Montogue

Microeconomics | Quiz ECN2

Producer Theory and Competitive Equilibria

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◆ Problem List

◆ Problems

Problem 1 (Modified from Serrano and Feldman, 2018)

A Simple Production Function – Direct Approach. Let the production function of a certain firm be $y = x^{1/2}$.

(a) Show that the production function is concave.

(b) Suppose the price of x is w = \$1. Find the firm's total cost curve $C(y)$, average cost curve $AC(y)$, and marginal cost curve $MC(y)$.

(c) Find the firm's supply curve $y^*(p)$.

(d) Suppose the price of y is p = \$15. Compute the firm's profit.

Problem 2 (Modified from Serrano and Feldman, 2018)

A Simple Production Function – Reverse Approach. Reconsider the production function $y = x^{1/2}$ introduced in the previous problem. Assume now that, for political reasons, it is not feasible to hire fewer than one unit of input. That is, assume that $x \geq 1$.

(a) Show that the inverse production function $x(y)$ is convex.

(b) Find the firm's marginal product $MP(x)$ and average product $AP(x)$.

(c) The price of y is p = \$15. Find the firm's value of marginal product $VMP(x)$ and value of average product $VAP(x)$.

(d) Find the firm's input demand curve $x^*(w)$.

(e) Suppose the price of x is w = \$1. Calculate the firm's profit.

Problem 3 (Modified from Kolmar and Hoffmann, 2018)

Parametric Analysis of a Production Function I. Assume we have the following production function,

$$y(l) = l^{\alpha}$$

where l is labor input and $\alpha > 0$.

(a) For which values of α does the production function have decreasing, constant, or increasing marginal product? Sketch the marginal product for α = 1/2, 1, and 3/2.

(b) Let wage rate $w > 0$. Determine the cost function *C(y)*, the marginal cost function *MC(y)*, and the average cost function *AC(y)* of the above production function.

(c) For which values of α are the average cost and the marginal cost functions increasing, decreasing, or constant? Letting $w = 1$, draw *AC(y)* and *MC(y)* for α = 1/2, 1, and 3/2.

Problem 4 (Modified from Kolmar and Hoffmann, 2018)

Parametric Analysis of a Production Function II. Assume we have the following production function:

$$y(l) = (l - \gamma)^{\beta}$$

with $\beta \in [1/2; 1]$ and $\gamma > 0$.

(a) Let wage rate $w > 0$. Determine the cost function *C(y)*, the marginal cost function *MC(y)*, and the average cost function *AC(y)* for the above production function.

(b) For which values of β are the average cost and the marginal cost functions increasing, decreasing, or constant? Let $w = \gamma = 1$, and draw both functions for β = 1/2 and 1.

(c) Consider the parametric production functions investigated in this problem and the previous one. Which of these two production functions is **not** compatible with a perfectly competitive market (depending on the values of α and β)?

Problem 5

Optimal Allocation of Capital and Labor I. A firm's production function is given by

$$q(k,l) = 80k^{0.25}l^{0.75}$$

Assume the rental rate is r = \$20 and the wage rate is w = \$40.

(a) Find the quantity of labor and capital that the firm should use in order to minimize the cost of producing 9840 units of output.

(b) What is the minimum cost?

Problem 6 (Modified from Nicholson and Snyder, 2016)

Optimal Allocation of Capital and Labor II. Suppose the production function for a certain good is

$$q(k,l) = 6kl - 3.6k^2 - 0.4l^2$$

where q represents the annual quantity of goods produced, k represents the annual capital input, and l represents the annual labor input.

(a) Suppose $k = 10$; graph the total and average productivity of labor curves. At what level of labor input does this average productivity reach a maximum? How many goods are produced at that point?

(b) Again assuming that $k = 10$, graph the marginal productivity of labor (MP_l) curve. At what level of labor input does $MP_l = 0$?

(c) Suppose capital inputs increased to $k = 20$. How would your answers to parts (a) and (b) change?

(d) Does the good production exhibit constant, increasing, or decreasing returns to scale?

Problem 7 (Modified from Nicholson and Snyder, 2016)

Short-Run and Long-Run Production I. A firm that produces chocolate bars has the following production function:

$$q = 4\sqrt{kl}$$

In the short run, the firm's amount of capital equipment is fixed at $k = 300$. The rental rate for k is $v = \$1$, and the wage rate for labor l is $w = \$5$.

Part 1. **(a)** Calculate the firm's short-run total cost curve. Also calculate the short-run average cost curve.

(b) What is the firm's short-run marginal cost function? What are the short-run cost, short-run average cost, and short-run marginal cost if the firm produces 120 chocolate bars? 240 chocolate bars? 480 chocolate bars? 960 chocolate bars?

(c) Graph the short-run average cost and short-run marginal cost for the firm, indicating the points found in part (b).

(d) Where does the short-run marginal curve intersect the short-run average cost curve? Explain why the short-run marginal cost curve will always intersect the short-run marginal cost curve at its lowest point.

Part 2. Suppose now that the capital used for producing chocolate bars is fixed at k_1 in the short run. **(e)** Calculate the firm's total costs as a function of output q, wage rate w, rental rate v, and k_1.

(f) Given q, w, and v, how should the capital stock be chosen to minimize total cost?

(g) Use the results from part (f) to calculate the long-run total cost of chocolate bar production.

Problem 8 (Modified from Nicholson and Snyder, 2016)

Short-Run and Long-Run Production II. An enterprising entrepreneur purchases two factories to produce widgets. Each factory produces identical products, and each has a production function given by

$$q_i(k,l) = \sqrt{k_i l_i} \quad ; \quad i = 1,2$$

The factories differ, however, in the amount of capital equipment each has. In particular, factory 1 has $k_1 = 49$, whereas factory 2 has $k_2 = 196$. Rental rates for k and l are given by $w = v = \$1$.

(a) If the entrepreneur wishes to minimize short-run total costs of widget construction, how should output be allocated between the two factories?

(b) Given that output is optimally allocated between the two factories, calculate the short-run total, average, and marginal cost curves. What is the marginal cost of the 500th widget? The 1000th widget? The 2000th widget?

(c) How should the entrepreneur allocate widget production between the two factories tin the long run? Calculate the long-run total, average, and marginal cost curves for widget production.

(d) How would your answer to part (c) change if both factories exhibited diminishing returns to scale?

Problem 9

Short-Run and Long-Run Production III. Suppose that a firm's fixed proportion production function is given by

$$q = \min(5k, 10l)$$

(a) Calculate the firm's long-run total, average, and marginal cost functions.

(b) Suppose that k is fixed at 10 in the long run. Calculate the firm's short-run total, average, and marginal cost functions.

(c) Let $v = \$3$ and $w = \$4$. Calculate the firm's long-run and short-run average and marginal cost curves.

Problem 10

Perfectly Competitive Equilibrium I. Suppose there are 200 identical firms in a perfectly competitive industry. Each firm has a short-run total cost function of the form

$$C(q) = \frac{1}{300}q^3 + 0.25q^2 + 6.25q + 15$$

(a) Calculate the firm's short-run supply curve with q as a function of market price (p).
(b) On the assumption that firms' output decisions do not affect their costs, calculate the short-run industry supply curve.
(c) Suppose market demand is given by $Q = 23{,}800 - 200p$. What will be the short-run equilibrium price-quantity combination? What are the profits of each firm in the equilibrium price-quantity combination?

Problem 11

Perfectly Competitive Equilibrium II. Suppose the market for microeconomics textbooks can be assumed to be constituted of many similar firms in a perfectly competitive environment. The firms operate with the same long-run cost function

$$C(q) = q^3 - \frac{1}{2}q^2 + \frac{99}{2}$$

(a) Derive the representative firm's market supply curve for microeconomics textbooks.
(b) If the market demand for microeconomics textbooks faced by the firms is $Q = 1140 - 10p$, calculate the long-run competitive equilibrium for this industry, indicating equilibrium price, industry output, number of firms in the market, and the individual firm's output and profit.

Problem 12 (Modified from Serrano and Feldman, 2018)

Exchange Economy with Cobb-Douglas Utility Functions. Alfred has one kilogram of ham ($x_a = 1$) and no cheese ($y_a = 0$), and Beto has a kilogram of cheese ($y_b = 1$) and no ham ($x_b = 0$). Both Alfred and Beto have interests described by Cobb-Douglas utility functions $U_a(x_a, y_a) = x_a^{\alpha} y_a^{1-\alpha}$ and $U_b(x_b, y_b) = x_b^{\alpha} y_b^{1-\alpha}$, respectively, with $\alpha \in (0;1)$.
(a) Show that the contract curve is the diagonal of the Edgeworth box.
(b) Show that the price ratio of goods x (ham) and y (cheese) at competitive equilibrium is

$$\frac{p_x}{p_y} = \frac{\alpha}{1-\alpha}$$

Problem 13 (Modified from Serrano and Feldman, 2018)

General Equilibrium. Consider an exchange economy with two goods, x and y, and two consumers, Alfred and Beto. Alfred's utility function is $U(x_A, y_A) = x_A y_A$ and his endowment (i.e., his starting amount of each good) is $\omega_A = (2,2)$. Beto's utility function is $U(x_B, y_B) = x_B y_B^2$ and his endowment is $\omega_B = (3,3)$. A third party suggests there may be a competitive equilibrium at $(x'_A, y'_A) = (4,1)$, $(x'_B, y'_B) = (1,4)$, with prices $p_x = p_y = \$1$.
Part 1. (a) Is the third party's suggested equilibrium allocation a Pareto improvement over the endowment? Explain.
(b) Write down Alfred's budget constraint given the third party's suggested prices. Solve for Alfred's optimal consumption bundle $(x_A^\star, y_A^\star)$.
(c) Write down Beto's budget constraint given the third party's suggested prices. Solve for Beto's optimal consumption bundle $(x_B^\star, y_B^\star)$.
(d) Is the third party right that these bundles and these prices make a competitive equilibrium? Explain.
Part 2. Reconsider now Alfred and Beto's economy. We shall solve this general equilibrium model. We are free to set one of the prices equal to 1. We will let good x be the numeraire good; that is, we will set $p_x = \$1$, and we will solve for the appropriate p_y.
(e) Write down Alfred's budget constraint. Solve for Alfred's optimal consumption bundle, $(x_A^\star, y_A^\star)$, with $x_A^\star$ and $y_A^\star$ expressed as functions of p_y.
(f) Write down Beto's budget constraint. Solve for Beto's optimal consumption bundle, $(x_B^\star, y_B^\star)$, with $x_B^\star$ and $y_B^\star$ expressed as functions of p_y.

(g) Write down the market-clearing (that is, total demand = total supply) condition for x. Using your answers from (e) and (f), rewrite the market-clearing condition as a function of p_y, and solve for p_y. Find the competitive equilibrium.

Problem **14** (Modified from Serrano and Feldman, 2018)

Production Economy with One Product. Paul's technology for production of bread (good x) is represented by $x = l^{1/2}$, where l is labor in hours per day. Paul can have his preferences for bread and labor represented by the utility function $U(x,l) = x - l/2$. Assume Paul is the only consumer of bread, and the owner of the only firm that produces bread. The price of bread is set at $1.

(a) Find the Pareto efficient allocation of this simple production economy.

(b) Derive the competitive equilibrium of this economy. Find Paul's consumption of bread, his labor supply, the market wage rate, and the firm's profits.

◆ Assume now that Paul's technology for producing bread has changed to $x = l^{2/3}$. His utility function remains unchanged at $U(x,l) = x - l/2$.

(c) Calculate the new Pareto efficient allocation.

(d) Derive the competitive equilibrium of this economy. Find Paul's consumption of bread, his labor supply, the market wage rate, and the firm's profits.

