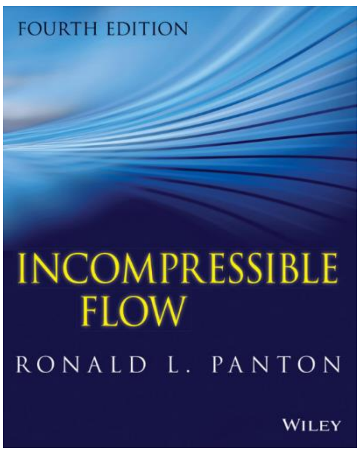
Montogue

Quiz FM114



Reviewed Solutions to Panton's *Incompressible Flow*, 4th Edition

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Chapter	Problems covered
7	7.1, 7.2, 7.5, 7.6, 7.7, 7.9, 7.11, 7.14, 7.18
8	8.1, 8.2, 8.3, 8.6, 8.8, 8.9, 8.11, 8.13
11	11.1, 11.2, 11.12, 11.14, 11.16, 11.21
12	12.1, 12.4, 12.13
13	13.1, 13.2, 13.3, 13.6, 13.12, 13.13, 13.15
18	18.2, 18.4, 18.6, 18.7, 18.10, 18.11

PROBLEMS – CHAPTER 7

Problem 7.1

Find the velocity profile for laminar flow in a round pipe with given fluid and pressure drop $\Delta p/L$.

Problem 7.2

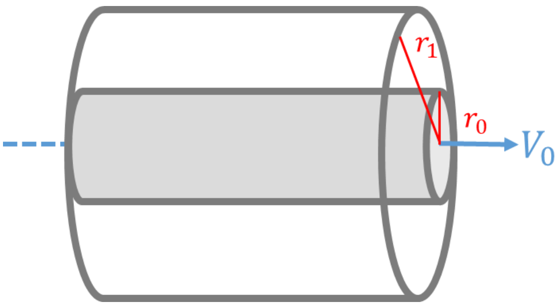
Using the answer from Problem 7.1, find the shear stress on the wall and the volume flow rate.

Problem 7.5

Consider the annulus formed between a rod of radius r_0 and a tube of radius r_1. Find the velocity profile for Couette flow where the inner rod is rotated with speed Ω. Neglect gravity. Do not assume that the gap is small compared to the radius.

Problem 7.6

For the same geometry as in Problem 7.5, but $\Omega = 0$, find the velocity profile if the rod is being pulled in the axial direction at a speed $v_z = V_0$. Neglect gravity and any pressure gradient.

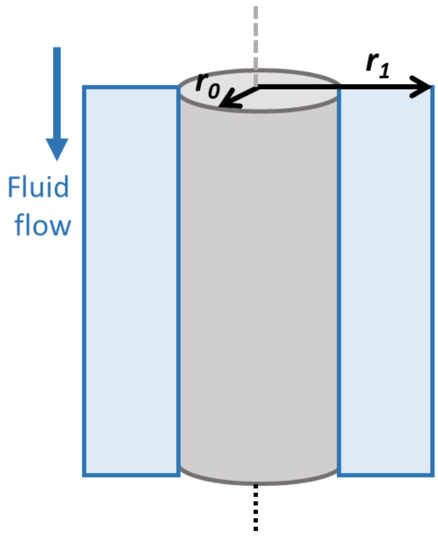


Problem 7.7

For the same geometry as in Problem 7.5, but $\Omega = 0$, find the velocity profile if a pressure gradient $\Delta P/L$ is applied in the direction of the rod axis. Neglect gravity.

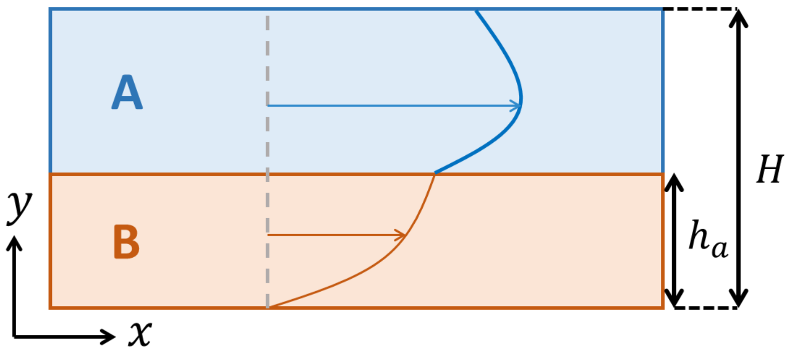
▶ Problem 7.9

A vertical pipe of radius r_0 has a film of liquid flowing downward on the outside. Find the velocity profile for a given film thickness and find an expression for the flow rate Q.



▶ Problem 7.11

A horizontal channel of height H has two fluids of different viscosities and densities flowing because of a pressure gradient. Find the velocity profiles if the height of the interface is h_a.



