cos y\, dy dx$$

$$\therefore I=\int_0^1\left[\left(x\sin y\right)\Big|_{y=0}^{y=x^2}\right]dx$$

$$\therefore I=\int_0^1 x\sin x^2 dx$$

$$\therefore I=\left[-\frac{1}{2}\cos\left(x^2\right)\right]\Bigg|_{x=0}^{x=1}=-\frac{1}{2}\left[\cos\left(1^2\right)-\cos\left(0\right)\right]$$

$$\therefore I=\frac{1}{2}\left[1-\cos\left(1\right)\right]\approx\boxed{0.23}$$

17 ➡ D

The curl of vector field **F** is given by

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial(\ )/\partial x & \partial(\ )/\partial y & \partial(\ )/\partial z\\ xy^2z^2 & x^2yz^2 & x^2y^2z\end{vmatrix}$$

$$\therefore\nabla\times\mathbf{F}=\left[\frac{\partial}{\partial y}\left(x^2y^2z\right)-\frac{\partial}{\partial z}\left(x^2yz^2\right)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}\left(x^2y^2z\right)-\frac{\partial}{\partial z}\left(xy^2z^2\right)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}\left(x^2yz^2\right)-\frac{\partial}{\partial y}\left(xy^2z^2\right)\right]\mathbf{k}$$

$$\therefore\nabla\times\mathbf{F}=\left(2x^2yz-2x^2yz\right)\mathbf{i}-\left(2xy^2z-2xy^2z\right)\mathbf{j}+\left(2xyz^2-2xyz^2\right)\mathbf{k}$$

$$\therefore\nabla\times\mathbf{F}=0\mathbf{i}-0\mathbf{j}+0\mathbf{k}$$

$$\therefore\boxed{\nabla\times\mathbf{F}_{(0,1,1)}=\mathbf{0}}$$

18 ➡ D

The vector field has the form $\boldsymbol{F}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$, but we know that $R=0$ for the field in question. In addition, P and Q do not vary in the z-direction, so

$$\frac{\partial R}{\partial x}=\frac{\partial R}{\partial y}=\frac{\partial R}{\partial z}=\frac{\partial P}{\partial z}=\frac{\partial Q}{\partial z}=0$$

As x increases, the x-component of each vector of **F** increases while the y-component remains constant, so $\partial P/\partial x>0$ and $\partial Q/\partial x=0$. Similarly, as y increases, the y-component of each vector increases while the x-component remains constant, so $\partial Q/\partial y>0$ and $\partial P/\partial y=0$. It follows that

$$\nabla\cdot\mathbf{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+0>0$$

and

$$\nabla\times\mathbf{F}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$\therefore\nabla\times\mathbf{F}=\left(0-0\right)\mathbf{i}+\left(0-0\right)\mathbf{j}+\left(0-0\right)\mathbf{k}=\mathbf{0}$$

19 ➡ C

Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a\le t\le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then, by the fundamental theorem for line integrals,

$$\int_C\nabla f\cdot d\mathbf{r}=f\left(\mathbf{r}\left(b\right)\right)-f\left(\mathbf{r}\left(a\right)\right)$$

In the present case, $a=0$ and

$$\mathbf{r}\left(t=0\right)=\left(0^2+1\right)\mathbf{i}+\left(0^3+0\right)\mathbf{j}=\mathbf{i}+0\mathbf{j}$$

In the given table, we read $f(1,0)=3$.
Then, with $b=1$,

$$\mathbf{r}\left(t=1\right)=\left(1^2+1\right)\mathbf{i}+\left(1^3+1\right)\mathbf{j}=2\mathbf{i}+2\mathbf{j}$$

In the given table, we read $f(2,2) = 9$. Lastly, the integral at hand is evaluated as

$$\int_C \nabla f \cdot d\mathbf{r} = f(2,2) - f(1,0) = 9 - 3 = \boxed{6}$$

20 ➡ A

The first step is to restate force field **F** in terms of the parametric variable *t:*

$$\mathbf{F} = t \times t^2\mathbf{i} + t^2\mathbf{j} - t^2 \times t\mathbf{k} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k}$$

Differentiating **r** with respect to *t*, we have

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\,\mathbf{j} + \mathbf{k}$$

Then, we compute the dot product

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 \times 1 + t^2 \times 2t - t^3 = 2t^3$$