Problem **15** (Modified from Serrano and Feldman, 2018)

Production Economy with Two Products. Paul the breadmaker has expanded his production to bread (x) and cookies (y). His inverse production function when accounting for the two goods is $l = x^2 + y^2 + 3xy$, where l is labor in hours per day. Paul's utility function has changed to $U(l,x,y) = 3xy/2 + x + y - l/2$. Suppose the market wage rate w is set at $1.

(a) Solve Peter's profit maximization problem. Derive his supply of bread and cookies.

(b) Solve Peter's utility maximization problem. Derive his demand for bread and cookies, and find his labor supply.

(c) Find the price of bread, p_x, and cookies, p_y.

Problem **16** (Modified from Nicholson and Snyder, 2016)

Production Possibility Frontier. Suppose the production possibility frontier for guns (x) and butter (y) is described by the equation

$$x^2 + 2y^2 = 900$$

(a) Graph this frontier.

(b) If individuals always prefer consumption bundles in which $y = 2x$, how much x and y will be produced?

(c) At the point described in part (b), what will be the rate of product transformation and hence what price ratio will cause production to take place at that point?

Problem **17** (Modified from Nicholson and Snyder, 2016)

Production Economy w/ and w/o Trade with the Outside World. Suppose that Robinson Crusoe produces and consumes fish (F) and coconuts (C). Assume that, during a certain period, he has decided to work 200 hours and is indifferent as to whether he spends this time fishing or gathering coconuts. Robinson's production for fish is given by

$$F = \sqrt{l_F}$$

and for coconuts by

$$C = \sqrt{l_C}$$

where l_F and l_C are the number of hours spent fishing or gathering coconuts. Consequently,

$$l_C + l_F = 200$$

Robinson Crusoe's utility for fish and coconuts is given by

$$U = \sqrt{F \times C}$$

(a) If Robinson cannot trade with the rest of the world, how will he choose to allocate his labor? What will the optimal levels of F and C be? What will his utility be? What will be the rate of product transformation (of fish for coconuts)?

(b) Suppose now that trade is opened and Robinson can trade fish and coconuts at a price ratio of $p_F/p_C = 2/1$. If Robinson continues to produce the quantities of *F* and *C* from part (a), what will he choose to consume once given the opportunity to trade? What will his new level of utility be?
(c) How would your answer to part (b) change if Robinson adjusts his production to take advantage of the world prices?
(d) Graph your results for parts (a), (b), and (c).

Problem 18 (Modified from Nicholson and Snyder, 2016)

Production Economy in Two Different Regions. In the country of Ruritania there are two regions, *A* and *B*. Two goods (*x* and *y*) are produced in both regions. Production functions for region *A* are given by

$$\begin{cases} x_A = \sqrt{l_x} \\ y_A = \sqrt{l_y} \end{cases}$$

Here, l_x and l_y are the quantities of labor devoted to *x* and *y* production, respectively. Total labor available in region *A* is 100 units; that is,

$$l_x + l_y = 100$$

Using a similar notation for region B, the production functions are given by

$$\begin{cases} x_B = \dfrac{1}{2}\sqrt{l_x} \\ y_B = \dfrac{1}{2}\sqrt{l_y} \end{cases}$$

There are also 100 units of labor available in region B,

$$l_x + l_y = 100$$

(a) Calculate the production possibility curves for regions A and B.
(b) What condition must hold if production in Ruritania is to be allocated efficiently between regions A and B (assuming labor cannot move from one region to another)?
(c) Calculate the production possibility curve for Ruritania (again assuming labor is immobile between regions). How much total *y* can Ruritania produce if total *x* output is 12?

Problem 19

Perfectly Competitive Equilibrium III: Multiple Types of Firms. A perfectly competitive market is constituted of three types of firms; each type of firm operates with a different cost function, as follows:

$$\begin{cases} \text{Type 1 firms: } C_1(q_1) = 2q_1^2 + 242 \\ \text{Type 2 firms: } C_2(q_2) = 3q_2^2 + 192 \\ \text{Type 3 firms: } C_3(q_3) = 4q_3^2 + 100 \end{cases}$$

The market demand function for the good is $Q = 1200 - 3p$. In the short run, there are 24 type 1 firms, 24 type 2 firms, and 16 type 3 firms. What is the equilibrium price of the good?

Problem 20 (Modified from Nicholson and Snyder, 2016)

Perfectly Competitive Equilibrium IV: Variable Wage Rate. In a perfectly competitive market, there are 800 identical firms producing teddy bears. Each firm operates with cost function *C(q,w)* given by

$$C(q,w) = q^2 + wq$$

where *q* is the firm's output and *w* is the wage rate of factory workers.
(a) For a constant wage rate *w* = \$8, what will be the firm's short-run supply curve? How many teddy bears will be produced at a price of \$12 each? How many teddy bears will be produced at a price of \$20 each?

(b) Assume now that the wages of factory workers depend on the total quantity of teddy bears produced, and that the form of this relationship is

$$w = 0.0025Q$$

where Q represents total industry output, which is 800 times the output of the typical firm. In this situation, show that the firm's marginal cost (and short-run supply) curve depends on Q. What is the industry supply curve? How much will be produced at a price of \$12? How much more will be produced at a price of \$20? What do you conclude about the shape of the short-run supply curve?

Problem 21 (Modified from Nicholson and Snyder, 2016)

Perfectly Competitive Equilibrium V: Short-Run and Long-Run Equilibria. A perfectly competitive industry has a large number of potential entrants. Each firm has an identical cost structure such that long-run average cost is minimized at an output of 25 units ($q_i = 25$). The minimum average cost is \$5 per unit. Total market demand is given by

$$Q = 3000 - 100p$$

(a) What is the industry's long-run supply schedule?
(b) What is the long-run equilibrium price? The total industry output? The output of each firm? The number of firms? The profits of each firm?
(c) The short-run total cost function associated with each firm's long-run equilibrium output is given by

$$C(q) = \frac{1}{4}q^2 - 8q + 156.25$$

Calculate the short-run average and marginal cost function. At what output level does short-run average cost reach a minimum?
(d) Calculate the short-run supply function for each firm and the industry short-run supply function.
(e) Suppose now that the market demand function shifts upward to $Q = 4500 - 100p$. Using this new demand curve, answer part (b) for the very short run when firms cannot change their outputs.
(f) In the short run, use the industry short-run supply function to recalculate the answers to (b).
(g) What is the new long-run equilibrium for the industry?

◆ Solutions

■ Problem 1

Part (a): Differentiating the production function *y(x)* a first time, we obtain

$$y'(x) = \frac{1}{2}x^{-1/2}$$

Differentiating *y(x)* a second time, we obtain

$$y''(x) = -\frac{1}{4}x^{-3/2}$$

Clearly, this result is such that $y''(x) < 0$, $\forall\, x > 0$. Thus, production function *y(x)* is concave in the interval $x \in (0; +\infty)$.
Part (b): The production function can be inverted to yield

$$y(x) = x^{1/2} \quad \rightarrow \quad x = y^2$$

If the price of x is $w = 1$, the cost function becomes

$$C(y) = wx = 1.0 \times y^2 = \boxed{y^2}$$

The average cost function is, in turn,

$$AC(y) = \frac{C(y)}{y} = \frac{y^2}{y} = \boxed{y}$$

and the marginal cost function is

$$MC(y)=\frac{dC(y)}{dy}=\frac{d}{dy}\left(y^2\right)=\boxed{2y}$$

Part (c): The supply curve is the marginal cost curve *MC(y)*, provided that 1) $p \geq$ *min AC(y)* (that is, the price *p* is greater than or equal to the minimum average cost *AC(y)*) and 2) *MC(y)* is rising. To verify condition 1, note that

$$\min AC(y)=AC(0)=0$$

To verify condition 2, note that the first derivative of marginal cost is positive and constant,

$$\frac{dMC(y)}{dy}=\frac{d}{dy}(2y)=2>0$$

hence, *MC(y)* is rising. With 1 and 2 satisfied, we conclude that, for $p \geq 0$, the supply curve is such that

$$p=MC(y)\quad\rightarrow\quad p=2y$$

$$\therefore \boxed{y^*(p)=\frac{1}{2}p}$$

Part (d): The profit function is

$$\pi(p)=p\times y-w\times x(y)=p\times\frac{1}{2}p-w\times\left(\frac{1}{2}p\right)^2$$

$$\therefore \pi(p)=\frac{1}{2}p^2-\frac{1}{4}wp^2$$

Substituting p = 15 and w = 1, we obtain

$$\pi(p)=\frac{1}{2}\times 15^2-\frac{1}{4}\times 1.0\times 15^2=\boxed{\$56.25}$$

■ Problem 2

Part (a): Inverting the production function yields $x = y^2$. Differentiating *x(y)* a first time, we obtain

$$x'(y)=2y$$

Differentiating *x(y)* a second time, we obtain

$$x''(y)=2$$

Clearly, this result is such that $x''(y) > 0$, $\forall\, y > 0$. Thus, *x(y)* is convex in the interval $y \in (0; +\infty)$.

Part (b): Differentiating the production function with respect to *x* gives the marginal product *MP(x)*,

$$MP(x)=\frac{dy(x)}{dx}=\boxed{\frac{1}{2\sqrt{x}}}$$

In turn, the average product *AP(x)* is

$$AP(x)=\frac{y(x)}{x}=\frac{\sqrt{x}}{x}=\boxed{\frac{1}{\sqrt{x}}}$$

Part (c): Multiplying *MP(x)* and *AP(x)* by the price *p* of *y* yields the value of marginal product *VMP(x)* and the value of average product *VAP(x)*, respectively,

$$VMP(x)=p\times MP(x)=15\times\frac{1}{2\sqrt{x}}=\boxed{\frac{15}{2\sqrt{x}}}$$

$$VAP(x)=p\times AP(x)=15\times\frac{1}{\sqrt{x}}=\boxed{\frac{15}{\sqrt{x}}}$$

Part (d): Here, we must use the assumption that $x \geq 1$. The input demand curve *x*(w)* equals the value of marginal product *VMP(x)* so long as 1) *max VAP*(*x*) $\geq w$

(that is, the maximum value of average product is greater than or equal to the price of *x*) and 2) *VMP(x)* is falling. To verify condition 1, note that

$$\max VAP(x) = VAP(1) = \frac{15}{\sqrt{1}} = 15 = w$$

To verify condition 2, note that the first derivative of average product is negative for all $x > 0$,

$$\frac{dVMP(x)}{dx} = \frac{d}{dy}\left(\frac{15}{2}x^{-1/2}\right) = -\frac{15}{4}x^{-3/2} < 0$$

Thus, for $w > 15$, the input demand curve is $x = 0$, and, for $w \leq 15$, the input demand curve follows from the value of marginal product *VMP(x)*,

$$w = VMP(x) \quad \rightarrow \quad w = \frac{15}{2\sqrt{x}}$$

$$\therefore \sqrt{x} = \frac{15}{2w}$$

$$\therefore \boxed{x^*(w) = \frac{225}{4w^2}}$$

Part (e): The profit function is

$$\pi(p) = p \times y(x) - w \times x = p \times \left(\frac{225}{4w^2}\right)^{1/2} - w \times \left(\frac{225}{4w^2}\right)$$

$$\therefore \pi(p) = \frac{15}{2}\frac{p}{w} - \frac{225}{4w^2}$$

Substituting $p = 15$ and $w = 1$, we obtain

$$\pi(p) = \frac{15}{2} \times \frac{15}{1.0} - \frac{225}{4 \times 1.0^2} = \boxed{\$56.25}$$

This result is identical to the one obtained at the end of the previous problem. Predictably, it turns out that optimizing the inverse production function leads to the same profit as when optimizing the direct production function.

