Montogue

Quiz GY202

Atmospheric and Oceanic Science

Advanced Problems

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Note: Use Ω = 7.3×10^{-5} rad/s as the rotational frequency of the Earth, unless stated otherwise.

PROBLEMS

Problem 1

In an unstable compressible troposphere:

A) Temperature decreases with height **faster** than the adiabatic lapse rate, and potential temperature **decreases** with height.

B) Temperature decreases with height **faster** than the adiabatic lapse rate, and potential temperature **increases** with height.

C) Temperature decreases with height **slower** than the adiabatic lapse rate, and potential temperature **decreases** with height.

D) Temperature decreases with height **slower** than the adiabatic lapse rate, and potential temperature **increases** with height.

Problem 2

A westerly zonal flow at 45° is forced to rise adiabatically over a north-south oriented mountain barrier. Before striking the mountain the westerly wind increases linearly toward the south at a rate of 20 m s^{-1}/1000 km. The crest of the mountain range is at the 85-kPa level and the tropopause, located at 30 kPa (300 mb), remains undisturbed by the forced ascent of the air. The surface pressure to the west of the mountain barrier is 100 kPa (1000 mb). True or false?

1.( ) The initial relative vorticity of the air is greater than 10^{-5} s^{-1}.

2.( ) As flow reaches the crest, it is deflected by 6 degrees of latitude towards the south. The relative vorticity at the end of the forced ascent trajectory has absolute value greater than 5.0×10^{-6} s^{-1}.

3.( ) If the current assumes a uniform speed of 8 m/s during its ascent to the crest, the radius of curvature of the streamlines at the crest has absolute value greater than 2400 km.

Problem 3.1

A homogeneous barotropic ocean of depth H = 4 km has a zonally symmetric geostrophic jet whose profile is given by the expression $\bar{u}_g = U \times \exp[-(y/L)^2]$, where U = 1 m/s and L = 250 km are constants. Compute the vertical velocity produced by convergence in the Ekman layer at the ocean bottom. What are the maximum values of velocity components $\bar{v}$ and $\bar{w}$ if eddy viscosity K = 10^{-3} m^{2}s^{-1} and Coriolis parameter f = 10^{-4} s^{-1}? (Assume that $\bar{w}$ and the viscous stress vanish at the surface.)

Problem 3.2

Using the approximate zonally averaged momentum equation $\partial\bar{u}/\partial t = f\bar{v}$, compute the spin-down time for the zonal jet described in the previous part.

▶ Problem 4

Using the linearized form of the vorticity equation,

$$\frac{D_h\zeta}{Dt} = -f_0\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) - \beta v$$

and the β-plane approximation, derive the Rossby wave speed for a homogeneous incompressible ocean of depth h. Assume a motionless basic state and small perturbations that depend only on x and t:

$$u = u'(x,t)\ ;\ v = v'(x,t)\ ;\ h = H + h'(x,t)$$

where H is the mean depth of the ocean. With the aid of the continuity equation for a homogeneous layer and the geostrophic wind relationship $v' = gf_0^{-1}\partial h'/\partial x$, show that the perturbation vorticity equation can be written in the form

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} - \frac{f_0^2}{gH}\right)h' + \beta\frac{\partial h'}{\partial x} = 0$$

and that $h' = h_0 exp[ik(x - ct)]$ is a solution provided that

$$c = -\frac{\beta}{\left(k^2 + \dfrac{f_0^2}{gH}\right)}$$

If the ocean is 3 km deep, what is the Rossby wave speed at latitude 45° for a wave of zonal wavelength equal to 800 km?

▶ Problem 5

The linearized form of the quasi-geostrophic vorticity equation can be written as follows:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\nabla^2\psi' + \beta\frac{\partial \psi'}{\partial x} = -f_0\nabla\cdot\mathbf{V}$$

Suppose that the horizontal divergence field is given by

$$\nabla\cdot\mathbf{V} = A\cos\left[k\left(x - ct\right)\right]$$

where A is a constant. Find a solution for the corresponding relative vorticity field. What is the phase relationship between vorticity and divergence? For what value of c does vorticity become infinite?

▶ Problem 6

Consider a thermally stratified fluid contained in a rotating annulus of inner radius 0.8 m, outer radius 1.0 m, and depth 0.1 m. The temperature at the bottom boundary is held constant at T_0. The fluid is assumed to satisfy the equation of state $\rho = \rho_0[1 - \varepsilon(T - T_0)]$, where ρ_0 is density at initial temperature T_0, ρ is density at temperature T, and ε is the thermal expansion coefficient. If the temperature increases linearly with height along the outer radial boundary at a rate of 1°C cm^{-1} and is constant with height along the inner radial boundary, determine the geostrophic velocity at the upper boundary for a rotation rate of Ω = 1 rad s^{-1}. Assume that the temperature depends linearly on radius at each level. Take $\rho_0 = 10^3$ kg/m^3 and $\varepsilon = 2\times10^{-4}$ K^{-1}.

A) u_g = 0.104 cm/s
B) u_g = 0.245 cm/s
C) u_g = 0.385 cm/s
D) u_g = 0.504 cm/s

▶ Problem 7

If the value of the β parameter equals zero, the critical wavelength for a baroclinic instability does not depend on the magnitude of the basic state thermal wind U_T. The growth rate, however, does depend on U_T. Indeed, the exponential growth rate is stated as $\alpha = kc_i$, where k is wavenumber and c_i designates the imaginary part of the phase speed. In the present case,

$$\alpha = kU_T \left(\frac{2\lambda^2 - k^2}{2\lambda^2 + k^2} \right)^{1/2}$$

where λ is the wavelength. This expression indicates that growth rate increases linearly with the mean thermal wind. Use the expression above to show that the maximum growth rate for baroclinic instability when $\beta = 0$ occurs for a squared wavenumber k^2 such that

$$k^2 = 2\lambda^2 \left(\sqrt{2} - 1 \right)$$

How long does it take the most rapidly growing wave to amplify by a factor of e^1 if $\lambda = \sqrt{2} \times 10^{-6}$ m^{-1} and $U_T = 20$ m/s?

▶ Problem 8

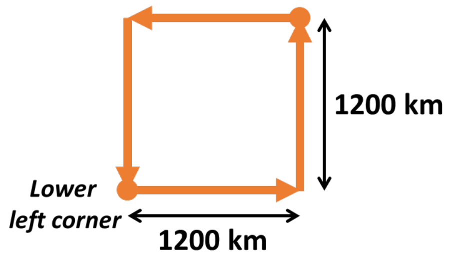
Regarding the theories of atmospheric and oceanic dynamics, are the following statements true or false?

1.( ) A region of the atmosphere has temperature of 47°C, pressure of 1010 hPa, and saturation-mixing ratio of 22 g/kg. Taking $c_p = 1000$ J/kg·K and $L_c = 2.5\times10^6$ J/kg as the latent heat of vaporization of air, the corresponding saturation lapse rate in this region is greater than 3.8 K/km.