▶ Problem 7.14

Consider the Rayleigh problem, but allow the plate velocity to be a function of time, $V_0(t)$. By differentiation show that the shear stress $\tau = \mu \partial u / \partial y$ obeys the same diffusion equation that the velocity does. Suppose that the plate is moved in such a way as to produce a constant surface shear stress. What are the velocity profile and the surface velocity for this motion?

▶ Problem 7.18

Oil, with specific gravity = 0.9, used in a viscous coupling has a kinematic viscosity of 30 centistokes (cSt) (10^{-6} m^2/s = 1 cSt). If the coupling has a 5-cm radius and 1-mm gap width, what difference in rotary speeds is needed to transmit a torque of 50 N·m? Is the same power produced by both the input and output shafts?

PROBLEMS – CHAPTER 8

▶ Problem 8.1

Rework the pump analysis using F, M, L, T as primary dimensions.

▶ Problem 8.2

Rework the pump analysis using M, S (speed), and T as primary variables.

▶ Problem 8.3

A list of variables for a problem has only one variable with the dimension of mass. In what two possible ways could the list be in error?

▶ Problem 8.6

The speed of a surface water (liquid) wave is thought to depend on the wave height, the wavelength, the depth of the water, and the acceleration of gravity. What would happen if density was also proposed to be an important parameter? Find a nondimensional form of the answer.

▶ Problem 8.8

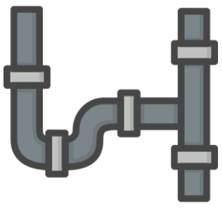
A windmill is designed to operate at 20 rpm in a 15-mph wind and produce 300 kW of power. The blades are 175 ft in diameter. A model 1.75 ft in diameter is to be tested at 90-mph wind velocity. What rotor speed should be used, and what power should be expected?

▶ Problem 8.9

A propeller is placed in a tank of chemicals to mix them together. The diameter is D, the rotation speed is N, and the power to turn the propeller is P. The fluid density is ρ and the viscosity is μ. Tests in water (ρ = 1000 kg/m^3, μ = 1.01×10^{-3} Pa·s) show that a propeller D = 225 mm rotating at 23 rev/s requires a driving power of 159 N·m/s. Calculate the speed and torque required to drive a dynamically similar propeller, 675 mm in diameter in air (ρ = 1.2 kg/m^3, μ = 1.86×10^{-5} Pa·s).

▶ Problem 8.11

The pressure at the end of a round pipe p_2 is a function of the initial pressure p_1, the density ρ, the average velocity V, the viscosity μ, the size of the wall roughness ε (length), the length of the pipe L, and the diameter d. By making several assumptions, find the simplest nondimensional relation that governs the pressure in incompressible flow.



▶ Problem 8.13

Consider the flow into and out of a stationary shock wave in a perfect gas (specified heat capacity ratio γ). The pressure p_2 downstream of a shock wave depends on the thermodynamic state ahead of the shock, p_1, T_1, and the initial flow velocity v_1. Find the nondimensional relation for the pressure p_2 using M, L, T, and degree as primary dimensions.

PROBLEMS – CHAPTER II

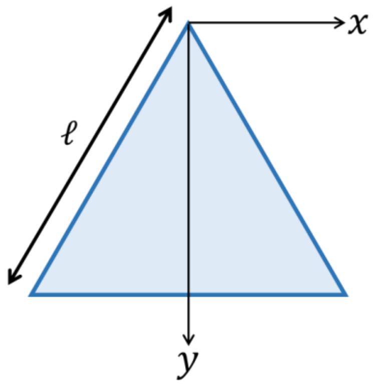
▶ Problem 11.1

The cross-section of a tube is an equilateral triangle with sides of length ℓ and a horizontal base. Flow in the tube is produced by an imposed pressure gradient dp/dz. Check that the velocity profile is given by

$$w(x,y)=\frac{1}{2\sqrt{3}\mu\ell}\left(-\frac{dp}{dz}\right)\left(y-\frac{\sqrt{3}}{2}\ell\right)\left(3x^2-y^2\right)$$

where the coordinate origin is at the apex of the triangle with y bisecting the angle and positive downward, and x is horizontal. Check that the flow rate is

$$Q=\frac{\sqrt{3}}{320}\frac{\ell^4}{\mu}\left(-\frac{dp}{dz}\right)$$



▶ Problem 11.2

Waves in shallow water induce an oscillatory motion that extends to the bottom. The motion is parallel to the bottom and sinusoidal. Estimate the thickness of the viscous effect caused by the no-slip condition at the bottom when the wave period is 5 s.

▶ Problem 11.12

Find the exact relations for the maximum velocity and its position (the core radius) as functions of time for the Oseen vortex.

▶ Problem 11.14

Air flows around a cylinder of 5-cm radius at 15 cm/s (the Reynolds number is 1000), with the free-stream velocity perpendicular to the axis. Find the dimensional u and v components of the velocity at a point 0.15 cm away from the surface and 0.5 cm away from the symmetry plane on the upwind side of the cylinder. Find the shear stress on the wall at a point 0.5 cm away from the symmetry plane.