■ Problem 3

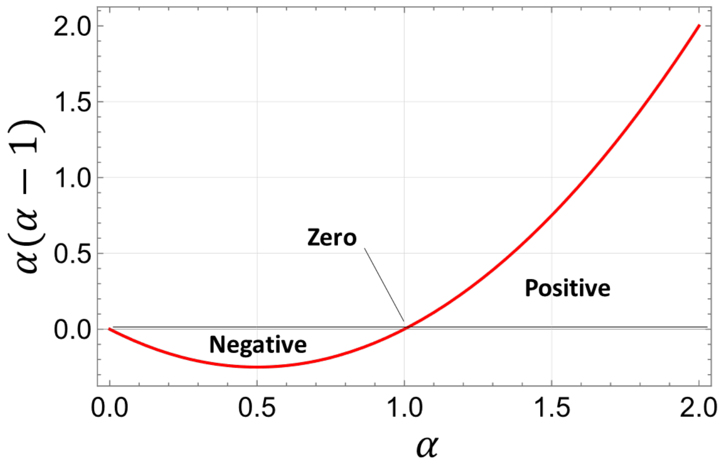
Part (a): Differentiating $y(l)$ with respect to input (in this case, labor l is the only input) gives the marginal product,

$$MP(l) = \frac{dy}{dl} = \alpha l^{\alpha-1}$$

The first derivative of the marginal product (that is, the second derivative of *y(l))* indicates whether the marginal product function is increasing, decreasing, or constant in *l*,

$$y''(l) = \frac{d}{dl}\left(\alpha l^{\alpha-1}\right) = \alpha(\alpha-1)l^{\alpha-2}$$

The sign of *y''(l)* depends on the product $\alpha(\alpha - 1)$, which plots as a parabola:



With reference to the graph, we conclude the following with regard to the behavior of the marginal product relatively to exponent α,

$$MP(y) \text{ is } \begin{Bmatrix} \text{Increasing} \\ \text{Constant} \\ \text{Decreasing} \end{Bmatrix} \text{ in } l \text{ if } \alpha \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 1$$

We can also extract information pertaining to the curvature of the marginal product by taking its second derivative – or, equivalently, the third derivative of *y*(*l*),

$$y'''(l) = \frac{d}{dl}\left(\alpha(\alpha-1)l^{\alpha-2}\right) = \alpha(\alpha-1)(\alpha-2)l^{\alpha-3}$$

Substituting $\alpha = 1/2$,

$$y'''(l) = 0.5\times(0.5-1)\times(0.5-2)l^{0.5-3} = 0.375l^{-2.5} > 0$$

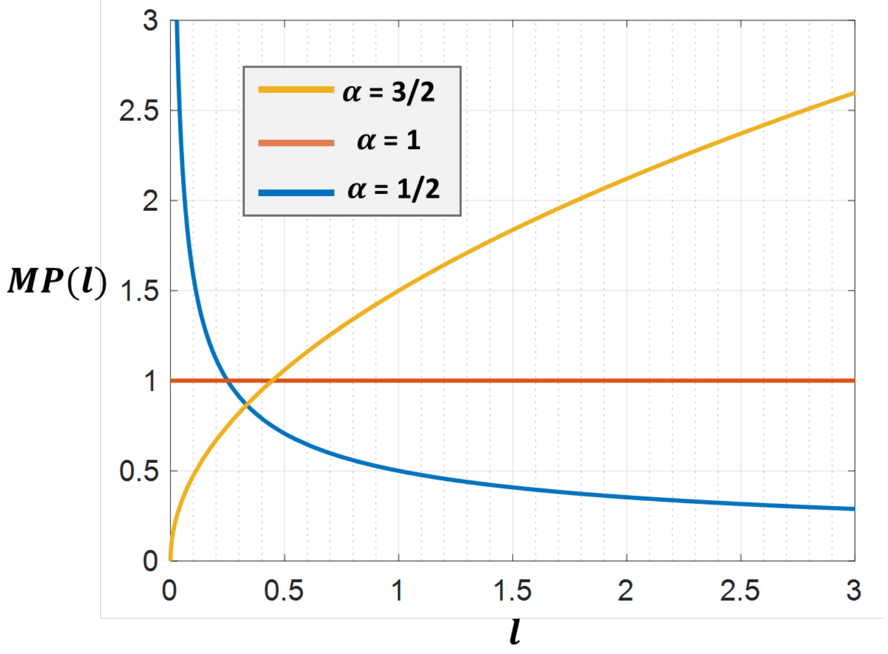
which means that for $\alpha = 1/2$ the marginal product is a strictly convex function. Substituting $\alpha = 1$,

$$y'''(l) = 1.0\times(1.0-1)\times(1.0-2)l^{1.0-3} = 0$$

which means that for $\alpha = 2$ the marginal product is a linear function. Finally, we substitute $\alpha = 3/2$,

$$y'''(l) = 1.5\times(1.5-1)\times(1.5-2)l^{1.5-3} = -0.375l^{-1.5} < 0$$

which means that for $\alpha = 3/2$ the marginal product is a strictly concave function. It remains to plot *MP(l)* for the three specified exponents α.



Part (b): To find the cost function, we multiply the wage rate by *l*(*y*) (i.e., the inverse of the production function),

$$C(y) = w\times l(y) = \boxed{wy^{1/\alpha}}$$

To find the marginal cost function, we differentiate *C(y)*, giving

$$MC(y) = C'(y) = \frac{w}{\alpha}y^{\frac{1}{\alpha}-1}$$

$$\therefore MC(y) = \boxed{\frac{w}{\alpha}y^{\left(\frac{1-\alpha}{\alpha}\right)}}$$

To find the average cost function, we divide *C(y)* by *y*,

$$AC(y) = \frac{C(y)}{y} = \frac{wy^{1/\alpha}}{y} = wy^{\frac{1}{\alpha}-1}$$

$$\therefore AC(y) = \boxed{wy^{\left(\frac{1-\alpha}{\alpha}\right)}}$$

The slope of the marginal cost function relatively to y is

$$MC'(y) = \frac{1-\alpha}{\alpha} \times \frac{w}{\alpha} \times y^{\left(\frac{1-2\alpha}{\alpha}\right)}$$

$$\therefore MC'(y) = \left(\frac{1-\alpha}{\alpha^2}\right) wy^{\left(\frac{1-2\alpha}{\alpha}\right)}$$

Similarly, the slope of the average cost function relatively to y is

$$AC'(y) = \left(\frac{1-\alpha}{\alpha}\right) wy^{\left(\frac{1-2\alpha}{\alpha}\right)}$$

The behaviors of marginal cost MC and average cost AC hinge on the sign of derivatives MC' and AC', which in turn depend on $1 - \alpha$, as highlighted above. In general, we have:

$$MC(y) \text{ and } AC(y) \text{ are } \begin{Bmatrix} \text{Increasing} \\ \text{Constant} \\ \text{Decreasing} \end{Bmatrix} \text{ in } l \text{ if } \alpha \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 1$$

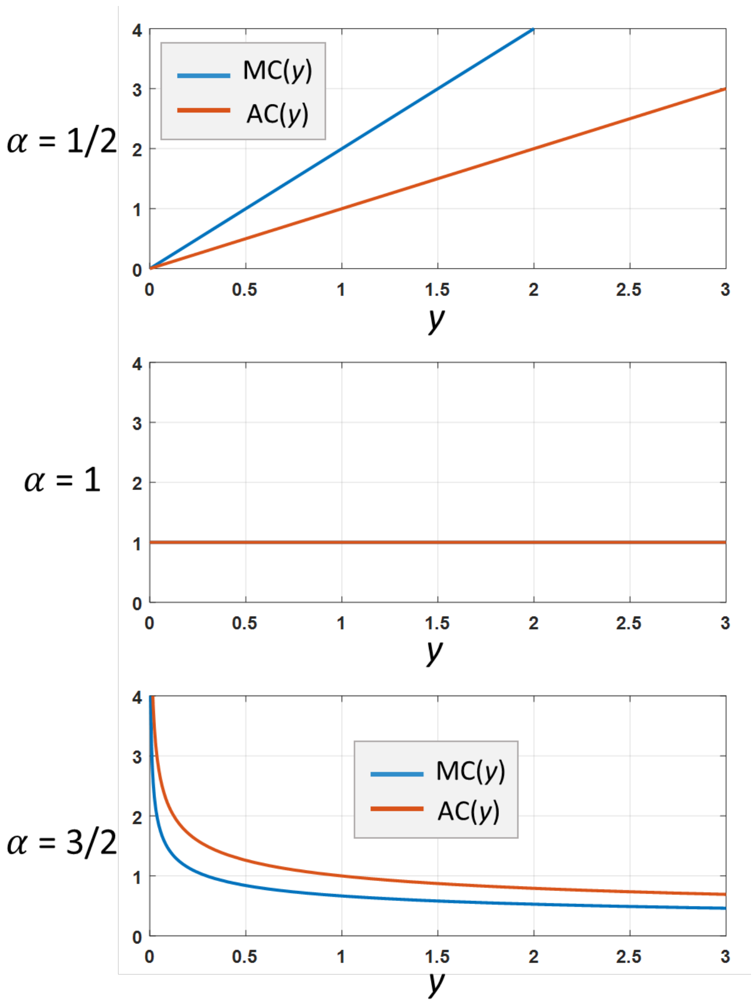
Part (c): Letting $w = 1$, the marginal cost function becomes

$$MC(y) = \frac{w}{\alpha} y^{\left(\frac{1-\alpha}{\alpha}\right)} = \frac{1}{\alpha} y^{\left(\frac{1-\alpha}{\alpha}\right)}$$

and the average cost is

$$AC(y) = wy^{\left(\frac{1-\alpha}{\alpha}\right)} = y^{\left(\frac{1-\alpha}{\alpha}\right)}$$

Notice that $AC(y) = \alpha \times MC(y)$. We proceed to plot $MC(y)$ and $AC(y)$ for $\alpha = 1/2$, 1, and 3/2.