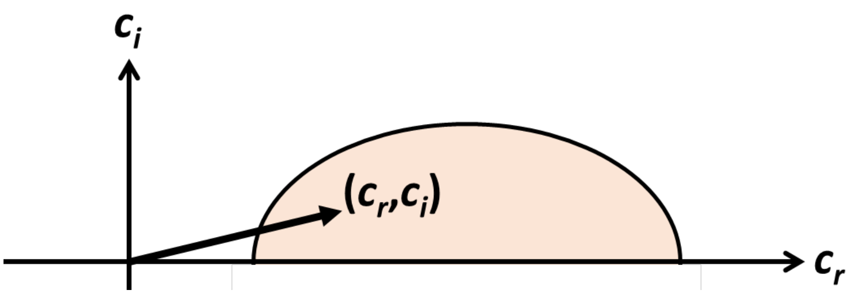
2.( ) The geostrophic wind speed around a low-pressure region is 16 m/s. The corresponding gradient wind speed, assuming a Coriolis parameter of 10^{-4} s^{-1} and a radius of curvature of 320 km, is greater than 11 m/s.

3.( ) A background jet stream of speed equal to 50 m/s meanders with 3000 km wavelength while centered at 30°N. The barotropic Rossby wave formed by the jet has phase speed relative to the ground greater than 38 m/s.

4.( ) Consider the rate of change of circulation about a square in the *xy*-plane with sides of 1200-km length if temperature increases eastward at a rate of 1°C/200 km and pressure increases northward at a rate of 1 mb/200 km. The pressure at the origin (the lower left corner; see illustration below) is 1100 mb. The rate of change of circulation under these conditions is greater than 10 m^{2}s^{-2}.



5.( ) Howard and Pedlosky showed that the complex phase velocity of a wave arising from barotropic instability must lie inside, or on, a semi-elliptical region on the wave speed plane (c_r, c_i), where $c = c_r + ic_i$, as illustrated below. Importantly, the ellipse in question always has eccentricity greater than 1.



6.( ) Barotropic instability is a possible energy source for some equatorial disturbances. Consider the following profile for an easterly jet near the equator,

$$\bar{u}(y) = -u_0 \sin^2\left[\ell(y - y_0)\right]$$

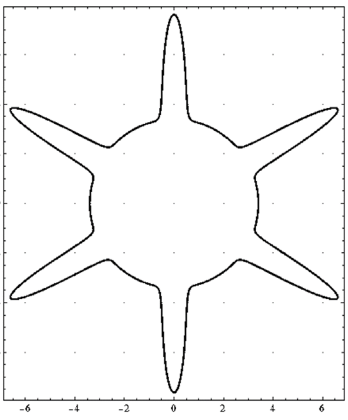
where u_0, y_0, and ℓ are constants and y is the distance from the equator. It can be shown that this profile will be barotropically unstable if $3u_0\ell^2 > \beta$, where β is the β-plane parameter.

7.( ) Raindrops can be accurately modelled as quasi-oblate spheroids. An oblate spheroid has separate horizontal and vertical symmetry and can have its shape described by the so-called axis ratio $\alpha_N = b/a$, where a and b are the lengths of the semi-major (horizontal) and semi-minor (vertical) axes, respectively. Empirical investigations show that the axis ratio of raindrops increases with drop size, indicating that drops become more round (less flat) as they become larger.

8.( ) Unlike raindrops, shapes of ice and snow crystals are not spheroids; rather, they take on geometries symmetric with respect to a central axis, and hence can be better represented by polar coordinates. One simple expression used for this purpose is

$$r = a\left[\sin^2(n\theta)\right]^b + c$$

where r and θ are the radial and angular coordinates, respectively, and a, b, and c are fitting constants. Parameter n determines the number of symmetric sides of the crystal, which is simply $2n$. Plotting the relation above with a = 4.22, b = 12, c = 3.4 and $2n$ = 6 yields a snowflake with the following shape.



9.( ) The kinematic viscosity of air at 288 K is 1.48×10^{-5} m^{2}s^{-1}, and a typical free-atmosphere turbulent dissipation rate at this temperature is 5×10^{-4} m^{2}s^{-3}. Given these values, the corresponding Kolmogorov scale in a cloud-free atmosphere is calculated to be greater than 2 mm.

10.( ) Assume that equivalent potential temperature, θ_e, can be used to model a saturated air parcel undergoing pseudoadiabatic ascent. Let θ_e be approximated by $\theta_e = \theta exp(L_c q_s/c_p T)$, where θ is potential temperature, L_c is the latent heat of vaporization, q_s is the saturation mixing ratio (i.e., the mass of water vapor per unit mass of dry air in a saturated parcel), c_p is the constant-pressure specific heat, and T is temperature. An air parcel at 920 mb with temperature 20°C is saturated (mixing ratio 16 g kg^{-1}). Taking c_p = 1000 J/kgK and $L_c = 2.5\times10^6$ J/kg as the latent heat of vaporization of water, the corresponding equivalent potential temperature is greater than 338 K.

11.( ) The time τ required to reach the peak mixed layer thickness in warm air mass genesis can be estimated with the implicit relation

$$\tau = c\left(\frac{\theta_0}{|g|\beta\gamma_0 e^{\beta\tau}}\right)^{1/3}$$

where c = 140 (dimensionless), θ_0 is near-surface temperature (K), g is gravitational acceleration (m/s^2), β is large-scale divergence (s^{-1}), γ_0 is the initial potential temperature (K/km) in the time profile $\gamma = \gamma_0 exp(\beta t)$. Using θ_0 = 20°C, $|g|$ = 9.81 m/s^2, $\beta = 10^{-6}$ s^{-1}, and γ_0 = 3.3 K/km, the equation above can be solved to yield a time for peak ML thickness greater than two and a half days.

12.( ) A 10-km thick cumulonimbus cloud over land has θ_v - $\hat{\theta}_v$ = 8 K, where θ_v denotes the potential virtual temperature of a rising parcel of air and $\hat{\theta}_v$ denotes the potential virtual temperature of the environment. The value of $\hat{\theta}_v$ between 0 and 10 km is 288 K. Using these parameters, the convective available potential energy (CAPE) for the cloud in question is calculated to be greater than 2800 m^2s^{-2}.

13.( ) As a follow-up to the previous statement, we may consider that CAPE is a measure of the energy available to accelerate a parcel vertically, and as such can be used to estimate the maximum vertical velocity of an air parcel with the relation $w = (2 \times CAPE)^{0.5}$. However, this equation tends to overestimate the vertical velocity achieved because mixing of environmental air with the thunderstorm updraft reduces the CAPE significantly. Assuming that only 10% of the CAPE obtained in the previous statement is converted to kinetic energy, we find a value of w greater than 20 m/s.

14.( ) Several models have been developed to explain the oscillatory nature of the El Niño-Southern Oscillation (ENSO). One early approach is the *delayed oscillator* model, in which the low-frequency interannual oscillations associated with ENSO are mainly explained by the reflection of Rossby waves at the western boundary of the Pacific. Importantly, reflection at the eastern boundary plays no role in the delayed oscillator model.

Recent research has indicated the existence of a novel ENSO-related phenomenon known as El Niño Modoki. It is distinguished from canonical El Niño, among other reasons, by a distinct sea surface temperature anomaly (SSTA) signature, with largest warming in the central tropical Pacific flanked by cooler SSTA on both sides.

15.( ) Researchers have gone on to produce a differential treatment of the features and effects of canonical ENSO and ENSO Modoki and found, for instance, that a large-scale warm SST anomaly in the extratropical North Pacific is seen during summer of an El Niño Modoki, while a large-scale cool anomaly prevails during summer of an El Niño. In spite of differences in definition, the societal impact patterns of canonical ENSO and ENSO Modoki in communities such as, say, the Pacific Northwest of the United States were found to be equivalent. ■ **(A black square indicates the end of a multi-paragraph statement.)**

The closest analogue to ENSO in the Atlantic Ocean is the so-called Atlantic Zonal Mode (AZM). The evolution of ENSO, in its developing phase, crucially depends on wind stress anomalies over the western and equatorial Pacific, the so-called westerly wind bursts and easterly wind surges. These wind events force equatorial Kelvin waves that influence thermocline depth and sea-surface temperatures (SST) to the east. Thus, there is a strong link between anomalous wind stress and SST.