▶ Problem 11.16

Integrate Eq. 11.8.8 once to find f', then again by parts to find Eq. 11.8.10.

▶ Problem 11.21

Write a computer program to evaluate the velocity pattern in a rectangular tube, Eq. 11.2.8. For a tube with aspect ratio $a = 0.4$ plot velocity profiles at appropriate x cross-sections. Make a contour plot of the velocity in the cross-section. How many terms in the summation of Eq. 11.2.8, for $a = 0.5$, are needed to obtain five decimal point accuracy in the velocity profile?

PROBLEMS – CHAPTER 12

▶ Problem 12.1

Find the streamfunction for the ideal flow toward a plane stagnation point. The velocity components are $u = ax$ and $v = -ay$. Plot several streamlines using equal increments in ψ.

▶ Problem 12.4

Consider a uniform stream from left to right with a speed U. Find the streamfunction for this flow in all four coordinate systems of Appendix D.

▶ Problem 12.13

The streamfunction for flow over a circular cylinder is

$$\psi = Ur\sin\theta\left(1 - \frac{r_0^2}{r^2}\right)$$

Find the pressure distribution on the surface.

PROBLEMS – CHAPTER 13

▶ Problem 13.1

A disk of radius R is spinning about its axis at a speed Ω. What is the vorticity of the particles at $r = 0$, $r = R/2$, and $r = R$?

▶ Problem 13.2

A Burgers vortex in cylindrical coordinates has the velocity components $v_r = -ar$, $v_z = 2az$, and $v_\theta = (\Gamma/2\pi r)[1 - exp(-ar^2/2v)]$. What is the vorticity field for this flow?

▶ Problem 13.3

Compute the vorticity for the von Kármán pump problem. Leave your answer in terms of the functions F, G, and their derivatives. What relations of F and G determined the fluid vorticity at the wall? Contrast with Problem 13.1.

▶ Problem 13.6

Find the equation of the "cross curve" that marks the path of two counterrotating line vortices as they approach a wall in an inviscid flow.

▶ Problem 13.12

Stuart (1967) vortices are an infinite row of vortices that undergo inviscid motion. Because the motion is inviscid, $\omega_z = F(\psi)$. Show that if $F(\psi) = exp(-2\psi)$ the inviscid equation $\nabla^2\psi = -\omega$ is satisfied by

$$\psi = \ln\left(C\cosh y + \sqrt{C^2 - 1}\cos x\right)$$

Here $1 \leq C \leq \infty$. Find the velocity components. With the use of a computer plot the streamlines for $C = 2$.

▶ Problem 13.13

Show that in Problem 13.12 the limit $C = 1$ gives the uniform shear layer $u = \partial\psi/\partial y = \tanh y$.

▶ Problem 13.15

Sullivan's vortex is an example where flow has a two-cell structure. The velocity profiles are

$$v_\theta = \frac{\kappa}{r}\frac{H(\eta^2)}{H(\infty)} \;;\; \text{where } \eta^2 \equiv \frac{ar^2}{2\upsilon}$$

$$v_r = -ar + \frac{6\upsilon}{r}\left[1 - \exp\left(-\eta^2\right)\right]$$

$$v_z = 2az\left[1 - 3\exp\left(-\eta^2\right)\right]$$

and

$$H\left(\eta^2\right) \equiv \int_0^{\eta^2} \exp[-t + 3\int_0^t \left(1 - e^{-s}\right)s^{-1}ds]dt$$

with $H(\infty) = 37.905$. Sketch the velocity profiles and the streamlines in the r-z plane. Use nondimensional variables.

PROBLEMS – CHAPTER 18

▶ Problem 18.2

For the stagnation point flow $F = Uz^2$, find the streamlines, potential lines, and the equations for the velocity components u, v.

▶ Problem 18.4

Verify that the streamlines from a doublet are given by Eq. 18.5.3. Find the velocity components v_r and v_θ for this flow.

▶ Problem 18.6

Find the streamline equations for a line source superimposed with a line vortex both located at the origin. Determine the pressure as a function of distance from the origin.

▶ Problem 18.7

A line source of strength m is parallel to a wall at a distance h. Find the pressure distribution on the wall where p_0 is the pressure at the stagnation point.

▶ Problem 18.10

Consider an elliptic cylinder of length five times the thickness. Find the complex potential and complex velocity for streaming flow without circulation past this object.

▶ Problem 18.11

Determine the pressure distribution over the surface in Problem 18.10 as a function of distance S from the stagnation point.