■ **Problem 4**

Part (a): The inverse of the given production function is

$$y(l) = (l-\gamma)^{\beta} \quad \rightarrow \quad l(y) = \gamma + y^{\frac{1}{\beta}}$$

so the cost function follows as

$$C(y) = w \times l(y) = w \times \left(\gamma + y^{\frac{1}{\beta}} \right)$$

$$\therefore C(y) = w\gamma + wy^{\frac{1}{\beta}}$$

Note that the first term is a constant and hence constitutes the fixed cost, whereas the second term varies with *y* and constitutes the variable cost.

$$C(y) = \underbrace{w\gamma}_{=FC} + \underbrace{wy^{\frac{1}{\beta}}}_{=VC(y)}$$

The marginal cost function follows by differentiating *C(y)*,

$$MC(y) = C'(y) = 0 + \frac{1}{\beta} wy^{\frac{1}{\beta}-1}$$

$$\therefore MC(y) = \boxed{\frac{w}{\beta} y^{\left(\frac{1-\beta}{\beta}\right)}}$$

To find the average cost function, we divide *C(y)* by the output *y*,

$$AC(y) = \frac{C(y)}{y} = \underbrace{\frac{w\gamma}{y}}_{=AFC(y)} + \underbrace{\frac{wy^{\frac{1}{\beta}}}{y}}_{=AVC(y)} = \boxed{\frac{w\gamma}{y} + wy^{\left(\frac{1-\beta}{\beta}\right)}}$$

Part (b): The slope of the marginal cost function is

$$MC'(y) = \frac{d}{dy}\left[\frac{w}{\beta} y^{\left(\frac{1-\beta}{\beta}\right)} \right] = \frac{1-\beta}{\beta} \frac{w}{\beta} y^{\left(\frac{1-2\beta}{\beta}\right)}$$

In turn, the slope of the average cost function is

$$AC'(y) = \frac{d}{dy}\left[\frac{w\gamma}{y} + wy^{\left(\frac{1-\beta}{\beta}\right)} \right] = -\frac{w\gamma}{y^2} + \left(\frac{1-\beta}{\beta} \right) wy^{\left(\frac{1-2\beta}{\beta}\right)}$$

Now, the slope of *MC(y)* with *β* = 0.5 is

$$MC'(y)\Big|_{\beta=1/2} = \frac{1-0.5}{0.5} \times \frac{w}{0.5} \times y^{\left(\frac{1-2\times 0.5}{0.5}\right)} = 2w > 0$$

This indicates that *MC(y; β* = 1/2) is a monotonically increasing function.
Next, the slope of *AC(y)* with *β* = 0.5 is

$$AC'(y)\Big|_{\beta=1/2} = -\frac{w\gamma}{y^2} + \left(\frac{1-0.5}{0.5} \right) wy^{\left(\frac{1-2\times 0.5}{0.5}\right)} = -\frac{w\gamma}{y^2} + w$$

Hence, *AC(y; β* = 1/2) is a non-monotonic function.
Next, the slope of *MC(y)* with *β* = 1 is

$$MC'(y)\Big|_{\beta=1} = \underbrace{\frac{1-1.0}{1.0}}_{=0} \times \frac{w}{1.0} \times y^{\left(\frac{1-2\times 1.0}{1.0}\right)} = 0$$

Hence, *MC(y; β* = 1) is a constant function.
Next, the slope of *AC(y)* with *β* = 1 is

$$AC'(y)\Big|_{\beta=1} = -\frac{w\gamma}{y^2} + \underbrace{\left(\frac{1-1.0}{1.0} \right)}_{=0} wy^{\left(\frac{1-2\times 1.0}{1.0}\right)} = -\frac{w\gamma}{y^2} < 0$$

Hence, *AC(y; β* = 1) is a monotonically decreasing function.

Part (c): Monotonically decreasing average costs are incompatible with perfect competition. Accordingly, the production function addressed in Problem 1, ($y(l)=l^{\alpha}$) with $\alpha > 1$ and the production function addressed in Problem 2 ($y(l)=(l-\gamma)^{\beta}$) with $\beta = 1$ are incompatible with perfect competition. In the first case, decreasing average costs result from the increase in productivity; the more labor one uses, the larger the marginal productivity of labor will be, i.e., the larger the additional output will be. In the second case, decreasing average costs result from the combination of constant marginal and fixed costs; marginal costs always correspond to average variable costs, which in turn correspond to average wage costs (*w*). This is true regardless of the output. However, the larger the output, the lower the average fixed costs (*AFC(y)*), because the fixed costs can be distributed among more units of output. Thus, total average costs (*AC(y)*) are monotonically decreasing for $y > 0$.

■ Problem 5

Parts (a,b). Method 1: Lagrange multipliers. The Lagrangian for the problem at hand is

$$L(k,l)=40k+20l+\lambda\left(9840-80k^{0.25}l^{0.75}\right)$$

The first-order conditions are

$$\frac{dL}{dk}=40-20k^{-0.75}l^{0.75}=0$$

$$\therefore k^{-0.75}l^{0.75}=2 \quad \text{(I)}$$

$$\frac{dL}{dl}=20-60k^{0.25}l^{-0.25}=0$$

$$\therefore k^{0.25}l^{-0.25}=\frac{1}{3} \quad \text{(II)}$$

$$\frac{dL}{d\lambda}=9840-80k^{0.25}l^{0.75}=0$$

$$\therefore k^{0.25}l^{0.75}=123 \quad \text{(III)}$$

Dividing (I) by (II),

$$\frac{k^{-0.75}l^{0.75}}{k^{0.25}l^{-0.25}}=\frac{2}{1/3}$$

$$\therefore \frac{l}{k}=6$$

$$\therefore l=6k \quad \text{(IV)}$$

Substituting in (III),

$$k^{0.25}l^{0.75}=20 \;\rightarrow\; k^{0.25}\times(6k)^{0.75}=123$$

$$\therefore 3.83k=123$$

$$\therefore k=\frac{123}{3.83}\approx 32.1$$

Substituting in (IV),

$$l=6k=6\times 32.1\approx 192.6$$

The corresponding cost is

$$C(k,l)=40k+20l$$

$$C(k=32.1,\ l=192.6)=40\times 32.1+20\times 192.6=\boxed{\$5136}$$

Thus, the firm should employ approximately 32 units of capital and 193 units of labor for a minimum cost of approximately $5140.

Method 2: Equating marginal products. Alternatively, we can make use of the ratio

$$\frac{MP_L}{w}=\frac{MP_K}{r} \quad \text{(V)}$$

where *MP* denotes marginal product, namely

$$MP_L = \frac{dq}{dl} = 60k^{0.25}l^{-0.25}$$

$$MP_K = \frac{dq}{dk} = 20k^{-0.75}l^{0.75}$$

so that, substituting in (V),

$$\frac{60k^{0.25}l^{-0.25}}{20} = \frac{20k^{-0.75}l^{0.75}}{40}$$

$$\therefore l = 6k$$

This is identical to (IV); substituting in (III) and solving for *k* should yield $k \approx 32.1$, just like in the first method. The optimized cost is also found to be the same.

■ Problem 6

Part (a): With an annual capital input *k* = 10, the total labor productivity is

$$TP_l(l) = 6\times 10\times l - 3.6\times 10^2 - 0.4l^2$$

$$\therefore TP_l(l) = 60l - 360 - 0.4l^2$$

and the average labor productivity is

$$AP_l(l) = \frac{TP_l(l)}{l} = 60 - \frac{360}{l} - 0.4l$$

To find the labor input for which AP_l is maximum, we differentiate it with respect to *l* and set the result to zero,

$$\frac{dAP_l(l)}{dl} = -0.4 + \frac{360}{l^2} = 0$$

$$\therefore l = \sqrt{\frac{360}{0.4}} = 30$$

Substituting in the productivity function, we obtain

$$q(k = 10, l = 30) = 6\times 10\times 30 - 3.6\times 10^2 - 0.4\times 30^2 = 1080$$

Part (b): The marginal labor productivity is

$$TP_l(l) = 60l - 360 - 0.4l^2 \quad \rightarrow \quad MP_l = \frac{dTP_l}{dl} = 60 - 0.8l$$

Setting this to zero and solving for *l*,

$$60 - 0.8l = 0$$

$$l = \frac{60}{0.8} = \boxed{75}$$

Part (c): With *k* = 20, the total labor productivity is

$$TP_l(l) = 6\times 20\times l - 3.6\times 20^2 - 0.4l^2$$

$$\therefore TP_l(l) = 120l - 1440 - 0.4l^2$$

and the average labor productivity is

$$AP_l(l) = \frac{TP_l(l)}{l} = 120 - \frac{1440}{l} - 0.4l$$

To find the labor input for which AP_l is maximum, we differentiate with respect to *l* and set the result to zero,

$$\frac{dAP_l(l)}{dl} = -0.4 + \frac{1440}{l^2} = 0$$

$$\therefore l = \sqrt{\frac{1440}{0.4}} = 60$$

Substituting in the productivity function, we obtain

$$q(k = 20, l = 60) = 6 \times 20 \times 60 - 3.6 \times 20^2 - 0.4 \times 60^2 = 4320$$

The marginal labor productivity is

$$TP_l(l) = 120l - 1440 - 0.4l^2 \quad \rightarrow \quad MP_l = \frac{dTP_l}{dl} = 120 - 0.8l$$

Setting this to zero and solving for l,

$$120 - 0.8l = 0$$

$$\therefore l = \frac{120}{0.8} = \boxed{150}$$

Part (d): Doubling k (the capital input changed from k = 10 to k = 20) and doubling l (the labor input changed from l = 30 to l = 60) caused the output q to increase four-fold from 1080 to 4320. Hence, the production function at hand exhibits increasing returns to scale.

■ Problem 7

Part (a): Since the firm's capital in the short-run is fixed at k = 100, we can write

$$q = 4\sqrt{kl} \quad \rightarrow \quad q = 4\sqrt{100 \times l}$$

$$\therefore q = 40\sqrt{l}$$

Solving for labor,

$$q = 40\sqrt{l} \quad \rightarrow \quad l = \frac{q^2}{1600}$$

Substituting in the cost-function equation,

$$C(k,l) = vk + wl = 1.0 \times 100 + 5.0 \times \frac{q^2}{1600}$$

$$\therefore \boxed{C(q) = 100 + \frac{q^2}{320}} \quad \text{(I)}$$

To find the short-run *average* cost curve, we divide through by q,

$$AC(q) = \frac{C(q)}{q} = \boxed{\frac{100}{q} + \frac{q}{320}} \quad \text{(II)}$$

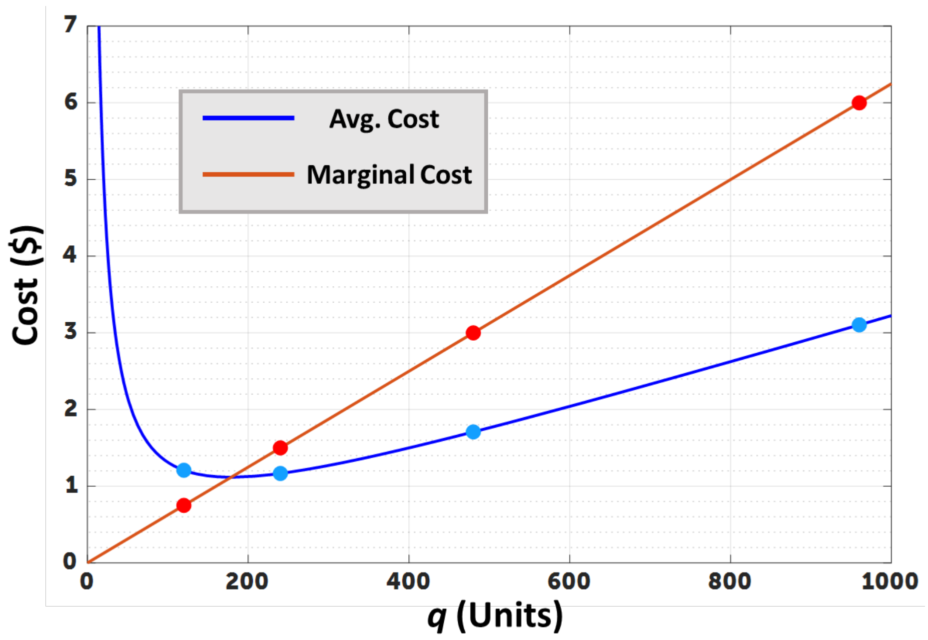
Part (b): To find the short-run marginal cost curve, we differentiate *C(q)* with respect to q, giving

$$C(q) = 100 + \frac{q^2}{320} \quad \rightarrow \quad MC(q) = \frac{dC(q)}{dq} = \boxed{\frac{q}{160}} \quad \text{(III)}$$

Equations (I), (II), and (III) yield the short-run total cost, average cost, and marginal cost, respectively, for a level of output q. We proceed to tabulate the results for q = 120, 240, 480, and 960; use Excel to save time and avoid tedious calculations.

q	*C(q)*	*AC(q)*	*MC(q)*
120	$140.00	$1.21	$0.75
240	$260.00	$1.17	$1.50
480	$740.00	$1.71	$3.00
960	$2,660.00	$3.10	$6.00

Part (c): The graph in question is shown on the next page. The blue line is the average cost equation (II); the red line is the marginal cost equation (III). The data points obtained in the foregoing table are shown as well.