Lastly, the work *W* is calculated as

$$W = \int_0^1 \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2\int_0^1 t^3 dt = 2 \times \frac{t^4}{4} = \boxed{\frac{1}{2}}$$

21 ➡ D

The Lagrangean function that describes this optimization problem is

$$L = 2x_1 + 3x_2 + \lambda\left(10 - 2x_1^2 - x_2^2\right)$$

The first-order conditions constitute the following system of equations:

$$\begin{cases} 2 - 4\lambda x_1 = 0 \;\; \text{(I)} \\ 3 - 10\lambda x_2 = 0 \;\; \text{(II)} \\ 10 - 2x_1^2 - 5x_2^2 = 0 \;\; \text{(III)} \end{cases}$$

Solving (II) for λ gives $\lambda = 3/(10x_2)$. Substituting in (I), we have

$$2 - 4\lambda x_1 = 0 \;\; \rightarrow \;\; 2 - 4 \times \frac{3}{10x_2} \times x_1 = 0$$

$$\therefore 2 - \frac{12x_1}{10x_2} = 0$$

$$\therefore 2 = \frac{12x_1}{10x_2}$$

$$\therefore x_1 = \frac{5}{3}x_2 \;\; \text{(IV)}$$

Substituting into (III) and solving for x_1,

$$10 - 2x_1^2 - 5x_2^2 = 0 \;\; \rightarrow \;\; 10 - 2 \times \frac{25}{9}x_2^2 - 5x_2^2 = 0$$

$$\therefore 10 - \frac{50}{9}x_2^2 - 5x_2^2 = 0$$

$$\therefore 2 - \frac{10}{9}x_2^2 - x_2^2 = 0$$

$$\therefore 2 - \frac{19}{9}x_2^2 = 0$$

$$\therefore \boxed{x_2 = \sqrt{\frac{18}{19}}}$$

Finally, we substitute x_2 into (IV) to obtain

$$x_1 = \frac{5}{3}x_2 = \boxed{\frac{5}{3}\sqrt{\frac{18}{19}}}$$

22 ➡ D

Replacing the complex numbers with trigonometric notation, we obtain:

$$Z=\left(\frac{1+i}{\sqrt{2}}\right)^8+\left(\frac{1-i}{\sqrt{2}}\right)^8=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^8+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)^8$$

$$\therefore Z=\left[\operatorname{cis}\left(\frac{\pi}{4}\right)\right]^8+\left[\operatorname{cis}\left(-\frac{\pi}{4}\right)\right]^8$$

$$\therefore Z=\operatorname{cis}\left(8\times\frac{\pi}{4}\right)+\operatorname{cis}\left[8\times\left(-\frac{\pi}{4}\right)\right]$$

$$\therefore Z=\operatorname{cis}(2\pi)+\operatorname{cis}(-2\pi)$$

$$\therefore Z=1+1=\boxed{2}$$

23 ➡ C

The function inside the limit is not defined at $z = i$ because the denominator becomes zero at this point. We can easily circumvent this problem by using L'Hospital's rule:

$$\lim_{z\to i}\frac{z^2+z+1-i}{z^2-3iz-2}\overset{\text{L'H}}{=}\lim_{z\to i}\frac{2z+1}{2z-3i}$$

$$\therefore \lim_{z\to i}\frac{2z+1}{2z-3i}=\frac{2\times i+1}{2\times i-3i}=\frac{2i+1}{-i}$$

$$\therefore \lim_{z\to i}\frac{2z+1}{2z-3i}=\boxed{-2+i}$$

24 ➡ D

Let $u = xy$ and $v = xy + x$. We proceed to compute the derivatives

$$\frac{\partial u}{\partial x}=y\quad;\quad\frac{\partial v}{\partial y}=x$$

so that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\quad\to\quad y=x$$

Next, we write

$$\frac{\partial u}{\partial y}=x\quad;\quad-\frac{\partial v}{\partial x}=-y-1$$

so that

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\quad\to\quad x=-y-1$$

Gleaning our results, the function has a derivative at a point whose coordinates (x,y) satisfy

$$\begin{cases}x=y\\x=-y-1\end{cases}$$

Substituting the first equality into the second and solving for x,

$$x=-x-1$$
$$\therefore 2x=-1$$
$$\therefore x=-\frac{1}{2}$$
$$\therefore y=x=-\frac{1}{2}$$

Thus, the function has a derivative at $z = -1/2 - i/2$. However, f is not analytic anywhere because its derivative does not exist in a domain, only at a certain point.