SOLUTIONS

P.7.1 ➡ ***Solution***

The problem essentially asks us to derive the Hagen-Poiseuille velocity paraboloid. We begin with the simplified Navier-Stokes equation for momentum in the z- (axial) direction:

$$-\frac{dp}{dz} = \mu\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}\right)$$

The pressure gradient is assumed constant and equal to $\Delta p/L$; we shall denote this ratio as $-k$:

$$k = \mu\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}\right)$$

Dividing through by viscosity:

$$k = \mu\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}\right) \rightarrow \frac{k}{\mu} = \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}$$

The expression above can be restated as

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}=\frac{d}{dr}\left(r\frac{du}{dr}\right)=\frac{k}{\mu}r$$

so that, integrating once:

$$\frac{du}{dr}=\frac{1}{2}\frac{k}{\mu}r+\frac{A}{r}$$

where *A* is an integration constant. Integrating a second time gives

$$u(r)=\frac{1}{4}\frac{k}{\mu}r^2+A\ln(r)+B \quad \text{(I)}$$

where *B* is an integration constant. One of the boundary conditions is that the velocity at the center of the tube must be finite, that is:

$$u(r=0)\neq\infty$$

This condition requires that *A* be equal to zero, otherwise the logarithm that multiplies this constant will tend to (negative) infinity and the velocity at the centerline will be unbounded. The other boundary condition is the no-slip condition, that is, *u(r = R)* = 0. Substituting in (I),

$$u(r=R)=\frac{1}{4}\frac{k}{\mu}R^2+0\times\ln(R)+B=0$$

$$\therefore B=-\frac{1}{4}\frac{k}{\mu}R^2$$

Then, replacing *B* in (I) brings to

$$u(r)=\frac{1}{4}\frac{k}{\mu}r^2+0\times\ln(r)-\frac{1}{4}\frac{k}{\mu}R^2$$

$$\therefore \underline{u(r)=\frac{k}{4\mu}\left(r^2-R^2\right)}$$

The expression above is the velocity profile for Hagen-Poiseuille flow.

P.7.2 ➡ ***Solution***

To find the wall shear, we evoke the shear stress-strain rate relationship

$$|\tau_w|=\left|\mu\frac{du}{dr}\bigg|_{r=R}\right|=\left|\cancel{\mu}\times\left(\frac{kR}{2\cancel{\mu}}\right)\right|$$

$$\therefore \boxed{|\tau_w|=\frac{\Delta pR}{2L}}$$

To find the flow rate, we integrate the velocity profile radially from the centerline to the wall:

$$Q=\int u\,dA=\int_0^R\frac{k}{4\mu}\left(r^2-R^2\right)2\pi r\,dr$$

$$\therefore Q=2\pi\times\frac{k}{4\mu}\times\int_0^R\left(r^2-R^2\right)r\,dr$$

$$\therefore Q=\frac{\pi k}{2\mu}\int_0^R\left(r^2-R^2\right)r\,dr$$

$$\therefore Q=\frac{\pi k}{2\mu}\int_0^R\left(r^3-R^2r\right)dr$$

$$\therefore Q=\frac{\pi k}{2\mu}\left[\frac{r^4}{4}-\frac{R^2r^2}{2}\right]\Bigg|_{r=0}^{r=R}$$

$$\therefore Q=\frac{\pi k}{2\mu}\left(\frac{R^4}{4}-\frac{R^4}{2}\right)$$

$$\therefore Q = \frac{\pi}{2\mu} \times \left(-\frac{\Delta p}{L}\right) \times \left(-\frac{R^4}{4}\right)$$

$$\therefore \boxed{Q = \frac{\pi \Delta p R^4}{8\mu L}}$$

This is the famous Poiseuille equation for flow in a cylindrical tube.

P.7.5 ➡ ***Solution***

The momentum equation for the system at hand is

$$\mu \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right] = 0$$

Integrating with respect to *r*,

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = C_1$$

where C_1 is an integration constant. Separating variables and integrating a second time,

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = C_1 \rightarrow \partial(r v_\theta) = C_1 r \partial r$$

$$\therefore \int \partial(r v_\theta) = C_1 \int r \partial r$$

$$\therefore r v_\theta = C_1 \frac{r^2}{2} + C_2$$

$$\therefore v_\theta = C_1 \frac{r}{2} + \frac{C_2}{r}$$

where C_2 is another integration constant. The boundary conditions are $v_\theta(r = r_0) = r_0\Omega$ and $v_\theta(r = r_1) = 0$. Applying the latter BC, we get

$$v_\theta(r = r_1) = C_1 \frac{r_1}{2} + \frac{C_2}{r_1} = 0$$

$$\therefore C_2 = -C_1 \frac{r_1^2}{2} \quad \text{(I)}$$

Substituting (I) in the expression for v_θ,

$$\therefore v_\theta = C_1 \frac{r}{2} + \frac{C_2}{r} = C_1 \frac{r}{2} - C_1 \frac{r_1^2}{2r} = \frac{C_1}{2}\left(r - \frac{r_1^2}{r}\right)$$

$$\therefore v_\theta = \frac{C_1 r_1}{2}\left(\frac{r}{r_1} - \frac{r_1}{r}\right) \quad \text{(II)}$$