Part (d): The average cost curve intersects the marginal cost curve at an output level q_0 such that

$$AC(q_0) = MC(q_0) \rightarrow \frac{100}{q_0} + \frac{q_0}{320} = \frac{q_0}{160}$$

$$\therefore \frac{100}{q_0} = \frac{q_0}{160} - \frac{q_0}{320}$$

$$\therefore \frac{100}{q_0} = \frac{q_0}{320}$$

$$\therefore q_0 = \sqrt{100 \times 320} \approx \boxed{179}$$

As long as the marginal cost of producing one more unit is below the average-cost curve, average costs will be falling. Similarly, if the marginal cost of producing one more unit is higher than the average cost, then average costs will be rising. Therefore, the short-run marginal cost curve must intersect the short-run average cost curve at its lowest point.

Part (e): Squaring the production function and solving for labor,

$$q = 4\sqrt{k_1 l} \rightarrow q^2 = 16k_1 l$$

$$\therefore l = \frac{q^2}{16k_1}$$

Substituting in the short-run cost,

$$C(q) = vk_1 + wl = \boxed{vk_1 + \frac{wq^2}{16k_1}}$$

Part (f): Differentiating *C(q)* from the previous part with respect to k_1 and setting the result to zero,

$$C(q) = vk_1 + \frac{wq^2}{16k_1} \rightarrow \frac{dC}{dk_1} = v - \frac{wq^2}{16k_1^2} = 0$$

$$\therefore v = \frac{wq^2}{16k_1^2}$$

$$\therefore k_1^2 = \frac{wq^2}{16v}$$

$$\therefore \boxed{k_1 = \frac{q}{4}\sqrt{\frac{w}{v}}}$$

Part (g): Substituting k_1 from part (f) into *C* from part (e), the long-run total cost of chocolate bar production becomes

$$C(q) = v \times \frac{q}{4}\sqrt{\frac{w}{v}} + \frac{wq^2}{16 \times \frac{q}{4}\sqrt{\frac{w}{v}}}$$

$$\therefore C(q)=\frac{q}{4}\sqrt{vw}+\frac{wq^2}{16}\frac{4}{q}\sqrt{\frac{v}{w}}$$

$$\therefore C(q)=\frac{q}{4}\sqrt{vw}+\frac{wq}{4}\sqrt{\frac{v}{w}}$$

$$\therefore C(q)=\frac{q}{4}\sqrt{vw}+\frac{q}{4}\sqrt{vw}$$

$$\therefore \boxed{C(q)=\frac{q\sqrt{vw}}{2}}$$

Problem 8

Part (a): The total output is $q = q_1 + q_2$, with

$$q_1=\sqrt{k_1 l_1}=\sqrt{49 l_1}=7\sqrt{l_1}$$

$$q_2=\sqrt{k_2 l_2}=\sqrt{196 l_2}=14\sqrt{l_2}$$

Solving for l_1 and l_2,

$$q_1=7\sqrt{l_1}\ \rightarrow\ l_1=\frac{q_1^2}{49}$$

$$q_2=14\sqrt{l_2}\ \rightarrow\ l_2=\frac{q_2^2}{196}$$

Noting that rental rates for capital and labor are v = \$1 and w = \$1, respectively, the short-run cost functions for the two factories are

$$C_1(k,l)=vk_1+wl_1=1.0\times 49+1.0\times\frac{q_1^2}{49}$$

$$\therefore C_1(k,l)=49+\frac{q_1^2}{49}\ \text{(I)}$$

$$C_2(k,l)=vk_2+wl_2=1.0\times 196+1.0\times\frac{q_2^2}{196}$$

$$\therefore C_2(k,l)=196+\frac{q_2^2}{196}\ \text{(II)}$$

The total short-run cost then becomes

$$C(q_1,q_2)=C_1(q_1)+C_2(q_2)=49+\frac{q_1^2}{49}+196+\frac{q_2^2}{196}=245+\frac{q_1^2}{49}+\frac{q_2^2}{196}$$

To minimize cost, we first set up a Lagrangian,

$$L(q_1,q_2,\lambda)=245+\frac{q_1^2}{49}+\frac{q_2^2}{196}+\lambda(q-q_1-q_2)$$

The first-order conditions are computed next,

$$\frac{dL}{dq_1}=\frac{2q_1}{49}-\lambda=0$$

$$\therefore \lambda=\frac{2q_1}{49}\ \text{(I)}$$

$$\frac{dL}{dq_2}=\frac{2q_2}{196}-\lambda=0$$

$$\therefore \lambda=\frac{q_2}{98}\ \text{(II)}$$

$$\frac{dL}{d\lambda}=q-q_1-q_2=0$$

$$\therefore q=q_1+q_2\ \text{(III)}$$

Equating (I) and (II) and solving for q_2,

$$\frac{q_2}{98}=\frac{2q_1}{49} \quad \rightarrow \quad q_2=4q_1$$

Substituting in (III),

$$q=q_1+q_2=q_1+4q_1=5q_1$$

$$\therefore \boxed{q_1=\frac{q}{5}}$$

Also,

$$q_2=4q_1=\boxed{\frac{4q}{5}}$$

In order to minimize short-run cost, 20% of the output should come from factory 1, and 80% should come from factory 2.

Part (b): The short-run cost curve was derived in part (a) and can be stated as

$$C(q_1,q_2)=245+\frac{q_1^2}{49}+\frac{q_2^2}{196}$$

Using the results from part (a), we can restate this in terms of the total output *q*,

$$C(q)=245+\frac{(q/5)^2}{49}+\frac{(4q/5)^2}{196}=\boxed{245+\frac{q^2}{245}}$$

This is the short-run cost curve. Differentiating it with respect to *q* gives the short-run marginal cost,

$$MC(q)=\frac{dC(q)}{dq}=\boxed{\frac{2q}{245}}$$

Dividing *C(q)* by *q* yields the short-run average cost,

$$AC(q)=\frac{C(q)}{q}=\boxed{\frac{245}{q}+\frac{q}{245}}$$

The marginal cost of the 500th widget is

$$MC(q=500)=\frac{2\times 500}{245}=\underline{\$4.08}$$

The marginal cost of the 1000th widget is

$$MC(q=1000)=\frac{2\times 1000}{245}=\underline{\$8.16}$$

The marginal cost of the 2000th widget is

$$MC(q=2000)=\frac{2\times 2000}{245}=\underline{\$16.33}$$

Part (c): In the long run, given constant returns to scale, location doesn't really matter because the producer can change the capital *k*. The entrepreneur could split evenly or produce all output in one location, etc.

Part (d): If there are decreasing returns to scale with identical production functions, then the entrepreneur should let each firm have equal share of production. *AC* and *MC* are not constant anymore, becoming increasing functions of *q*.

■ Problem 9

Part (a): In the long run, no input should be wasted. Hence, $5k = 10l = q$, implying $k = 2l = q/5$. Letting *v* and *w* denote rent rate and wage rate, respectively, the cost function becomes

$$C=vk+wl=v\times(2l)+w\times l=v\times\frac{q}{5}+w\times\frac{q}{10}$$

$$\therefore C=v\times\frac{2q}{10}+w\times\frac{q}{10}$$

$$\therefore \boxed{C = \frac{q}{10}(2v + w)}$$

To find the long-run average cost function, we divide through by *q*, giving

$$AC = \frac{C(q)}{q} = \boxed{\frac{(2v + w)}{10}}$$

Differentiating *C(q)* yields the long-run marginal cost function,

$$MC = C'(q) = \boxed{\frac{(2v + w)}{10}}$$

Part (b): With *k* set to 10, the production function becomes

$$q = \min(5k, 10l) = \min(5 \times 10, 10l) = \min(50, 10l)$$

There are two cases to consider. First, with *l* < 5, the production function is

$$q = \min(50, 10l) = 10l$$

and the cost function becomes

$$C(q) = vk + wl = \boxed{10v + \frac{wq}{10}}$$

Dividing through by *q* gives the average cost function,

$$AC(q) = \frac{C(q)}{q} = \boxed{\frac{10v}{q} + \frac{w}{10}}$$

Differentiating with respect to *q* gives the marginal cost,

$$MC(q) = C'(q) = \boxed{\frac{w}{10}}$$

If *l* ≥ 5, then *q* = 50. It is impossible to produce more than 50 units in the long run. Hence, *C(q)* = *AC(q)* = *MC(q)* = ∞ for *q* > 50. Lastly, right at *q* = 50, we have the same formula for total cost as that when *l* < 5, namely

$$C(q) = 10k + \frac{wq}{10}$$

Note, however, that marginal cost *MC(q)* is not defined at *q* = 50 because the lateral limits as we approach *q* = 50 from the left or the right are different, and hence there is no well-defined *C′*(*q* = 50). That said, the average cost is well-defined and given by the same expression as that for *l* < 5,

$$AC(q) = \frac{10k}{q} + \frac{w}{10}$$

Part (c): Let *v* = 3 and *w* = 4. In the long run, the average cost *AC(q)* with these rates is

$$AC\Big|_{(v=\$3, w=\$4)} = \frac{(2 \times 3 + 4)}{10} = \boxed{\$1.0}$$

Since the expression for long-run marginal cost is identical to *AC*, it follows that

$$MC = AC\Big|_{(v=\$3, w=\$4)} = \boxed{\$1.0}$$

In the short run, assuming *q* < 50, the average cost with the given rates is

$$AC\Big|_{(v=\$3, w=\$4)} = \frac{10 \times 3}{q} + \frac{4}{10}$$

$$\therefore AC\Big|_{(v=\$3, w=\$4)} = \boxed{\frac{30}{q} + \frac{2}{5}}$$

Lastly, the marginal cost is

$$MC\big|_{(v=\$3,w=\$4)}=\frac{4}{10}=\boxed{\$0.40}$$

Problem 10

Part (a): Differentiating to obtain the marginal cost, we have

$$\overline{MC}=\frac{dC(q)}{dq}=\frac{1}{100}q^2+0.5q+6.25$$

Equating this to price *p* and manipulating,

$$p=\frac{1}{100}q^2+0.5q+6.25$$

$$\therefore 100p=q^2+50q+625$$

$$\therefore 100p=(q+25)^2$$

$$\therefore \sqrt{100p}=(q+25)$$

$$\therefore 10\sqrt{p}=q+25$$

$$\therefore \boxed{q=10\sqrt{p}-25}$$

Part (b): Here, all we have to do is multiply the individual short-run supply curve obtained in part (a) by the number of firms,

$$Q=200q=200\times\left(10\sqrt{p}-25\right)=\boxed{2000\sqrt{p}-5000}$$

Part (c): Equating *Q* from part (b) to the given market demand function,

$$2000\sqrt{p}-5000=23{,}800-200p$$

Using Mathematica to speed things up, we have

```
In[683]:= Solve[2000 * √p - 5000 == 23800. - 200 * p, p]
Out[683]=
{{p → 64.}}
```

(The dot after 23,800 ensures that Mathematica will print a numerical result instead of a symbolic one.) Thus, $p \approx 64$. The corresponding aggregate supply *Q* is

$$Q=23{,}800-200\times 64=\boxed{11{,}000}$$

Substituting $p = 64$ in the firm supply curve obtained in part (a), we obtain

$$q=10\times\sqrt{64}-25\approx\boxed{55}$$

Each firm will supply 55 units of the product. The corresponding cost is

$$C(q=55)=\frac{1}{300}\times 55^3+0.25\times 55^2+6.25\times 55+15\approx \$1670$$