16.( ) The same such relationship holds in the equatorial Atlantic, as westerly wind events in this ocean are always followed by warm SST anomalies. ■

17.( ) The following information describes the inflow of calcium ion from a river into the oceans. Assuming that the only source of Ca^{2+} is the river water in question, the residence time of calcium in the ocean is found to be greater than 2 million years.

Calcium ion concentration In seawater	0.41 g/kg
Mass of seawater in the oceans	1.4×10^{21} kg
Calcium ion concentration By weight in river water	12.5 g/kg
Average salt content of river water by percent mass	0.12%
Annual river runoff Into the oceans	3.6×10^{16} kg/yr

18.( ) In the so-called Sverdrup balance, the meridional transport of the entire water column is directly related and locally determined by the divergence of the wind stress.

▶ Problem 9

Suppose the drag coefficient for wind flow on a horizontal plane located z_R above the ground can be estimated as

$$C_D = \frac{\kappa^2}{\left[\ln\left(z_R/z_0\right)\right]^2}$$

where $\kappa \approx 0.41$ is the von Kármán constant, z_R is the reference height, which in this problem we take as 10 m, and z_0 is the roughness height, which we take as 1 m. Suppose next that the friction or shear velocity u^* is related to the wind flow velocity u_{z_0} at the reference height by the simple expression

$$\left(u^*\right)^2 = C_D\left(u_{z_0}\right)^2$$

Assuming a wind speed of 9 m/s at 10-m altitude and a wind density equal to 1.2 kg/m^3, compute the wind shear at the surface level.

A) τ_0 = 0.412 N/m^2
B) τ_0 = 1.02 N/m^2
C) τ_0 = 2.11 N/m^2
D) τ_0 = 3.07 N/m^2

▶ Problem 10

Suppose that the relative vorticity at the top of the Ekman layer at 15°N is $\zeta = 2\times10^{-5}$ s^{-1}. Let the eddy viscosity coefficient be $K_m = 10$ m^{2}s^{-1}, and the water vapor mixing ratio at the top of the Ekman layer be 12 g kg^{-1}. Estimate the precipitation rate owing to moisture convergence in the Ekman layer. Use $\rho = 1$ kg/m^3 and $\rho_w = 1000$ kg/m^3 as estimates of the densities of air and water, respectively.

A) *Precipitation* = 3.77 mm/day
B) *Precipitation* = 7.54 mm/day
C) *Precipitation* = 9.11 mm/day
D) *Precipitation* = 12.0 mm/day

→ Problem 11.1

Find the vertical velocity fluctuation for a Kelvin wave of zonal wavenumber 1, phase speed 40 m/s, and zonal velocity perturbation amplitude 5 m/s. Let the squared buoyancy frequency $N^2 = 4\times10^{-4}$ s^{-2}.

→ Problem 11.2

For the situation of Problem 11.1, compute the vertical momentum flux $M \equiv \rho_0\overline{u'w'}$. Show that M is constant with height.

→ Problem 11.3

Determine the form for the vertical velocity perturbation for the mixed Rossby-gravity wave corresponding to the u', v', and Φ' perturbations given by equations (12.32) of Holton (1992).

→ Problem 11.4

For a Rossby-gravity wave of zonal wavenumber 4 and phase speed 20 m/s, determine the latitude at which the vertical momentum flux $M \equiv \rho_0\overline{u'w'}$ is a maximum.

▶ Problem 12

Calculate the number of flashes per hour in a cylindrical cloud of radius equal to 0.8 km and thickness 2 km when two populations of particles, with number concentrations of 750 and 0.05 particles cm^{-3}, respectively, are present. Assume that the collision kernel between the populations is 10^{-4} cm^3 particle^{-1} s^{-1} and that the coalescence efficiency is 40 percent. Use 3.33×10^{-14} as the charge separation per collision and 3000 V/cm as threshold electric field strength.

A) *No. flashes per hour* = 5.05 flashes/hr
B) *No. flashes per hour* = 10.1 flashes/hr
C) *No. flashes per hour* = 14.9 flashes/hr
D) *No. flashes per hour* = 20.1 flashes/hr

SOLUTIONS

P.1 Solution

Regarding the relationship between atmospheric stability and variability of temperature with increasing height, we can state the general rules

$$\left.\begin{matrix}\text{Unstable}\\ \text{Neutral}\\ \text{Stable}\end{matrix}\right\}\text{if}\left(\frac{dT}{dz}\right)\left\{\begin{matrix}<-\Gamma_d\\ =-\Gamma_d\\ >-\Gamma_d\end{matrix}\right.$$

where Γ_d is the adiabatic lapse rate. Similarly, the relationship between changes in potential temperature and atmospheric stability is such that

$$\left.\begin{matrix}\text{Unstable}\\ \text{Neutral}\\ \text{Stable}\end{matrix}\right\}\text{if}\left(\frac{d\theta}{dz}\right)\left\{\begin{matrix}<0\\ =0\\ >0\end{matrix}\right.$$

- The correct answer is **C**.

P.2 Solution

1. True. Since the air motion has a westerly component only, the initial vorticity is

$$\zeta_0=-\frac{\partial u}{\partial y}=-\frac{20}{1000\times10^3}=\boxed{2.0\times10^{-5}\ \text{s}^{-1}}$$

2. False. First note that, for adiabatic flow, the potential vorticity

$$\frac{f+\zeta}{\delta p}$$

is conserved, so we may write

$$\frac{f_0+\zeta_0}{\delta p_0}=\frac{f_1+\zeta_1}{\delta p_1}$$

Here, $f_0 = 2\Omega sin\theta_0 = 2 \times (7.3\times10^{-5}) \times sin(45°) = 1.03\times10^{-4}$ s^{-1}, δp_0 = 100 – 30 = 70 kPa, $\zeta_0 = 2.0\times10^{-5}$ s^{-1}, $f_1 = 2\Omega sin\theta_1 = 2 \times (7.3\times10^{-5}) \times sin(45° – 6°)$ = 9.19×10^{-5} s^{-1}, and δp_1 = 85 – 30 = 55 kPa. Solving for vorticity ζ_1 brings to

$$\frac{f_0+\zeta_0}{\delta p_0}=\frac{f_1+\zeta_1}{\delta p_1}\rightarrow\frac{1.03\times10^{-4}+2.0\times10^{-5}}{70\times10^3}=\frac{9.19\times10^{-5}+\zeta_1}{55\times10^3}$$

$$\therefore\boxed{\zeta_1=-3.11\times10^{-6}\ \text{s}^{-1}}$$

3. True. For uniform current, $\zeta_1 = V/R$, where V = 8 m/s as given and $\zeta_1 = -3.11\times10^{-6}$, so

$$\zeta_1=\frac{V}{R}\rightarrow R=\frac{V}{\zeta_1}$$

$$\therefore R=\frac{8}{-3.11\times10^{-6}}=-2.57\times10^6\ \text{m}=\boxed{-2570\,\text{km}}$$

The negative sign indicates anticyclonic curvature.