Applying the remaining BC, in turn,

$$v_\theta(r = r_0) = \frac{C_1 r_1}{2}\left(\frac{r_0}{r_1} - \frac{r_1}{r_0}\right) = r_0\Omega$$

$$\therefore C_1 = \frac{2 r_0 \Omega}{r_1\left(\frac{r_0}{r_1} - \frac{r_1}{r_0}\right)}$$

Substituting C_1 in (II) yields the velocity distribution we're looking for:

$$\therefore v_\theta = \frac{C_1 r_1}{2}\left(\frac{r}{r_1} - \frac{r_1}{r}\right) = \frac{r_1}{2} \times \frac{2 r_0 \Omega}{r_1\left(\frac{r_0}{r_1} - \frac{r_1}{r_0}\right)} \times \left(\frac{r}{r_1} - \frac{r_1}{r}\right)$$

$$\therefore \boxed{v_\theta(r) = r_0 \Omega \frac{\left(\frac{r}{r_1} - \frac{r_1}{r}\right)}{\left(\frac{r_0}{r_1} - \frac{r_1}{r_0}\right)}}$$

P.7.6 ➡ *Solution*

In this case, we are concerned with the z-momentum equation, not the θ-momentum equation. Accordingly, we write

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=0$$

Integrating once brings to

$$r\frac{\partial v_z}{\partial r}=C_1$$

where C_1 is an integration constant. Separating variables and integrating a second time,

$$r\frac{\partial v_z}{\partial r}=C_1 \rightarrow \partial v_z = C_1\frac{1}{r}\partial r$$
$$\therefore \int \partial v_z = C_1 \int \frac{1}{r}\partial r$$
$$\therefore v_z = C_1 \ln r + C_2 \;\; \text{(I)}$$

where C_2 is another integration constant. The boundary conditions are $v_z(r = r_0) = V_0$ and $v_z(r = r_1) = 0$. Applying the latter BC yields

$$v_z = C_1 \ln r_1 + C_2 = 0$$
$$\therefore C_2 = -C_1 \ln r_1$$

so that, substituting in (I),

$$v_z = C_1 \ln r + C_2 = C_1 \ln r - C_1 \ln r_1$$
$$\therefore v_z = C_1 \ln\left(r/r_1\right) \;\; \text{(II)}$$

Applying the remaining BC, in turn, we obtain

$$v_z\left(r=r_0\right)=C_1\ln\left(r_0/r_1\right)=V_0$$
$$\therefore C_1=\frac{V_0}{\ln\left(r_0/r_1\right)}$$

so that, substituting in (II),

$$v_z = C_1\ln\left(r/r_1\right)=V_0\frac{\ln\left(r/r_1\right)}{\ln\left(r_0/r_1\right)}$$

That is,

$$\boxed{v_z\left(r\right)=V_0\frac{\ln\left(r/r_1\right)}{\ln\left(r_0/r_1\right)}}$$

P.7.7 ➡ *Solution*

The formulation is similar to that of Problem 7.5, but a pressure gradient term, *dp/dz*, is added to the equation:

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=\frac{dp}{dz}$$

Integrating once:

$$r\frac{\partial v_z}{\partial r}=\frac{1}{\mu}\frac{dp}{dz}\frac{r^2}{2}+C_1$$

(Note that the pressure gradient $dp/dz = \Delta p/L$ is constant.) Dividing through by r and integrating again:

$$r\frac{\partial v_z}{\partial r}=\frac{1}{\mu}\frac{dp}{dz}\frac{r^2}{2}+C_1 \rightarrow \frac{\partial v_z}{\partial r}=\frac{1}{\mu}\frac{dp}{dz}\frac{r}{2}+\frac{C_1}{r}$$

$$\therefore \int \partial v_z = \int\left(\frac{1}{\mu}\frac{dp}{dz}\frac{r}{2}+\frac{C_1}{r}\right)\partial r$$

$$\therefore v_z = \frac{1}{\mu}\frac{dp}{dz}\frac{r^2}{4} + C_1 \ln(r) + C_2 \quad \text{(I)}$$

The available boundary conditions are no-slip at $r = r_0$ and $r = r_1$. Applying the former, it follows that

$$v_z(r = r_0) = \frac{1}{\mu}\frac{dp}{dz}\frac{r_0^2}{4} + C_1 \ln(r_0) + C_2 = 0$$

$$\therefore C_2 = -\frac{1}{\mu}\frac{dp}{dz}\frac{r_0^2}{4} - C_1 \ln(r_0)$$

so that, substituting in (I):

$$v_z = \frac{1}{\mu}\frac{dp}{dz}\frac{r^2}{4} + C_1 \ln(r) + -\frac{1}{\mu}\frac{dp}{dz}\frac{r_0^2}{4} - C_1 \ln(r_0)$$

$$v_z = \frac{1}{4\mu}\frac{dp}{dz}\left(r^2 - r_0^2\right) + C_1 \ln(r/r_0) \quad \text{(II)}$$