Dividing this by *q* gives the average cost

$$\overline{AC}=\frac{C(q=55)}{55}\approx \$30.4$$

Lastly, the individual firm profits are

$$\pi=\left(p-\overline{AC}\right)q=(64-30.4)\times 55=\boxed{\$1848}$$

Problem 11

Part (a): As in Problem 10, the representative firm's supply curve is obtained by equating marginal cost to price and solving for *q*. However, we can follow Serrano and Feldman (2018) and be a bit more rigorous, first checking that **(1)** price $p \geq$ *min AC(q)* (that is, price must be greater than minimum average cost), and **(2)** marginal cost *MC(q)* is rising. We proceed to differentiate *AC(q)* and *MC(q)*,

$$\overline{AC}(q)=\frac{C(q)}{q}=q^2-\frac{1}{2}q+\frac{99}{2q}$$

$$\overline{MC}(q)=\frac{dC(q)}{dq}=3q^2-q$$

To minimize *AC(q)*, we take its first derivative, set it to zero, and solve for *q*,

$$\frac{d\overline{AC}}{dq}=2q-\frac{1}{2}-\frac{99}{2q^2}=0$$

```
In[23]:= Solve[2 * q - 1/2 - 99/(2 * q^2) == 0, q]

Out[23]= {{q → 3}, {q → 1/8 (-11 - i √407)}, {q → 1/8 (-11 + i √407)}}
```

As shown in the Mathematica snippet, the real value of *q* at which average cost is minimized is *q* = 3. We can check that this is indeed a minimum by finding the second derivative of *AC(q)*, substituting *q* = 3, and examining the sign of the result:

```
In[24]:= AC[q_] := q^2 - 1/2 q + 99/(2 * q);

In[26]:= D[AC[q], {q, 2}] /. q → 3

Out[26]= 17/3
```

Since *AC''(q* = 3) = 17/3 > 0, the point in question is indeed a local minimum. The corresponding average cost at *q* = 3 is

$$\min\overline{AC}=\overline{AC}(3)=3^2-\frac{1}{2}\times 3+\frac{99}{2\times 3}=24$$

This is all we need to check with regard to condition (1). Turning to condition (2), we can ascertain that marginal cost is rising by taking its first derivative and substituting *q* = 3,

$$\overline{MC}(q)=3q^2-q \quad\rightarrow\quad \frac{d\overline{MC}}{dq}=6q-1$$

$$\therefore \left.\frac{d\overline{MC}(q)}{dq}\right|_{q=3}=6\times 3-1=17>0$$

Thus, marginal cost is rising for *q* > 3. We conclude that for *p* < 24, the representative firm's supply curve is *q* = 0. In turn, for *p* > 24, the supply curve is obtained by setting *MC* = *p* and solving for output *q*,

$$\overline{MC}(q)=p \quad\rightarrow\quad 3q^2-q=p$$

$$\therefore 3q^2-q-p=0$$

$$\therefore \boxed{q=\frac{1+\sqrt{1+12p}}{6}}$$

Part (b): In the long run, the number of firms adjusts itself to drive the market to the zero-profit equilibrium. The long-run market supply curve is horizontal at *p* = *min AC(y)* = 24. Each firm produces

$$q=\frac{1+\sqrt{1+12\times 24}}{6}=\boxed{3}$$

Substituting *p* = 24 into the demand curve yields

$$Q=1140-10\times 24=\boxed{900}$$

Therefore, there are 900/3 = 300 firms in the market.

Problem 12

Part (a): The contract curve is the locus of the equality $MRS^a = MRS^b$, that is, the curve obtained by equating the marginal rates of substitution of Alfred and Beto. The marginal rate of substitution for either Alfred and Beto is given by $MRS = MU_{x,i}/MU_{y,i}$, where subscript $i = \{a, b\}$ and MU denotes marginal utility. The marginal utilities are

$$MU_{x,a} = \alpha x_a^{\alpha-1} y_a^{1-\alpha}$$

$$MU_{y,a} = (1-\alpha) x_a^{\alpha} y_a^{-\alpha}$$

$$MU_{x,b} = \alpha x_b^{\alpha-1} y_b^{1-\alpha}$$

$$MU_{y,b} = (1-\alpha) x_b^{\alpha} y_b^{-\alpha}$$

so that

$$MRS_a = MRS_b \rightarrow \frac{MU_{x,a}}{MU_{y,a}} = \frac{MU_{x,b}}{MU_{y,b}}$$

$$\therefore \frac{\alpha x_a^{\alpha-1} y_a^{1-\alpha}}{(1-\alpha) x_a^{\alpha} y_a^{-\alpha}} = \frac{\alpha x_b^{\alpha-1} y_b^{1-\alpha}}{(1-\alpha) x_b^{\alpha} y_b^{-\alpha}}$$

$$\therefore \frac{\alpha y_a}{(1-\alpha) x_a} = \frac{\alpha y_b}{(1-\alpha) x_b}$$

$$\therefore \frac{y_a}{x_a} = \frac{y_b}{x_b} \quad \text{(I)}$$

This equality has four unknowns. To obtain more equations, we set the market clearing conditions,

$$x_a + x_b = 1 \rightarrow x_a = 1 - x_b$$

$$y_a + y_b = 1 \rightarrow y_a = 1 - y_b$$

Substituting these in (I),

$$\frac{y_a}{x_a} = \frac{y_b}{x_b} \rightarrow \frac{1-y_b}{1-x_b} = \frac{y_b}{x_b}$$

$$\therefore x_b - x_b y_b = y_b - x_b y_b$$

$$\therefore \boxed{x_b = y_b}$$

Thus, the contract curve is the diagonal of the Edgeworth box.

Part (b): The competitive equilibrium is on the contract curve. From part (a), we have $y_g = x_g$ on the contract curve. It follows that, at the competitive equilibrium,

$$\frac{p_x}{p_y} = MRS_a = MRS_b = \frac{\alpha \cancel{y_b}}{(1-\alpha)\cancel{x_b}} = \boxed{\frac{\alpha}{1-\alpha}}$$

Problem 13

Part (a): Alfred's utility with the initial endowment $\omega_A = (2,2)$ is

$$U(x_A = 2, y_A = 2) = 2 \times 2 = 4$$

Alfred's utility with the third party's suggested equilibrium is

$$U(x'_A = 4, y'_A = 1) = 4 \times 1 = 4$$

Beto's utility with the initial endowment $\omega_B = (3,3)$ is

$$U(x_B = 3, y_B = 3) = 3 \times 3^2 = 27$$

Beto's utility with the third party's suggested equilibrium is

$$U(x'_B = 1, y'_B = 4) = 1 \times 4^2 = 16$$

Under the suggested equilibrium, Alfred's utility is unchanged, while Beto's utility decreases from 27 to 16. Accordingly, the third party's suggested equilibrium is not a Pareto improvement over the endowment.

Part (b): With prices $p_x = p_y = 1$, Alfred's budget constraint is $x_A + y_A = 4$. Equating the ratio of marginal utilities to the ratio of prices and substituting in the budget line gives the optimal consumption bundle $(x''_A, y''_A) = (2,2)$.

Part (c): With prices $p_x = p_y = 1$, Beto's budget constraint is $x_B + y_B = 6$. Equating the ratio of marginal utilities to the ratio of prices and substituting in the budget line gives the optimal consumption bundle $(x_B'', y_B'') = (2,4)$.
Part (d): The third party's suggested equilibrium allocation and price vector do not constitute a competitive equilibrium. Consumers are not maximizing their utilities at this allocation, in that there is excess supply for *x* and excess demand for *y*. This suggests that in equilibrium, either p_x should drop below 1 or p_y should rise above 1.
Part (e): Alfred's budget constraint is

$$x_A + p_y y_A = 2 + 2p_y \quad \text{(I)}$$

To find his desired consumption bundle, we equate the marginal rate of substitution to the prices ratio,

$$\overline{MRS}_A = \frac{y_A}{x_A} = \frac{1}{p_y}$$

$$\therefore x_A = p_y y_A$$

Substituting x_A in (I) brings to

$$p_y y_A + p_y y_A = 2 + 2p_y$$

$$\therefore 2p_y y_A = 2 + 2p_y$$

$$\therefore y_A^* = \frac{1 + p_y}{p_y}$$

Combining this with the budget constraint gives $x_A^* = (1 + p_y)$.
Part (f): Beto's budget constraint is

$$x_B + p_y y_B = 3 + 3p_y \quad \text{(II)}$$

To find his desired consumption bundle, we equate the marginal rate of substitution to the prices ratio,

$$\overline{MRS}_B = \frac{y_B}{2x_B} = \frac{1}{p_y}$$

$$\therefore x_B = \frac{1}{2} p_y y_B$$

Substituting in (II) yields

$$\frac{1}{2} p_y y_B + p_y y_B = 3 + 3p_y$$

$$\therefore \frac{3}{2} p_y y_B = 3 + 3p_y$$

$$\therefore y_B^* = \frac{2(1 + p_y)}{p_y}$$

Combining this with the budget constraint gives $x_B^* = (1 + p_y)$.
Part (g): Equating the demand for *x* to the supply for *x*, we have

$$x_A^* + x_B^* = x_A^0 + x_B^0$$

$$\therefore (1 + p_y) + (1 + p_y) = 2 + 3$$

$$\therefore 2 + 2p_y = 5$$

$$\therefore \boxed{p_y = \frac{3}{2}}$$

Given the price p_y, we can solve for Alfred's competitive equilibrium bundle, namely

$$x_A^* = 1 + p_y = 1 + \frac{3}{2} = \frac{5}{2}$$

$$y_A^* = \frac{1 + 3/2}{3/2} = \frac{5}{3}$$

$$\boxed{(x_A^*, y_A^*) = \left(\frac{5}{2}; \frac{5}{3}\right)}$$

Similarly, Beto's competitive equilibrium bundle is

$$x_B^* = 1 + p_y = 1 + \frac{3}{2} = \frac{5}{2}$$

$$y_B^* = \frac{2\times(1+3/2)}{3/2} = \frac{10}{3}$$

$$\boxed{(x_B^*, y_B^*) = \left(\frac{5}{2}; \frac{10}{3}\right)}$$

Problem 14

Part (a): The tangency condition is that the marginal rate of substitution of bread for labor, $MRS_{l,x}$, should equal the marginal product of labor in the production of bread, MP_l, so that

$$-MRS_{l,x} = MP_l$$

$$\therefore -\left(-\frac{1}{2}\right) = \frac{1}{2\sqrt{l}}$$

$$\therefore \sqrt{l} = 1$$

$$\therefore \boxed{l = 1}$$

Substituting in the production function, $x = \sqrt{l} = \sqrt{1} = 1$.

Part (b): We first use the utility-maximizing tangent condition to compute the wage rate w,

$$-MRS_{l,x} = \frac{w}{p} \quad \rightarrow \quad -\left(-\frac{1}{2}\right) = \frac{w}{1}$$

$$\therefore w = \frac{1}{2}$$

Paul's breadmaking firm must solve the profit maximization problem

$$\max_l \pi = px - wl = 1.0\sqrt{l} - \frac{1}{2}l$$

Differentiating and setting the result to zero,

$$\frac{d\pi}{dl} = \frac{1}{2\sqrt{l}} - \frac{1}{2} = 0$$

$$\therefore \frac{1}{2\sqrt{l}} = \frac{1}{2}$$

$$\therefore \sqrt{l} = 1$$

$$\therefore \boxed{l = 1}$$

Substituting $l = 1$ in the profit function,

$$\pi(l) = 1.0\sqrt{l} - \frac{1}{2}l = 1.0\times\sqrt{1} - \frac{1}{2}\times 1 = \boxed{\frac{1}{2}}$$

In summary, $x = 1$, $l = 1$, $p = 1$, $w = 1/2$, and $\pi = 1/2$.