P.3 Solution

Problem 3.1: Recalling that vertical velocity at the top of the Ekman layer is proportional to geostrophic vorticity, we write

$$\overline{w}_{\text{De}}=\zeta_g\left(\frac{K}{2f}\right)^{1/2}=-\frac{\partial\overline{u}_g}{\partial y}\left(\frac{K}{2f}\right)^{1/2}$$

The derivative in the rightmost side can be found from the velocity profile we were given,

$$\overline{u}_g=U\exp\left[-\left(\frac{y}{L}\right)^2\right]\rightarrow\frac{\partial\overline{u}_g}{\partial y}=\left(\frac{2yU}{L^2}\right)\exp\left(-\frac{y^2}{L^2}\right)$$

(Note that ordinary derivative notation would also be appropriate here, because the only variable in the velocity profile equation is *y*.) Thus,

$$\overline{w}_{\mathrm{De}} = \left(\frac{2yU}{L^2}\right)e^{-\left(\frac{y}{L}\right)^2}\left(\frac{K}{2f}\right)^{1/2} \quad \text{(I)}$$

The maximum $\overline{w}_{De}$ occurs at *y* for which $\partial\overline{w}_{\mathrm{De}}/\partial y = 0$, which can be solved to yield $y = L/\sqrt{2}$. Substituting in (I) brings to

$$\left(\overline{w}_{\mathrm{De}}\right)_{\max} = \left[\frac{2\times\dfrac{\left(250\times10^3\right)}{\sqrt{2}}\times1.0}{\left(250\times10^3\right)^2}\right]\times e^{-\left(\frac{L/\sqrt{2}}{L}\right)^2}\left(\frac{10^{-3}}{2\times10^{-4}}\right)^{1/2} = 7.67\times10^{-6}\ \text{m/s}$$

$$\therefore \boxed{\left(\overline{w}_{\mathrm{De}}\right)_{\max} = 7.67\times10^{-3}\ \text{mm/s}}$$

Now, appealing to the equation of continuity and solving for $\bar{v}$, we obtain

$$\int_0^H\left(\frac{\partial\overline{v}}{\partial y}\right)dz = \overline{w}_{\mathrm{De}} \quad \left(\begin{array}{c}\text{Integration bounds are chosen}\\ \text{because } H \gg \text{De}\end{array}\right)$$

$$\therefore \overline{v} = \frac{1}{H}\int\overline{w}_{\mathrm{De}}dy = \frac{U}{H}\left(\frac{K}{2f}\right)^{1/2}e^{-(y/L)^2} \quad \text{(II)}$$

$$\therefore \overline{v}_{\max} = \frac{1.0}{4000}\times e^{-\left(\frac{L/\sqrt{2}}{L}\right)^2}\left(\frac{10^{-3}}{2\times10^{-4}}\right)^{1/2} = 3.39\times10^{-6}\ \text{m/s}$$

$$\therefore \boxed{\overline{v}_{\max} = 0.339\ \text{mm/s}}$$

Problem 3.2: We were told to use the simplified momentum equation

$$\frac{\partial\overline{u}_g}{\partial t} = f\overline{v}$$

which, replacing $\bar{v}$ with (II) in the previous part, becomes

$$\frac{\partial\overline{u}_g}{\partial t} = f\frac{1}{H}\left(\frac{K}{2f}\right)^{1/2}\times\underbrace{Ue^{-(y/L)^2}}_{=\overline{u}_g} = \frac{f}{H}\left(\frac{K}{2f}\right)^{1/2}\overline{u}_g$$

$$\therefore \frac{\partial\overline{u}_g}{\partial t} = \tau^{-1}\overline{u}_g$$

where spin-down time τ is

$$\tau = \frac{H}{f}\left(\frac{2f}{K}\right)^{1/2} = \frac{4000}{10^{-4}}\times\left(\frac{2\times10^{-4}}{10^{-3}}\right)^{1/2} = 1.79\times10^7\ \text{sec}$$

$$\therefore \tau = 1.79\times10^7\ \text{sec}\times\frac{1}{86{,}400}\frac{\text{day}}{\text{sec}} = \boxed{207\ \text{days}}$$

P.4 ➡ *Solution*

From the quasi-geostrophic vorticity equation, we may write

$$\frac{\partial\zeta_g}{\partial t} + u\frac{\partial\zeta_g}{\partial x} + v\frac{\partial\zeta_g}{\partial y} + \beta v_g + f_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

For conditions of the present problem, the linearized version is

$$\frac{\partial\zeta_g'}{\partial t} + \beta v_g' + f_0\frac{\partial u'}{\partial x} = 0$$

but

$$v_g' = \frac{g}{f_0}\frac{\partial h'}{\partial x}$$

and

$$\zeta_g' = \frac{g}{f_0}\frac{\partial^2 h'}{\partial x^2}$$

From the perturbation equation

$$\frac{\partial h'}{\partial t} + \bar{u}\frac{\partial h'}{\partial x} + H\frac{\partial u'}{\partial x} = 0 \rightarrow \frac{\partial u'}{\partial x} = -\frac{1}{H}\frac{\partial h'}{\partial t}$$

we then have

$$\frac{\partial}{\partial t}\left(\frac{g}{f_0}\frac{\partial^2 h'}{\partial x^2}\right) + \beta\frac{g}{f_0}\frac{\partial h'}{\partial x} - \frac{f_0}{H}\frac{\partial h'}{\partial t} = 0$$

which readily reduces to the form given in the problem statement. Letting *h′ = Aexp[ik(x – ct)]* brings to

$$-ikc\left[-k^2 - \left(\frac{f_0^2}{gH}\right)\right] + ik\beta = 0 \rightarrow \boxed{c = -\frac{\beta}{k^2 + \frac{f_0^2}{gH}}}$$

Now, the numerical values we need are

$$\beta = \frac{2\Omega\cos\theta_0}{r_0} = \frac{2\times\left(7.3\times10^{-5}\right)\times\cos 45^{\circ}}{6.37\times10^3} = 1.62\times10^{-11}$$

and

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{800\times10^3} = 7.85\times10^{-6}\ \mathrm{m}^{-1}$$

Lastly,

$$c = -\frac{1.62\times10^{-11}}{\left[\left(7.85\times10^{-6}\right)^2 + \frac{\left(10^{-4}\right)^2}{9.81\times3000}\right]} = \boxed{-0.261\,\mathrm{m/s}}$$

P.5 ➡ *Solution*

Assume a solution of the form ψ = *Bsin[k(x – ct)]*. Substituting in the vorticity equation brings to

$$\left(\frac{\partial}{\partial t}\right)\nabla^2\psi + \bar{u}\left(\frac{\partial}{\partial x}\right)\nabla^2\psi + \beta\left(\frac{\partial\psi}{\partial x}\right) = \left(k^3c - \bar{u}k^2 + \beta k\right)B\cos\left[k\left(x-ct\right)\right]$$
$$= -f_0A\cos\left[k\left(x-ct\right)\right]$$

Solving for *B*,

$$B = -\frac{f_0A}{\left[\left(c-\bar{u}\right)k^3 + \beta k\right]}$$

so that

$$\nabla^2\psi = -\frac{f_0Ak}{\left(c-\bar{u}\right)k^2 + \beta}\sin\left[k\left(x-ct\right)\right]$$

Vorticity becomes infinite when the denominator in the right-hand side is zero; that is,

$$\left(c-\bar{u}\right)k^2 + \beta = 0 \rightarrow \boxed{c = \bar{u} - \frac{\beta}{k^2}}$$

which is the free Rossby speed. For $(c - \bar{u}) > -\beta/k^2$, vorticity *leads* divergence by $\pi/2$ radians, which is to say that maximum velocity is to the east of maximum divergence. For $(c - \bar{u}) < -\beta/k^2$, vorticity *lags* divergence by $\pi/2$ radians.