25 ➡ A

Firstly, we restate the integral as

$$I = \int_i^1 \tilde{z}dz = \int_{0,1}^{1,0} (x - iy)(dx + i\ dy)$$

$$\therefore I = \int_{x=0}^{x=1} xdx + \int_{y=1}^{y=0} ydy + i\int_{y=1}^{y=0} xdy - i\int_{x=0}^{x=1} ydx$$

$$\therefore I = \frac{1}{2} - \frac{1}{2} + i\int_{y=1}^{y=0} xdy - i\int_{x=0}^{x=1} ydx$$

$$\therefore I = i\int_{y=1}^{y=0} xdy - i\int_{x=0}^{x=1} ydx$$

Then, using the substitutions $y = (x - 1)^2$ and $dy = 2(x - 1)dx$,

$$I = i\int_{y=1}^{y=0} x \times 2(x-1)dx - i\int_{x=0}^{x=1} (x-1)^2\, dx$$

$$\therefore I = 2i\int_0^1 (x^2 - x)dx - i\int_0^1 (x-1)^2\, dx$$

$$\therefore I = 2i\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_{x=0}^{x=1} - i\left[\frac{(x-1)^3}{3}\right]\Bigg|_{x=0}^{x=1}$$

$$\therefore I = 2i\left(\frac{1}{3} - \frac{1}{2}\right) - i \times \frac{1}{3} = \boxed{-\frac{2i}{3}}$$

26 ➡ B

Firstly, note that $z = 3$ is a pole of order 2. Accordingly, the corresponding residue is given by

$$\text{Residue} = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\left((z-a)^n f(z)\right)\right]_{z=a}$$

$$\therefore \text{Residue} = \frac{1}{(2-1)!} \times \frac{d}{dz}\left[(z-3)^2 \times \frac{1}{(z+3)^2(z-3)^2}\right]_{z=3}$$

$$\therefore \text{Residue} = 1 \times \left[\frac{-2}{(z+3)^3}\right]_{z=3}$$

$$\therefore \text{Residue} = 1 \times \left[\frac{-2}{(3+3)^3}\right] = -\frac{2}{216} = \boxed{-\frac{1}{108}}$$

27 ➡ B

The probability of getting an odd number is 3/6 or 1/2. Likewise, the probability of getting an even number is 3/6 or 1/2. The probability that an odd number will follow an even number is 1/2 × 1/2 = 1/4.

28 ➡ D

The effect on the mean of a dataset when we subtract the same value from each entry is to reduce the old mean by that value (i.e., $\mu_{x-k} = \mu_x - k$). Because the original mean was 25, and every data point was reduced by 25, the mean of the modified dataset will be 25 – 25 = 0. In turn, the effect on the standard deviation of a dataset as each term is divided by the same value is to divide the standard deviation by that value, so that $\sigma_{x/k} = \sigma_x/k = 6/6 = 1$. The student should be aware that the process of subtracting the mean from each term and dividing by the SD creates a set of z-scores

$$z_x = \frac{x - \overline{x}}{s}$$

so that any complete set of z-scores has a mean of 0 and a standard deviation of 1. The shape is normal since any linear transformation of a normal distribution will still be normal.

29 ➡ B

Following the problem statement, the number X of defective balls in a box is Poisson-distributed with rate parameter $\lambda = n \times p = 200 \times 0.01 = 2$. To find the percentage of boxes that will contain 2 or more defectives, we write

$$P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$$

$$\therefore P(X \geq 2) = 1 - \left(\frac{e^{-2} \times 2^0}{0!} + \frac{e^{-2} \times 2^1}{1!}\right)$$

$$\therefore P(X \geq 2) = 1 - (0.135 + 0.271)$$

$$\therefore P(X \geq 2) = 0.594 = \boxed{59.4\%}$$

30 ➡ C

Let A and B denote the events 'number is a multiple of 3' and 'number is a multiple of 5', respectively. Since there are 100 multiples of 3 in the set spanning 1 to 300, we may write $P(A) = 100/300 = 1/3$. Further, because there are 60 multiples of 5 spanning 1 to 300, we have $P(B) = 60/300 = 1/5$. Also, there are 20 numbers that multiply both 3 and 5, giving $P(A \cap B) = 20/300 = 1/15$. The probability that a number divides 3 or 5 then becomes