Part (c): As before, the Pareto efficient allocation is that the indifference curve should be tangent to the production function, giving

$$-MRS_{l,x} = MP_l$$

$$\therefore -\left(-\frac{1}{2}\right) = \frac{d}{dl}\left(l^{2/3}\right) = \frac{2}{3l^{1/3}}$$

$$\therefore \frac{1}{2} = \frac{2}{3l^{1/3}}$$

$$\therefore l^{1/3} = \frac{4}{3}$$

$$\therefore \boxed{l = \frac{64}{27}}$$

Substituting in the production function,

$$x = l^{2/3} = \left(\frac{64}{27}\right)^{\frac{2}{3}} = \boxed{\frac{16}{9}}$$

Part (d): As in part (b), the wage rate continues to be $w = 1/2$. Paul's firm seeks to solve the profit maximization problem

$$\max_{l} \pi = px - wl = 1.0l^{2/3} - \frac{1}{2}l$$

Differentiating and setting the result to zero,

$$\frac{d\pi}{dl} = \frac{2}{3l^{1/3}} - \frac{1}{2} = 0$$

$$\therefore \frac{2}{3l^{1/3}} = \frac{1}{2}$$

$$\therefore l^{1/3} = \frac{4}{3}$$

$$\therefore \boxed{l = \frac{64}{27}}$$

Substituting in the profit function,

$$\pi(l) = l^{2/3} - \frac{1}{2}l = \left(\frac{64}{27}\right)^{\frac{2}{3}} - \frac{1}{2} \times \left(\frac{64}{27}\right) = \boxed{\frac{16}{27}}$$

Summarizing, $x = 16/9$, $l = 64/27$, $p = 1$, $w = 1/2$, and $\pi = 16/27$.

■ Problem 15

Part (a): With the wage normalized to 1, Peter's firm obtains a profit π given by

$$\pi = p_x x + p_y y - l$$

Replacing l with the given inverse production function,

$$\pi = p_x x + p_y y - \left(x^2 + y^2 + 3xy\right)$$

$$\therefore \pi = p_x x + p_y y - x^2 - y^2 - 3xy$$

Differentiating with respect to x gives the first-order condition

$$\frac{\partial \pi}{\partial x} = p_x - 2x - 3y = 0 \quad \text{(I)}$$

Differentiating with respect to y gives the first-order condition

$$\frac{\partial \pi}{\partial y} = p_y - 2y - 3x = 0 \quad \text{(II)}$$

Solving (I) for y,

$$p_x - 2x - 3y = 0$$

$$\therefore y = \frac{p_x - 2x}{3}$$

Substituting in (II),

$$p_y - 2 \times \left(\frac{p_x - 2x}{3}\right) - 3x = 0$$

Multiplying through by 3 and rearranging,

$$3p_y - 2 \times \left(p_x - 2x\right) - 9x = 0$$

$$\therefore 3p_y - 2p_x + 4x - 9x = 0$$

$$\therefore 3p_y - 2p_x = 5x$$

$$\therefore x = \frac{1}{5}\left(3p_y - 2p_x\right) \quad \text{(III)}$$

Substituting x into (II),

$$p_y - 2y - 3 \times \frac{1}{5}\left(3p_y - 2p_x\right) = 0$$

$$\therefore 5p_y - 10y - 3(3p_y - 2p_x) = 0$$
$$\therefore 5p_y - 10y - 9p_y + 6p_x = 0$$
$$\therefore 6p_x - 4p_y = 10y$$
$$\therefore 3p_x - 2p_y = 5y$$
$$\therefore y = \frac{1}{5}(3p_x - 2p_y) \text{ (IV)}$$

Equations (III) and (IV) are the supply functions for bread and cookies, respectively.
Part (b): Paul the price-taking consumer chooses x, y, l to maximize $U(x,y,l)$ subject to the budget constraint

$$p_x x + p_y y = l + \pi$$

Paul takes p_x, p_y, and profit π as given. To solve this utility maximization problem, we use the budget constraint to solve for l and then substitute back into the utility function. We then maximize

$$U(l,x,y) = \frac{3}{2}xy + x + y - \frac{1}{2}l = \frac{3}{2}xy + x + y - \frac{1}{2}\times(p_x x + p_y y - \pi)$$
$$\therefore U(l,x,y) = \frac{3}{2}xy + x + y - \frac{1}{2}p_x x - \frac{1}{2}p_y y + \frac{1}{2}\pi$$

This leads to the following first-order conditions,

$$\frac{\partial U}{\partial x} = \frac{3}{2}y + 1 - \frac{1}{2}p_x = 0$$
$$\frac{\partial U}{\partial y} = \frac{3}{2}x + 1 - \frac{1}{2}p_y = 0$$

Solving the latter for x,

$$\frac{3}{2}x + 1 - \frac{1}{2}p_y = 0$$
$$3x + 2 - p_y = 0$$
$$x = \frac{p_y - 2}{3} \text{ (V)}$$

Solving the former for y,

$$\frac{3}{2}y + 1 - \frac{1}{2}p_x = 0$$
$$3y + 2 - p_x = 0$$
$$y = \frac{p_x - 2}{3} \text{ (VI)}$$

Equations (V) and (VI) are the demand functions for x and y. (The demands for x and y in this exercise are independent of income because of the quasi-linearity in l. Moreover, we have an interior solution because the supply of labor is not assumed to have any bounds. In general, demand functions will be more complicated functions of prices and profit than in this exercise.)
Part (c): To find an equilibrium, we solve the system of linear equations comprised of (III), (IV), (V), and (VI) for the four variables x, y, p_x, and p_y; we can speed things up using Mathematica,

```
In[44]:= Solve[{X == (3 * py - 2 * px) / 5, Y == (3 * py - 2 * px) / 5, X == (py - 2) / 3, Y == (px - 2) / 3}, {X, Y, px, py}]
Out[44]= {{X → 1, Y → 1, px → 5, py → 5}}
```

As shown, $x = y = 1$ and $p_x = p_y = 5$. Substituting in the inverse production function gives the corresponding labor demand l,

$$l = x^2 + y^2 + 3xy = 1^2 + 1^2 + 3\times1\times1 = \boxed{5}$$

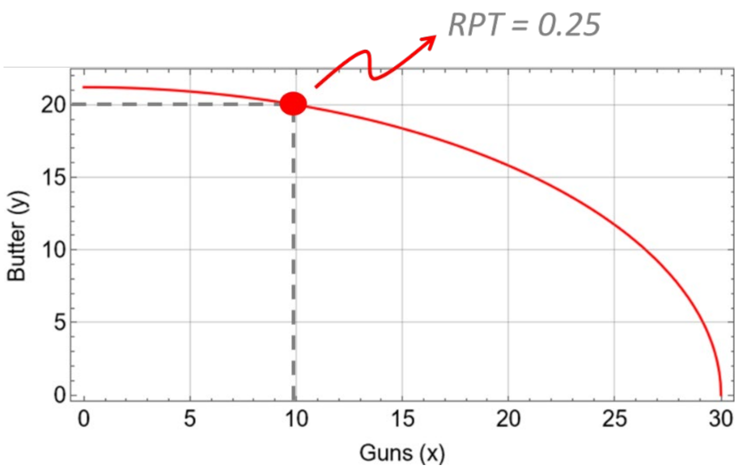
firm profit π at equilibrium is

$$\pi = p_x x + p_y y - l = 5\times1 + 5\times1 - 5 = \boxed{\$5}$$

and, for the consumer, the right-hand side of the budget equation is \$10. The supply of labor is found by using the budget equation, which yields supply equal to 5 (also equal to the firm's demand). The competitive equilibrium in this one-consumer economy can also be found by first finding the Pareto optimal allocation and then using the first welfare theorem.

Problem 16

Part (a): The frontier plots as an elliptical arc on the *xy*-plane.



Part (b): Substituting $y = 2x$ in the PPF and solving for x,

$$x^2 + 2y^2 = 900 \quad \rightarrow \quad x^2 + 2\times(2x)^2 = 900$$

$$\therefore 9x^2 = 900$$

$$\therefore x = \sqrt{\frac{900}{9}} = \boxed{10}$$

so that

$$y = 2x = 2\times 10 = \boxed{20}$$

Under the given consumption bundle preferences, the industry will produce 10 units of guns and 20 units of butter.

Part (c): Differentiating the given PPF implicitly, we have

$$x^2 + 2y^2 = 900 \quad \rightarrow \quad 2x + 4y\frac{dy}{dx} = 0$$

$$\therefore 4y\frac{dy}{dx} = -2x$$

$$\therefore \frac{dy}{dx} = -\frac{x}{2y}$$

$$\therefore -RPT = \frac{f_x}{f_y} = -\frac{10}{2\times 20} = -0.25$$

The price ratio will therefore be $p_x/p_y = 1/4$.

Problem 17

Part (a): Noting that $l_F = F^2$ and $l_C = C^2$, the production possibility frontier has the form

$$l_C + l_F = 200 \quad \rightarrow \quad C^2 + F^2 = 200 \;\; \text{(I)}$$

Differentiating with respect to F gives the rate of product transformation

$$2C\frac{dC}{dF} + 2F = 0$$

$$\therefore -2C\frac{dC}{dF} = 2F$$

$$\therefore \overline{RPT} = -\frac{dC}{dF} = \frac{F}{C}$$

The marginal rate of substitution follows from the marginal utilities,

$$\overline{MRS} = \frac{U_f}{U_c} = \frac{\sqrt{C}\big/\left(2\sqrt{F}\right)}{\sqrt{F}\big/\left(2\sqrt{C}\right)} = \frac{C}{F}$$

Equating *RPT* to *MRS*,

$$\overline{RPT} = \overline{MRS} \quad \rightarrow \quad \frac{F}{C} = \frac{C}{F}$$
$$\therefore F = C$$

Substituting in the PPF (equation (I)) brings to

$$C^2 + F^2 = 200 \quad \rightarrow \quad C^2 + C^2 = 200$$
$$\therefore 2C^2 = 200$$
$$\therefore \boxed{C = F = 10}$$

The corresponding utility is

$$U = \sqrt{F \times C} = \sqrt{10 \times 10} = \boxed{10}$$

The RPT follows as

$$\overline{RPT} = \frac{F}{C} = \frac{10}{10} = \boxed{1}$$

Part (b): Setting the marginal rate of substitution to the given price ratio, we have

$$\overline{MRS} = \frac{p_f}{p_c} = 2 = \frac{C}{F} \quad \rightarrow \quad C = 2F$$

Since Robinson continues to produce the same quantities of *F* and *C* as in the previous part, the total production is 30 (of which *F* = 10 and *C* = 2*F* = 2 × 10 = 20) and the budget constraint is

$$2F + C = 30$$

Substituting the condition for utility maximization *C* = 2*F*,

$$2F + 2F = 30$$
$$\therefore 4F = 30$$
$$\therefore \boxed{F = 7.5}$$

and

$$C = 2F = 2 \times 7.5 = \boxed{15}$$

The updated utility is

$$U = \sqrt{F \times C} = \sqrt{7.5 \times 15} = \boxed{\sqrt{112.5}} > 10$$

Thus, there is substantial improvement in utility relatively to part (a).
Part (c): To adjust to world prices, we equate the specified price ratio to the rate of product transformation, giving

$$\frac{p_f}{p_c} = 2 = \frac{F}{C} \quad \rightarrow \quad F = 2C$$

Substituting in the PPF and solving for *C*,

$$C^2 + F^2 = 200 \quad \rightarrow \quad C^2 + 4C^2 = 200$$
$$\therefore 5C^2 = 200$$
$$\therefore C = \sqrt{\frac{200}{5}} = 2\sqrt{10}$$