P.6 ➡ ***Solution***

The equation to use is

$$\frac{\partial u_g}{\partial z} = +\left(\frac{\varepsilon g}{2\Omega}\right)\frac{\partial T}{\partial y} \quad \text{(I)}$$

where y is the distance from the inner wall. If the temperature is to increase linearly with z as stated, we may write

$$T = T_0 + \left(\frac{y}{L}\right)\left[\left(\frac{dT}{dz}\right)z\right]$$

and dT/dZ is the thermal gradient along the outer wall. Deriving the relation above with respect to y and substituting in (I), we get

$$\frac{\partial u_g}{\partial z} = +\left(\frac{\varepsilon g}{2\Omega}\right)\frac{dT}{dz}\left(\frac{z}{L}\right)$$

Integrating from $z = 0$ to $z = H = 0.1$ m, we obtain the geostrophic velocity u_g,

$$\frac{\partial u_g}{\partial z} = +\left(\frac{\varepsilon g}{2\Omega}\right)\frac{dT}{dz}\left(\frac{z}{L}\right) \rightarrow u_g = \frac{\varepsilon g}{2\Omega}\left(\frac{dT}{dz}\right)\frac{1}{L}\int_0^H zdz$$

$$\therefore u_g = \frac{\varepsilon g}{2\Omega}\left(\frac{dT}{dz}\right)\frac{H^2}{2L}$$

$$\therefore u_g = \frac{\left(2\times 10^{-4}\right)\times 9.81}{2\times 1.0}\times 100\times\frac{0.1^2}{2\times 0.2} = 0.00245\,\text{m/s}$$

$$\therefore \boxed{u_g = 0.245\,\text{cm/s}}$$

♦ The correct answer is **B**.

P.7 ➡ ***Solution***

The maximum exponential growth rate of course occurs for the wavenumber k at which $\partial\alpha/\partial k = 0$. Plugging this relation into Mathematica, we have

$$\text{In[348]:= } \texttt{Simplify}\left[\text{D}\left[k * U_T * \left(\frac{2\lambda^2 - k^2}{2\lambda^2 + k^2}\right)^{1/2}, k\right]\right]$$

$$\text{Out[348]= } -\frac{\left(k^4 + 4k^2\lambda^2 - 4\lambda^4\right)U_T}{\sqrt{-\frac{k^2 - 2\lambda^2}{k^2 + 2\lambda^2}}\left(k^2 + 2\lambda^2\right)^2}$$

Setting the resulting derivative to zero means that the factor $k^4 + 4\lambda^2k^2 - 4\lambda^4$ in the numerator must equal zero. (Alternatively, $U_T = 0$, but this is trivial.) Solving this biquadratic equation for k yields

$$\text{In[349]:= } \texttt{Solve}\left[k^4 + 4k^2\lambda^2 - 4\lambda^4 == 0, k\right]$$

$$\text{Out[349]= } \left\{\left\{k \to -\sqrt{2}\sqrt{-\lambda^2 - \sqrt{2}\lambda^2}\right\}, \left\{k \to \sqrt{2}\sqrt{-\lambda^2 - \sqrt{2}\lambda^2}\right\}, \left\{k \to -\sqrt{2}\sqrt{-\lambda^2 + \sqrt{2}\lambda^2}\right\}, \left\{k \to \sqrt{2}\sqrt{-\lambda^2 + \sqrt{2}\lambda^2}\right\}\right\}$$

The solution we aim for is

$$k_{\max} = \sqrt{2\lambda^2\left(\sqrt{2} - 1\right)}$$

as we were supposed to show. Substituting k_{max} into the equation for α gives the corresponding maximum growth rate $\alpha_{\max}$,

$$\text{In[351]:= } \texttt{FullSimplify}\left[k * U_T * \left(\frac{2\lambda^2 - k^2}{2\lambda^2 + k^2}\right)^{1/2} \text{ /. } k \to \sqrt{2 * \lambda^2 * \left(\sqrt{2} - 1\right)}\right]$$

$$\text{Out[351]= } -\left(-2 + \sqrt{2}\right)\sqrt{\lambda^2}\,U_T$$

That is,

$$\alpha_{\max} = \sqrt{2}\lambda U_T\left(\sqrt{2}-1\right)$$

Substituting U_T = 20 m/s and $\lambda = \sqrt{2}\times 10^{-6}$ m yields the maximum growth rate

$$\alpha_{\max} = \sqrt{2}\times\left(\sqrt{2}\times 10^{-6}\right)\times 20\times\left(\sqrt{2}-1\right) = 2.34\times 10^{-5}\ \mathrm{s}^{-1}$$

The time required to achieve amplification by a factor of e^1 is simply $\alpha_{\max}^{-1}$, or

$$t = 1/\alpha_{\max} = \frac{1}{2.34\times 10^{-5}} = 42,700\,\mathrm{s} = \boxed{11.9\,\mathrm{hours}}$$

P.8 ➡ *Solution*

1.True. Simply apply the usual equation

$$\Gamma_s = \frac{g}{c_p}\left(\frac{1+\dfrac{r_s L_v}{RT}}{1+\dfrac{0.622L_v^2 r_s}{c_p RT^2}}\right) = \frac{9.81}{1000}\times\left[\frac{1+\dfrac{\left(22\times 10^{-3}\right)\times\left(2.5\times 10^{6}\right)}{287\times 320}}{1+\dfrac{0.622\times\left(2.5\times 10^{6}\right)^2\times\left(22\times 10^{-3}\right)}{1000\times 287\times 320^2}}\right] = 0.00401\,\mathrm{K/m}$$

$$\therefore \boxed{\Gamma_s = 4.01\,\mathrm{K/km}}$$

2.True. Simply apply the usual equation

$$V_{gr} = \frac{fR}{2}\left(\sqrt{1+\frac{4V_g}{fR}}-1\right)$$

$$\therefore V_{gr} = \frac{10^{-4}\times\left(320\times 10^{3}\right)}{2}\times\left[\sqrt{1+\frac{4\times 16}{10^{-4}\times\left(320\times 10^{3}\right)}}-1\right] = \boxed{11.7\,\mathrm{m/s}}$$

3.True. The phase speed we aim for is given by (see also Problem 5)

$$c = \bar{u}-\frac{\beta}{k^2}$$

Here, $\bar{u}$ = 50 m/s as given; the β parameter is

$$\beta = \frac{2\Omega\cos\theta_0}{r_0} = \frac{2\times\left(7.3\times 10^{-5}\right)\times\cos 30^{\circ}}{6370\times 10^{3}} = 1.98\times 10^{-11}\ \mathrm{m}^{-1}\,\mathrm{s}^{-1}$$

The wavenumber k is

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{3000\times 10^{3}} = 2.09\times 10^{-6}\ \mathrm{m}^{-1}$$

Finally,

$$c = 50-\frac{1.98\times 10^{-11}}{\left(2.09\times 10^{-6}\right)^2} = \boxed{45.5\,\mathrm{m/s}}$$

4.False. The rate of change of circulation is given by

$$\frac{DC}{Dt} = -\oint RTd\ln p$$

but, since *ln p* changes only northward, the integral simplifies to

$$\frac{DC}{Dt} = -R\ln\left[\frac{\left(p_0+\delta p\right)}{p_0}\right]\delta T$$

For a distance of 1200 km and a northward pressure gradient of 1 mb/200 km, we clearly have δp = 6 mb. For the same distance and a temperature gradient of 1°C/200 km, we obtain δT = 6°C. Substituting in the relation above brings to

$$\frac{DC}{Dt}=-287\times\ln\left[\frac{(1100+6)}{1100}\right]\times 6=\boxed{-9.37\,\mathrm{m}^2/\mathrm{s}^{-2}}$$

5.False. As any student with some background on the theory of barotropic instability should know, Howard's theorem requires that the tip of the vector (c_r, c_i) fall within the half-circle (not a half-ellipse with eccentricity greater than 1) constructed from the minimum and maximum velocities of the ambient shear flow.