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore P(A \cup B) = \frac{1}{3} + \frac{1}{5} - \frac{1}{15} = \boxed{\frac{7}{15}}$$

31 ➡ C

This is a straightforward application of Bayes' theorem:

$$P(\text{It rained/EMU won}) = \frac{P(\text{It rained})P(\text{EMU won/It rained})}{P(\text{It rained})P(\text{EMU won/It rained}) + P(\text{No rain})P(\text{EMU won/No rain})}$$

$$\therefore P(\text{It rained/EMU won}) = \frac{\frac{3}{5} \times \frac{3}{10}}{\frac{3}{5} \times \frac{3}{10} + \frac{2}{5} \times \frac{7}{10}} = \boxed{\frac{9}{23}}$$

32 ➡ C

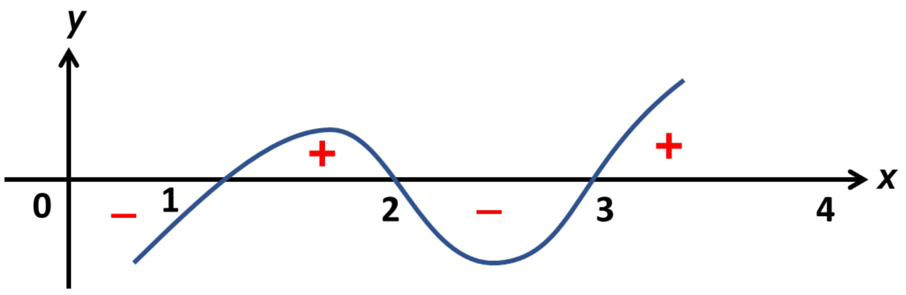
Let $f(x)$ denote the function whose function we were given. Then, by the trapezoidal rule,

$$\int_1^3 f(x)\,dx = \frac{h}{2} \times [y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})]$$

$$\therefore \int_1^3 f(x)\,dx = \frac{1}{2}(y_0 + 2y_2 + y_3) = \frac{1}{2} \times (1 + 2 \times 4 + 9) = \boxed{9}$$

33 ➡ A

A polynomial equation flips sign at every root (that is, whenever it crosses the x-axis), as illustrated below. Knowing that 1, 2 and 3 are roots of the polynomial at hand, we can state that $f(1) < 0$ (or > 0) and $f(4) > 0$ (or < 0), so that $f(1) \times f(4) < 0$.



34 ➡ B

As usual, the formula for Lagrange interpolation is

$$f(x)=\left[\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1 + (\ldots)+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n\right]$$

In the present case, we have three points, hence the interpolation is of order 2:

$$f(x)=\frac{(x-2)(x-4)}{(1-2)(1-4)}\times 2+\frac{(x-1)(x-4)}{(2-1)(2-4)}\times 4+\frac{(x-1)(x-2)}{(4-1)(4-2)}\times 14$$

$$\therefore f(x=3)=\frac{(3-2)(3-4)}{(1-2)(1-4)}\times 2+\frac{(3-1)(3-4)}{(2-1)(2-4)}\times 4+\frac{(3-1)(3-2)}{(4-1)(4-2)}\times 14$$

$$\therefore f(x=3)=\frac{1\times(-1)}{(-1)\times(-3)}\times 2+\frac{2\times(-1)}{1\times(-2)}\times 4+\frac{2\times 1}{3\times 2}\times 14$$

$$\therefore f(x=3)=-\frac{2}{3}+4+\frac{14}{3}$$

$$\therefore \boxed{f(x=3)=8}$$

35 ➡ A

Let the equation we want to solve be cast as a function $f(x)$ = *sin(x)* − x^2, so that $f'(x)$ = *cos(x)* −2x. Then, by the Newton-Raphson method, the abscissa x_{n+1} to be used in the (n + 1)-th iteration is

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{\sin(x_n)-x_n^2}{\cos(x_n)-2x_n}$$

With x_0 = 1, we get, in the first iteration,

$$x_1=x_0-\frac{\sin(x_0)-x_0^2}{\cos(x_0)-2x_0}=1-\frac{\sin(1)-1^2}{\cos(1)-2\times 1}=0.8914$$

With x_2 = 0.8914, we get, in the second iteration,

$$x_2=0.8914-\frac{\sin(0.8914)-0.8914^2}{\cos(0.8914)-2\times 0.8914}=\boxed{0.8770}$$

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