Resorting to the second boundary condition:

$$v_z(r = r_1) = \frac{1}{4\mu}\frac{dp}{dz}\left(r_1^2 - r_0^2\right) + C_1 \ln(r_1/r_0) = 0$$

$$\therefore C_1 = -\frac{1}{4\mu}\frac{dp}{dz}\frac{\left(r_1^2 - r_0^2\right)}{\ln(r_1/r_0)}$$

Substituting in (II) and manipulating:

$$v_z = \frac{1}{4\mu}\frac{dp}{dz}\left(r^2 - r_0^2\right) + C_1 \ln(r/r_0)$$

$$\therefore v_z = \frac{1}{4\mu}\frac{dp}{dz}\left(r^2 - r_0^2\right) - \frac{1}{4\mu}\frac{dp}{dz}\frac{\left(r_1^2 - r_0^2\right)}{\ln(r_1/r_0)}\ln(r/r_0)$$

$$\therefore \boxed{v_z(r) = \frac{1}{4\mu}\frac{dp}{dz}\left[\left(r^2 - r_0^2\right) - \left(r_1^2 - r_0^2\right)\frac{\ln(r/r_0)}{\ln(r_1/r_0)}\right]}$$

P.7.9 ➡ *Solution*

Let z denote the downward direction. We assume that velocity component v_z is a function of radial distance r only, while components v_r and v_θ are both assumed to be zero. The pertaining momentum equation is

$$\mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) + \rho g = 0$$

where ρ is the density of the fluid and g is gravitational acceleration. Integrating once, we obtain:

$$\mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) + \rho g = 0 \rightarrow \mu \frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\rho g r$$

$$\therefore r\frac{dv_z}{dr} = -\frac{\rho g}{\mu}\frac{r^2}{2} + C_1$$

Dividing through by r:

$$\frac{dv_z}{dr} = -\frac{\rho g}{\mu}\frac{r}{2} + C_1 \frac{1}{r}$$

Integrating a second time:

$$\int dv_z = -\int \frac{\rho g}{\mu}\frac{r}{2}dr + \int C_1 \frac{1}{r}dr$$

$$\therefore v_z = -\frac{\rho g}{4\mu}r^2 + C_1 \ln r + C_2 \quad \text{(I)}$$

The first BC is no-slip at the surface of the pipe, that is, $v_z(r = r_0) = 0$. Substituting in (I) and solving for constant C_2, we have

$$v_z(r=r_0) = -\frac{\rho g}{4\mu}r_0^2 + C_1 \ln r_0 + C_2 = 0$$

$$\therefore C_2 = +\frac{\rho g}{4\mu}r_0^2 - C_1 \ln r_0$$

Replacing C_2 in (I):

$$v_z = -\frac{\rho g}{4\mu}r^2 + C_1 \ln r + \frac{\rho g}{4\mu}r_0^2 - C_1 \ln r_0$$

$$\therefore v_z = -\frac{\rho g}{4\mu}\left(r^2 - r_0^2\right) + C_1 \ln\left(r/r_0\right) \quad \text{(II)}$$

The other BC is that there is no shear at the surface of the liquid film ($r = r_1$); that is,

$$\left.\frac{dv_z}{dr}\right|_{r=r_1} = 0 \rightarrow \left.\left(-\frac{\rho g}{2\mu}r + \frac{1}{r}C_1\right)\right|_{r=r_1} = 0$$

$$\therefore -\frac{\rho g}{2\mu}r_1 + \frac{1}{r_1}C_1 = 0$$

$$\therefore C_1 = \frac{\rho g}{2\mu}r_1^2$$

Lastly, we substitute C_1 in (II) to obtain

$$\boxed{v_z(r) = -\frac{\rho g}{4\mu}\left(r^2 - r_0^2\right) + \frac{\rho g}{2\mu}r_1^2 \ln\left(r/r_0\right)}$$

To compute the flow rate, we integrate the velocity profile from the inner radius r_0 at the bottom of the film to the outer radius r_1 at the surface of the film:

$$Q = \int_{r_0}^{r_1}\left[v_z(z)\right] \times 2\pi r dr$$

To speed things up, we set up this integral in Mathematica:

```
In[1238]:=
vz[r_] := -(ρ*g)/(4 μ) * (r^2 - r0^2) + (ρ*g)/(2 μ) * r1^2 * Log[r / r0];

In[1237]:=
Integrate[vz[r] * 2 * Pi * r, {r, r0, r1}, Assumptions → r1 > r0 > 0]

Out[1237]=
-(g π ρ (r0^4 - 4 r0^2 r1^2 + 3 r1^4 - 4 r1^4 Log[r1/r0]))/(8 μ)
```