6.False. For barotropic instability, the following inequality must hold,

$$\beta-\frac{d^2\overline{u}}{dy^2}<0 \quad \text{(I)}$$

where

$$\overline{u}(y)=-u_0\sin^2\left[\ell(y-y_0)\right]$$

which can be differentiated once to yield

$$\frac{d\overline{u}}{dy}=-2u_0\ell\sin\left[\ell(y-y_0)\right]\cos\left[\ell(y-y_0)\right]$$

or, differentiating a second time,

$$\frac{d^2\overline{u}}{dy^2}=-2u_0\ell^2\left\{\sin^2\left[\ell(y-y_0)\right]-\cos^2\left[\ell(y-y_0)\right]\right\}$$

$$\therefore\frac{d^2\overline{u}}{dy^2}=2u_0\ell^2\cos\left[2\ell(y-y_0)\right]$$

This derivative will be maximum when the rightmost cosine term equals unity, giving

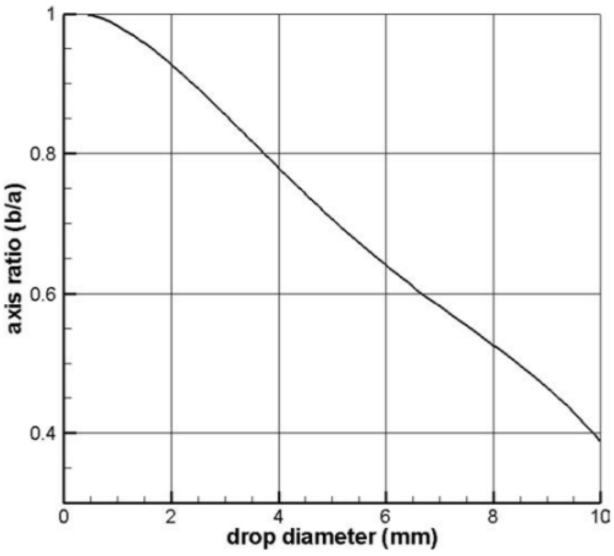
$$\left.\frac{d^2\overline{u}}{dy^2}\right|_{\max}=2u_0\ell^2$$

so that, substituting in (I),

$$\beta-2u_0\ell^2<0\rightarrow\underline{2u_0\ell^2>\beta}$$

Barotropic instability for this velocity profile will exist if the product $2u_0\ell^2$ exceeds the β parameter.

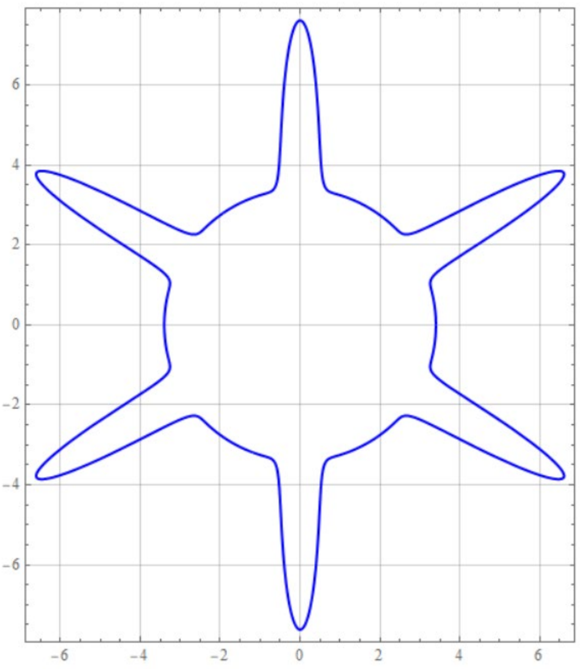
7.False. The following graph shows the variation of axis ratio with drop diameter. As can be seen, axis ratio decreases with drop size, which suggests that droplets become flatter, not more round, as diameter increases.



8.True. One way to proceed is to apply Mathematica's command *PolarPlot*,

```
In[446]:= PolarPlot[4.22 (Sin[3 θ]^2)^12 + 3.4, {θ, 0, 2 π}, PlotTheme → "Scientific",
  PlotStyle → RGBColor[0, 0, 1], Frame → True, GridLines → Automatic]
```

Out[446]=



The polar plot obtained matches the shape shown in the statement.

9.False. The Kolmogorov scale is given by the usual formula

$$\eta_k = \left(\frac{v_a^3}{\varepsilon_d}\right)^{1/4} = \left[\frac{\left(1.48\times10^{-5}\right)^3}{5.0\times10^{-4}}\right]^{1/4} = 1.60\times10^{-3}\text{ m} = \boxed{1.60\,\text{mm}}$$

This result indicates that the smallest eddies in an atmosphere at 288 K are about 1.6 mm in diameter.

10.True. Referring to the expression for θ_e,

$$\theta_e \approx \theta \exp\left(\frac{L_c q_s}{c_p T}\right)$$

and noting that

$$\theta = T\left(\frac{p_s}{p}\right)^{R/c_p}$$

We substitute and expand,

$$\theta_e = \theta \exp\left(\frac{L_c q_s}{c_p T}\right) \rightarrow \ln\theta_e = \ln\theta + \ln\left[\exp\left(\frac{L_c q_s}{c_p T}\right)\right]$$

$$\therefore \ln\theta_e = \ln\theta + \frac{L_c q_s}{c_p T}$$

$$\therefore \ln\theta_e = \ln T + \frac{R}{c_p}\ln\left(\frac{p_s}{p}\right) + \frac{L_c q_s}{c_p T}$$

$$\therefore \theta_e = \exp\left[\ln T + \frac{R}{c_p}\ln\left(\frac{p_s}{p}\right) + \frac{L_c q_s}{c_p T}\right]$$

$$\therefore \theta_e = \exp\left[\ln 293 + \frac{287}{1000}\ln\left(\frac{1000}{920}\right) + \frac{\left(2.5\times10^6\right)\times\left(16\times10^{-3}\right)}{1000\times293}\right] = \boxed{344\,\text{K}}$$

11.True. Substituting the appropriate variables into the equation given, we have

$$\tau = 140\times\left(\frac{293}{|9.81|\times10^{-6}\times\left(3.3\times10^{-3}\right)\times e^{10^{-6}\tau}}\right)^{1/3}$$

$$\therefore \tau = 292{,}000\left(e^{-10^{-6}\tau}\right)^{1/3}$$

This transcendental equation can be easily solved with Mathematica's *FindRoot* command, using an initial guess of, say, τ = 1000 s,

```
In[374]:= FindRoot[τ - 292000. * (Exp[-10^-6 * τ])^1/3, {τ, 1000}]
Out[374]= {τ → 267124.}
```

That is, $\tau \approx 267{,}000$ s. The maximum airmass thickness will be attained within approximately 3.09 days.