Consequently,

$$F = 2C = 2 \times 2\sqrt{10} = 4\sqrt{10}$$

Then, the budget constraint is now

$$2F + C = 10\sqrt{10}$$

Using the utility maximization condition *C* = 2*F* as stated in part (b),

$$2F + C = 10\sqrt{10} \quad \rightarrow \quad 2F + 2F = 10\sqrt{10}$$
$$\therefore F = \frac{10}{4}\sqrt{10} = \boxed{2.5\sqrt{10}}$$

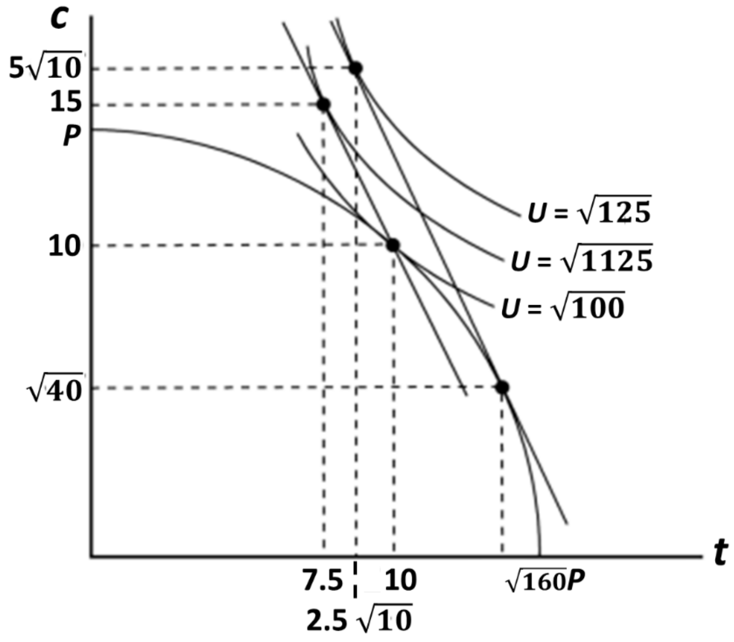
and

$$C = 2F = 2 \times 2.5\sqrt{10} = \boxed{5\sqrt{10}}$$

The updated utility is

$$U = \sqrt{F \times C} = \sqrt{2.5\sqrt{10} \times 5\sqrt{10}} = \boxed{\sqrt{125}} > \sqrt{112.5}$$

Thus, there is further improvement in utility relatively to part (b) because of the ability to specialize in production.
Part (d): The graph is shown below.



■ Problem 18

Part (a): Squaring the given production functions for region A, we have $l_x = x_A^2$ and $l_y = y_A^2$. Substituting these relationships into the given labor equation brings to

$$l_x + l_y = 100$$

$$\therefore \boxed{x_A^2 + y_A^2 = 100}$$

In region B, we have $l_x = 4x_B^2$ and $l_y = 4y_B^2$, so that

$$l_x + l_y = 100$$

$$\therefore 4x_B^2 + 4y_B^2 = 100$$

$$\therefore \boxed{x_B^2 + y_B^2 = 25}$$

Part (b): Optimally, the rates of product transformation should be equal in the two regions.
Part (c): For both regions, the RPT is given by x_i/y_i, $i = \{A,B\}$. Equating the RPTs and manipulating,

$$\frac{x_A}{y_A} = \frac{x_B}{y_B}$$

$$\therefore y_A = x_A\left(\frac{y_B}{x_B}\right)$$

$$\therefore y_A^2 = x_A^2\left(\frac{y_B^2}{x_B^2}\right) \quad \text{(I)}$$

But

$$x_A^2 + y_A^2 = 4\left(x_B^2 + y_B^2\right)$$

so that, substituting from (I),

$$x_A^2 + x_A^2\left(\frac{y_B^2}{x_B^2}\right) = 4\left(x_B^2 + y_B^2\right)$$

$$\therefore x_A^2\left(1 + \frac{y_B^2}{x_B^2}\right) = 4\left(x_B^2 + y_B^2\right)$$

$$\therefore x_A^2\left(1+\frac{y_B^2}{x_B^2}\right)=4x_B^2\left(1+\frac{y_B^2}{x_B^2}\right)$$
$$\therefore x_A^2=4x_B^2$$
$$\therefore x_A=2x_B$$

Returning to (I) with this result,

$$y_A^2=x_A^2\left(\frac{y_B^2}{x_B^2}\right)=4\cancel{x_B^2}\times\frac{y_B^2}{\cancel{x_B^2}}=4y_B^2$$
$$\therefore y_A=2y_B$$

Using the subscript *t* to denote total production, we may write, for good *x*,

$$x_t=x_A+x_B=2x_B+x_B=3x_B$$
$$\therefore x_t^2=9x_B^2$$

and similarly for good *y*,

$$y_t=y_A+y_B=2y_B+y_B=3y_B$$
$$\therefore y_t^2=9y_B^2$$

so that

$$x_t^2+y_t^2=9x_B^2+9y_B^2=9\underbrace{\left(x_B^2+y_B^2\right)}_{=25}$$
$$\therefore \boxed{x_t^2+y_t^2=225}$$

The equation above is the PPC of Ruritania. Setting $x_t = 12$ and solving for y_t, we obtain

$$x_t^2+y_t^2=225 \;\rightarrow\; y_t=\sqrt{225-x_t^2}$$
$$\therefore y_t=\sqrt{225-12^2}=\boxed{9}$$

If the total output of *x* is 12, then Ruritania can produce 9 units of *y*.

■ Problem 19

Setting marginal cost to price for each of the three firms and solving for the firms' individual output, we may write

$$MC_{\text{firm 1}}=4q_1=p \;\rightarrow\; q_1=\frac{1}{4}p$$
$$MC_{\text{firm 2}}=6q_2=p \;\rightarrow\; q_2=\frac{1}{6}p$$
$$MC_{\text{firm 3}}=8q_3=p \;\rightarrow\; q_3=\frac{1}{8}p$$

The market supply then becomes

$$S=24q_1+24q_2+16q_3=\frac{24}{4}p+\frac{24}{6}p+\frac{16}{8}p=12p$$

Equating this to the demand *Q* and solving for market price,

$$S=Q \;\rightarrow\; 12p=1200-3p$$
$$\therefore 15p=1200$$
$$\therefore p=\frac{1200}{15}=\boxed{\$80}$$

■ Problem 20

Part (a): Given the cost function

$$C(q,w)=q^2+wq$$

we differentiate to obtain the marginal cost

$$\overline{MC}=\frac{dC}{dq}=2q+w=2q+8$$

Setting this to price *p* and solving for firm supply *q*,

$$2q + 8 = p$$
$$\therefore 2q = p - 8$$
$$\therefore q = 0.5p - 4$$

For a perfectly competitive market of $N = 800$ firms,

$$Q = \sum_{i=1}^{N} q_i = 400p - 3200$$

The number of teddy bears produced at a price of \$12 is

$$Q = 400 \times 12 - 3200 = \boxed{1600}$$

Similarly, the number of teddy bears produced at a price of \$20 is

$$Q = 400 \times 20 - 3200 = \boxed{4800}$$

Part (b): With $w = 0.0025Q$, the marginal cost obtained in part (a) must be restated as

$$\overline{MC} = 2q + w = 2q + 0.0025Q$$

As before, we set this to price p and solve for individual firm supply q,

$$2q + 0.0025Q = p$$
$$\therefore 2q = p - 0.0025Q$$
$$\therefore q = 0.5p - 0.00125Q$$

The industry supply curve follows as

$$Q = \sum_{i=1}^{N} q_i = 400p - Q$$
$$\therefore 2Q = 400p$$
$$\therefore Q = 200p$$

The number of teddy bears produced at a price of \$12 is

$$Q = 200 \times 12 = \boxed{2400}$$

Similarly, the number of teddy bears produced at a price of \$20 is

$$Q = 200 \times 20 = \boxed{4000}$$

Comparing $Q = 400p - 3200$ from part (a) and $Q = 200p$ from part (b), we see that supply is more steeply sloped in the case where expanded output bids up wages.

■ Problem 21

Part (a): The long-run supply curve is horizontal at $p = MC = AC = \$5$.
Part (b): Setting $p = MC = \$5$ in the demand function, the industry output is

$$Q = 3000 - 100p = 3000 - 100 \times 5 = \boxed{2500}$$

If each firm is to supply 25 units, then the number of firms is $N = 2500/25 = 100$. The long-run profits are zero because average cost equals price in the long-run.
Part (c): Differentiating *C(q)* gives the marginal cost function,

$$\overline{MC} = \frac{dC(q)}{dq} = \frac{1}{2}q - 8$$

The average cost function is, in turn,

$$\overline{AC} = \frac{C(q)}{q} = \frac{1}{4}q - 8 + \frac{156.25}{q}$$

The average cost reaches a minimum at $\overline{AC} = \overline{MC}$, so that, solving for q,

$$\overline{MC} = \overline{AC} \quad \rightarrow \quad \frac{1}{2}q - 8 = \frac{1}{4}q - 8 + \frac{156.25}{q}$$

$$\therefore \frac{1}{4}q = \frac{156.25}{q}$$

$$\therefore q^2 = 4 \times 156.25$$

$$\therefore q = \sqrt{4 \times 156.25} = \boxed{25}$$

Thus, average cost is minimum when each firm produces an output of 25 units.

Part (d): Setting *MC* = *p* and solving for an individual firm's output *q*, we get

$$\overline{MC} = p \quad \rightarrow \quad \frac{1}{2}q - 8 = p$$

$$\therefore \frac{1}{2}q = p + 8$$

$$\therefore \boxed{q = 2p + 16}$$

Accounting for all *N* = 100 firms, the industry supply curve becomes

$$Q = \sum_{i=1}^{N} q_i = \boxed{200p + 1600}$$

Part (e): The demand is now *Q* = 4500 – 100*p*. With *Q* set at 2500 (i.e., the result of part (b)) because firms cannot adjust their outputs in the very short run, the market price becomes

$$2500 = 4500 - 100p$$

$$\therefore 100p = 2000$$

$$\therefore p = \$20$$

With each firm producing 25 units as in part (a), the updated profits are

$$\pi = \left(p - \overline{AC}\right)q = (20 - 5) \times 25 = \boxed{\$375}$$

Part (f): Equating the updated market demand function given in part (e) and the industry supply function of part (d), we have

$$200p + 1600 = 4500 - 100p$$

$$\therefore 300p = 2900$$

$$\therefore p = \$9.67$$

The industry output is updated as

$$Q = 4500 - 100 \times 9.67 = \boxed{3533}$$

Dividing this by the number of firms, we find that each firm will supply *q* = 3533/100 = 35.33 units, and the updated average cost is

$$\overline{AC} = \frac{1}{4} \times 35.33 - 8 + \frac{156.25}{35.33} = \$5.26$$

It follows that the short-run profits will be

$$\pi = \left(p - \overline{AC}\right)q = (9.67 - 5.26) \times 35.33 = \boxed{\$155.8}$$

Part (g): In the long run, the market price is expected to converge to the minimum average cost *AC* = \$5 again, and supply is expected to stabilize at

$$Q = 4500 - 100 \times 5 = 4000$$

The output of each firm should become

$$q = 2p + 16 = 2 \times 5.0 + 16 = 26$$

for a total of *N* = 4000/26 ≈ 154 firms. Profits tend to zero, $\pi \rightarrow 0$.

◆ References

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