(Note the tiny negative sign at the beginning of output [1237].) Rearranging the expression above such that $r_1/r_0 = \beta$, we have

$$Q = -\frac{\pi \rho g r_0^4}{8\mu}\left[\frac{1}{8} - \frac{1}{2}\left(\frac{r_1}{r_0}\right)^2 + \frac{3}{8}\left(\frac{r_1}{r_0}\right)^4 - \frac{1}{2}\left(\frac{r_1}{r_0}\right)^4 \ln\left(\frac{r_1}{r_0}\right)\right]$$

$$\therefore \boxed{Q = -\frac{\pi \rho g r_0^4}{\mu}\left[\frac{1}{8} - \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 - \frac{1}{2}\beta^4 \ln\left(\frac{r_1}{r_0}\right)\right]}$$

P.7.11 ➡ ***Solution***

Let A denote the upper fluid and B denote the lower fluid. Before attempting to solve the Navier-Stokes equations for this pair of fluids, we will specify the pertinent boundary conditions. Firstly, flow velocity of fluid A at the free surface should equal zero:

$$v_A(y=H) = 0 \;;\; \text{(BC1)}$$

Secondly, the flow velocities of fluids A and B should be equal at the interface:

$$v_A(y=h_a) = v_B(y=h_a) \;;\; \text{(BC2)}$$

Thirdly, the shear stresses should also match at the interface:

$$\mu_A \frac{dv_A}{dy}\left(y = h_a\right) = \mu_B \frac{dv_B}{dy}\left(y = h_a\right) \text{ ; (BC3)}$$

Lastly, we employ a no-slip condition with respect to the surface beneath fluid B:

$$v_B\left(y = 0\right) = 0 \text{ ; (BC4)}$$

We can now write the momentum equation for fluid A:

$$\mu_A \frac{d^2 v_A}{dy^2} = \frac{dp}{dx}$$

Integrating once and noting that the pressure gradient *dp*/*dx* is assumed constant, we get:

$$\mu_A \frac{dv_A}{dy} = \frac{dp}{dx} y + C_1 \text{ (I)}$$

Proceeding similarly with fluid B:

$$\mu_B \frac{dv_B}{dy} = \frac{dp}{dx} y + C_2 \text{ (II)}$$

Now, (BC3) can be used to equate (I) and (II) at the interface:

$$\cancel{\frac{dp}{dx} h_a} + C_1 = \cancel{\frac{dp}{dx} h_a} + C_2$$

$$\therefore C_1 = C_2$$

Let us denote C_1 or C_2 as K. Integrating (I) a second time brings to

$$\mu_A v_A = \frac{dp}{dx}\frac{y^2}{2} + Ky + C_3 \text{ (III)}$$

Integrating (II) a second time, in turn, we obtain

$$\mu_B v_B = \frac{dp}{dx}\frac{y^2}{2} + Ky + C_4 \text{ (IV)}$$

Now, substituting (BC1) into (III),

$$\mu_A v_A = \frac{dp}{dx}\frac{H^2}{2} + KH + C_3 = 0$$

$$\therefore C_3 = -\frac{dp}{dx}\frac{H^2}{2} - KH$$

so that, replacing C_3 in (III),

$$\mu_A v_A = \frac{dp}{dx}\frac{y^2}{2} + Ky - \frac{dp}{dx}\frac{H^2}{2} - KH$$

$$\mu_A v_A = \frac{1}{2}\frac{dp}{dx}\left(y^2 - H^2\right) + K\left(y - H\right) \text{ (V)}$$

Then, substituting (BC4) into (IV),

$$\mu_B v_B = \underbrace{\frac{dp}{dx}\frac{0^2}{2}}_{=0} + \underbrace{K \times 0}_{=0} + C_4 = 0$$

$$\therefore C_4 = 0$$

Thus, we can restate (IV) as

$$\mu_B v_B = \frac{dp}{dx}\frac{y^2}{2} + Ky + \underbrace{C_4}_{=0}$$

$$\mu_B v_B = \frac{dp}{dx}\frac{y^2}{2} + Ky \text{ (VI)}$$

From (V) and (VI), in conjunction with the remaining boundary condition (BC3), we have

$$\frac{1}{2}\frac{dp}{dx}\left(y^2-H^2\right)+K\left(y-H\right)=\frac{\mu_A}{\mu_B}\left(\frac{dp}{dx}\frac{y^2}{2}+Ky\right)$$

which can be solved for the remaining constant *K*,

$$K=-\frac{1}{2}\frac{dp}{dx}\frac{\left[H^2-\left(1-\frac{\mu_A}{\mu_B}\right)h_a^2\right]}{\left[H-\left(1-\frac{\mu_A}{\mu_B}\right)h_a\right]}$$

Thus, the velocity profile for fluid A is given by (V):

$$v_A\left(y\right)=\frac{1}{2\mu_A}\frac{dp}{dx}\left\{\left(y^2-H^2\right)-\left(y-H\right)\frac{\left[H^2-\left(1-\frac{\mu_A}{\mu_B}\right)h_a^2\right]}{\left[H-\left(1-\frac{\mu_A}{\mu_B}\right)h_a\right]}\right\}$$