12.False. In general, the *CAPE* can be determined with the relation

$$\text{CAPE} = g\int_{z_{LFC}}^{z_{LNB}} \left(\frac{\theta_v - \hat{\theta}_v}{\hat{\theta}_v}\right) dz = g\int_0^{10{,}000} \left(\frac{8.0}{288}\right) dz = \boxed{2730\,\text{m}^2\text{s}^{-2}}$$

13.True. *CAPE* was calculated to be 2730 m^2s^{-2}. Observing that only 10% of this quantity is converted to kinetic energy, we obtain the vertical velocity

$$w = \sqrt{2\times CAPE_{\text{eff}}} = \sqrt{2\times(0.1\times 2730)} = \boxed{23.4\,\text{m/s}}$$

14.True. In the delayed oscillator model, the western Pacific is assumed to be an inactive region for air-sea interaction, whereas ocean wave reflection is unimportant at the eastern boundary.

15.Debatable. Canonical El Niño and El Niño Modoki are associated with different meteorological disturbances, especially in the extratropics, and thus affect communities differently. For example, Behera and Yamagata (see reference below) observe that the persistent summer drought in the western United States is caused not only by below-normal rainfall, but also by above-normal temperature in El Niño Modoki summers. The surface temperature related to El Niño Modoki is warmer than normal in the western states, and cooler normal in the central and eastern states. However, the El Niño-related temperature in most areas of the United States, except for the southeastern and northwestern states, is basically cooler than normal.

Reference: Behera and Yamagata (*in* Behera, 2021).

16.False. The relationship between wind stress and SST is not as clear in the AZM system. For example, westerly wind events can be followed by *cool* SST anomalies and vice versa. This has led some investigators to speak of *noncanonical* AZM events, in the same spirit as the phenomenological variations in ENSO that led workers to create alternative designations for odd weather behavior in the Pacific (e.g., "ENSO Modoki").

17.True. We first compute the rate of addition of calcium ion,

$$\text{Rate of calcium ion addition} = \text{River runoff} \times \begin{pmatrix}\%\,\text{Salinity} \\ \text{of river water}\end{pmatrix} \times \begin{pmatrix}\text{Conc.Calcium} \\ \text{in river water}\end{pmatrix}$$

$$\therefore \text{Rate of Ca}^{2+}\text{ ion addition} = 3.6\times10^{16}\times\frac{0.12}{100}\times\frac{12.5}{1000} = 5.4\times10^{11}\text{ kg/yr}$$

The amount of calcium present in seawater is estimated as

$$\text{Calcium mass in seawater} = \begin{pmatrix}\text{Calcium conc.} \\ \text{in seawater}\end{pmatrix} \times \begin{pmatrix}\text{Mass of seawater} \\ \text{in ocean}\end{pmatrix}$$

$$\therefore \text{Ca}^{2+}\text{ mass in seawater} = \frac{0.41}{1000}\times\left(1.4\times10^{21}\right) = 5.74\times10^{17}\text{ kg}$$

The residence time follows as

$$\text{Residence time} = \frac{\text{Ca}^{2+}\text{ mass in seawater}}{\text{Rate of Ca}^{2+}\text{ ion addition}} = \frac{5.74\times10^{17}\text{ kg}}{5.4\times10^{11}\text{ kg/yr}} = \boxed{1.06\times10^{6}\text{ yr}}$$

18.False. Actually, the Sverdrup balance reads

$$\beta\int_{-H}^{0} v\,dz = \text{curl}\left(\frac{\boldsymbol{\tau}}{\rho_0}\right)$$

where β is the northward spatial derivative of the Coriolis frequency f, v is vertical velocity, $\boldsymbol{\tau}$ is the shear stress vector, and ρ_0 is the average water density. Integration is carried out over the entire water column *H*, including the wind-stirred layer. As the equation above indicates, meridional transport in the Sverdrup relation is governed by the curl, not divergence, of wind stress. The Sverdrup relation is a strong result because it indicates a simple dependence of meridional transport on local wind stress curl, excluding any dependency on large-scale distribution of wind stress and, perhaps most importantly, on the nature of density stratification in the ocean.

P.9 ➡ *Solution*

The drag coefficient is

$$C_D = \frac{0.41^2}{\left[\log(10/1.0)\right]^2} = 0.0317$$

The shear velocity is

$$u* = \sqrt{C_D\left(u_{z_0}\right)^2} = \sqrt{0.0317\times 9.0^2} = 1.60\,\text{m/s}$$

Lastly, we appeal to the definition of shear velocity to obtain

$$u* = \sqrt{\frac{\tau_0}{\rho}} \rightarrow \tau_0 = \rho u*^2$$

$$\therefore \tau_0 = 1.2\times 1.60^2 = \boxed{3.07\,\text{N/m}^2}$$

♦ The correct answer is **D**.

P.10 ➡ *Solution*

The moisture convergence into an atmospheric column is

$$P = \rho w q$$

where ρ is the density of air, *q* is the water-vapor mixing ratio at the top of the Ekman layer, and *w* is the vertical velocity (Ekman pumping), which can be stated as

$$w = \zeta\left(\frac{K_m}{2f_0}\right)^{1/2}$$

so that

$$P = \rho w q \rightarrow P = \rho\zeta\left(\frac{K_m}{2f_0}\right)^{1/2} q$$

$$\therefore P = \rho\zeta\left(\frac{K_m}{2\times 2\Omega\sin\theta_0}\right)^{1/2} q$$

$$\therefore P = 1.0\times\left(2.0\times 10^{-5}\right)\times\left[\frac{10}{4\times\left(7.3\times 10^{-5}\right)\times\sin 15^\circ}\right]^{1/2}\times\left(12\times 10^{-3}\right) = 8.73\times 10^{-5}\,\text{kg m}^{-2}\,\text{s}^{-1}$$

Dividing by the density of water gives the precipitation rate,

$$\frac{P}{\rho_w} = \frac{8.73\times 10^{-5}}{1000} = 8.73\times 10^{-8}\,\text{m/s}$$

$$\therefore \frac{P}{\rho_w} = \left(8.73\times 10^{-8}\right)\frac{\text{m}}{\text{s}}\times 86{,}400\frac{\text{s}}{\text{day}}\times\frac{1000}{1}\frac{\text{mm}}{\text{m}} = \boxed{7.54\,\text{mm/day}}$$

♦ The correct answer is **B**.

P.11 ➡ *Solution*

Problem 11.1: With a zonal velocity perturbation *u'* = 5 m/s and a phase speed *c* = 40 m/s, we first compute the geopotential Φ',

$$\Phi' = u'\frac{v}{k} = u'c = 5.0 \times 40 = 200\,\mathrm{m^2 s^{-2}}$$

The vertical velocity perturbation can be found by solving the following relation for *w*,

$$v\left(m - \frac{i}{2H}\right)\widehat{\Phi} + \widehat{w}N^2 = 0 \rightarrow w' \approx -\frac{vm}{N^2}\Phi' \quad \text{(I)}$$

where we have used the fact that vertical wavenumber *m* is much greater than *i*/2*H*. Further, *m* is determined as

$$m^2 = \frac{N^2}{c^2} = \frac{4.0 \times 10^{-4}}{40^2} = 2.5 \times 10^{-7}$$
$$\therefore m = 5.0 \times 10^{-4}$$

so that, substituting in (I),

$$w' = -\frac{vm}{N^2}\Phi' = -\frac{\frac{40}{6.37 \times 10^6} \times \left(5.0 \times 10^{-4}\right)}{4.0 \times 10^{-4}} \times 200 = \boxed{1.57 \times 10^{-3}\ \mathrm{m/s}}$$

(where we have used the radius of the Earth to compute v.)

Problem 11.2: We begin by restating the momentum flux as

$$M \equiv \rho_0 \overline{u'w'} = e^{-z^*/H} \times \langle u'w' \rangle \quad \text{(II)}$$

where zonal velocity disturbance *u'* is given by

$$u' = U_0 \cos(kx) e^{-z^*/2H}$$

whereas vertical velocity disturbance *w'* is (see also the first two results of the previous part)

$$w' = \left(\frac{c^2 km}{N^2}\right) U_0 \cos(kx) e^{-z^*/H}$$

Substituting in (II) brings to

$$M = \frac{U_0^2 c^2 km}{N^2}\langle \cos^2 kx \rangle$$

The mean value of *cos²kx* is 0.5; other variables were given or calculated in the previous part. Accordingly,

$$M = \frac{U_0^2 c^2 km}{N^2}\langle \cos^2 kx \rangle = \frac{5.0^2 \times 40^2 \times \frac{1.0}{6.37 \times 10^6} \times \left(5.0 \times 10^{-4}\right)}{4.0 \times 10^{-4}} \times 0.5 = 0.00393$$
$$\therefore \boxed{M = 3.93 \times 10^{-3}\ \mathrm{m^2 s^{-1}}}$$

Problem 11.3: In Problem 11.1 we established the vertical velocity perturbation

$$w' = -\frac{vm}{N^2}\Phi'$$

However, from Chapter 12 of Holton (1992),

$$\Phi' = ivy \times \exp\left(-\frac{\beta |m| y^2}{2N}\right)$$

so that

$$\underline{w' = -\frac{iv^2 m}{N^2} y \exp\left(-\frac{\beta |m| y^2}{2N}\right)} \quad \text{(III)}$$

Problem 11.4: For starters, the vertical momentum flux $M = \rho_0 \overline{u'w'}$ is dependent on the average of the product of w', which is given by (III) in the previous part, and u', which is stated as

$$u' = i|m|\nu N^{-1} y \exp\left(-\frac{\beta|m|y^2}{2N}\right) \quad \text{(IV)}$$

The average of the product of (III) and (IV) has general form

$$\langle u'w' \rangle = \text{Constant} \times y^2 \exp\left(-\frac{\beta|m|y^2}{N}\right) \quad \text{(V)}$$

so that, differentiating and setting the result to zero,

$$\frac{\partial \langle u'w' \rangle}{\partial y} = 0 \rightarrow \frac{2\exp\left(-\frac{\beta|m|y^2}{N}\right) y\left(N - \beta|m|y^2\right)}{N} = 0$$

Here, wavenumber $|m|$ is given by

$$|m| = N\nu^{-2}\left(\beta + \nu k\right)$$

so that, setting the factor in red to zero, substituting $|m|$, and solving for y,

$$y_{\max} = \frac{\pm \nu}{\beta\left(1 + \frac{k\nu}{\beta}\right)^{1/2}}$$

Substituting $\nu = ck = cs/a$ and $\beta \approx 2\Omega/a$ and simplifying,

$$y_{\max} = \pm \frac{cs}{2\Omega\left(1 + \frac{s\nu}{2\Omega}\right)^{1/2}}$$

Substituting zonal wavenumber s = 4.0, phase speed c = –20 m/s, and frequencies $\nu = -20 \times 4.0/(6.37\times10^6) = 1.26\times10^{-5}\ \text{s}^{-1}$ and $\Omega = 7.3\times10^{-5}\ \text{s}^{-1}$, we obtain

$$y_{\max} = \frac{-20\times4.0}{2\times\left(7.3\times10^{-5}\right)\times\left[1 - \frac{4.0\times\left(1.26\times10^{-5}\right)}{2\times\left(7.3\times10^{-5}\right)}\right]^{1/2}} = -677{,}000\,\text{m}$$

$$\therefore |y_{\max}| = 677\,\text{km}$$

(The term inside the square root in the denominator has a negative sign because $\nu < 0$.) The corresponding latitude is

$$|\theta| \approx \frac{|y_{\max}|}{a} = \frac{677}{6370} = 0.106\,\text{rad} = \boxed{6.07^\circ}$$

Thus, maximum vertical momentum flux occurs at about six degrees of latitude north or south.

P.12 ➡ *Solution*

The volume of the cylindrical cloud is $V_c = \pi \times 800^2 \times 2000 = 4.02\times10^9$ m^3 = 4.02×10^{15} cm^3. The solution is continued by finding the charge separation rate $dQ_{b,c}/dt$, which is given by

$$\frac{dQ_{\text{b,c}}}{dt} = B_{1,2} n_1 n_2 \Delta Q_{1,2} V_c$$

The first term in this equation is the rate coefficient for bounceoff $B_{1,2}$, which is in turn given by the relation $B_{1,2} = (1 - E)K$; here, E = 0.4 is the dimensionless coalescence efficiency and $K = 10^{-4}$ cm^3 particle^{-1} s^{-1} is the collision kernel. Next in the equation are factors n_1 = 750 partic./cm^3 and n_2 = 0.05 partic./cm^3, which represent the number concentrations of hydrometeor particles; ΔQ is the charge separation per collision, which has an average value of 3.33×10^{-14} coulomb/collision; lastly, V_c is the cloud volume, as determined just now. Substituting the pertaining data into the equation at hand gives

$$\frac{dQ_{\mathrm{b,c}}}{dt}=(1-0.4)\times10^{-4}\,\frac{\mathrm{cm}^3}{\mathrm{partic.}\times\mathrm{sec}}\times750\,\frac{\mathrm{partic}}{\mathrm{cm}^3}\times0.05\,\frac{\mathrm{partic}}{\mathrm{cm}^3}\times$$
$$3.3\times10^{-14}\,\frac{\mathrm{C}}{\mathrm{collision}}\times4.02\times10^{15}\ \mathrm{cm}^3=0.298\,\mathrm{C/s}$$

The rate of change of the in-cloud electric field strength is, in turn,

$$\frac{dE_f}{dt}=\frac{2k_c}{Z_c\sqrt{Z_c^2+R_c^2}}\frac{dQ_{b,c}}{dt}$$

where $k_c = 8.99\times10^{11}$ V·cm/C is the electrostatic constant, $Z_c = 2.0\times10^5$ cm is cloud thickness, and $R_c = 0.8\times10^5$ is cloud radius, giving

$$\frac{dE_f}{dt}=\frac{2\times\left(8.99\times10^{11}\,\dfrac{\mathrm{V\cdot cm}}{\mathrm{C}}\right)}{2.0\times10^5\ \mathrm{cm}\times\sqrt{\left(2.0\times10^5\ \mathrm{cm}\right)^2+\left(0.8\times10^5\ \mathrm{cm}\right)^2}}\times0.298\,\mathrm{C/s}=12.4\,\mathrm{V/cm\cdot s}$$

Lastly, with a threshold electric field strength E_{th} = 3000 V/cm, the number of intracloud flashes per hour in the cloud is calculated to be

$$\frac{1}{E_{\mathrm{th}}}\frac{dE_f}{dt}=\frac{1}{3000\,\mathrm{V/cm\cdot flash}}\times12.4\,\frac{\mathrm{V}}{\mathrm{cm\cdot s}}=0.00413\,\mathrm{flashes/s}=\boxed{14.9\,\mathrm{flashes/hr}}$$

♦ The correct answer is **C**.

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