The velocity profile for fluid B, in turn, is given by (VI):

$$v_B\left(y\right)=\frac{1}{2\mu_B}\frac{dp}{dx}\left\{y^2-y\frac{\left[H^2-\left(1-\frac{\mu_A}{\mu_B}\right)h_a^2\right]}{\left[H-\left(1-\frac{\mu_A}{\mu_B}\right)h_a\right]}\right\}$$

P.7.14 ➡ ***Solution***

We manipulate the diffusion equation as follows,

$$\frac{\partial v_x}{\partial t}=\upsilon\frac{\partial^2 v_x}{\partial y^2}\rightarrow\frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial y}\right)=\upsilon\frac{\partial^2}{\partial y^2}\left(\frac{\partial v_x}{\partial y}\right)\ \text{(I)}$$

Then, noting that from the shear stress-strain rate relationship,

$$\tau=\mu\frac{\partial v_x}{\partial y}$$

we may restate (I) as

$$\frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial y}\right)=\upsilon\frac{\partial^2}{\partial y^2}\left(\frac{\partial v_x}{\partial y}\right)\rightarrow\frac{1}{\mu}\frac{\partial}{\partial t}\left(\tau\right)=\frac{\upsilon}{\mu}\frac{\partial^2}{\partial y^2}\left(\tau\right)$$

$$\therefore\frac{\partial\tau}{\partial t}=\upsilon\frac{\partial^2\tau}{\partial y^2}$$

Thus, the shear τ follows the same diffusion law that applies to velocity. Accordingly, we can cast equation (7.7.13) in terms of shear:

$$\tau=\left[1-\text{erf}\left(\eta\right)\right]\left(-\tau_0\right)$$

$$\therefore\frac{\partial v_x}{\partial y}=-\frac{\tau_0}{\mu}\left[1-\text{erf}\left(\frac{y}{2\sqrt{\upsilon t}}\right)\right]=-\frac{\tau_0}{\mu}\text{erfc}\left(\frac{y}{2\sqrt{\upsilon t}}\right)$$

Separating variables and integrating,

$$\frac{\partial v_x}{\partial y}=-\frac{\tau_0}{\mu}\text{erfc}\left(\frac{y}{2\sqrt{\upsilon t}}\right)\rightarrow\int\partial v_x=-\frac{\tau_0}{\mu_0}\int\text{erfc}\left(\frac{y}{2\sqrt{\upsilon t}}\right)dy$$

$$\therefore v_x=-\frac{\tau_0}{\mu}\int_{\infty}^{y}\text{erfc}\left(\frac{y}{2\sqrt{\upsilon t}}\right)d\left(\frac{y}{2\sqrt{\upsilon t}}\right)\times 2\sqrt{\upsilon t}$$

Let $y/(2\sqrt{\nu t})=\xi$, so the expression above can be rewritten as

$$v_x=-\frac{2\tau_0\sqrt{\upsilon t}}{\mu}\int_{\infty}^{\xi}\text{erfc}\left(\xi\right)d\xi$$

The integral above can be evaluated with Mathematica:

```
In[1245]:=
Integrate[Erfc[ξ], {ξ, ∞, ξ}, Assumptions → ξ > 0]
Out[1245]=
-e^(-ξ^2)/Sqrt[π] + ξ Erfc[ξ]
```

That is,

$$v_x = -\frac{2\sqrt{\upsilon t}\tau_0}{\mu}\int_{\infty}^{\xi}\operatorname{erfc}(\xi)\,d\xi = -\frac{2\sqrt{\upsilon t}\tau_0}{\mu}\left[\xi\operatorname{erfc}(\xi) - \frac{\exp(-\xi^2)}{\sqrt{\pi}}\right]$$

Replacing ξ and transferring the "–" sign to the expression in square brackets:

$$v_x = \frac{2\sqrt{\upsilon t}\tau_0}{\mu}\left[-\frac{y}{2\sqrt{\upsilon t}}\operatorname{erfc}\left(\frac{y}{2\sqrt{\upsilon t}}\right) + \frac{1}{\sqrt{\pi}}\exp\left(-\frac{y^2}{4\upsilon t}\right)\right]$$

Substituting $y = 0$ gives an expression for the surface velocity:

$$V_0 = \frac{2\sqrt{\upsilon t}\tau_0}{\mu}\times\left[\underbrace{-\frac{0}{2\sqrt{\upsilon t}}\operatorname{erfc}\left(\frac{0}{2\sqrt{\upsilon t}}\right)}_{=0} + \frac{1}{\sqrt{\pi}}\underbrace{\exp\left(-\frac{0^2}{4\upsilon t}\right)}_{=1}\right]$$

$$\therefore \boxed{V_0 = \frac{2\tau_0}{\mu}\sqrt{\frac{\upsilon t}{\pi}}}$$

P.7.18 ➡ *Solution*

Let the difference in rotary speeds we aim for be denoted as ΔΩ. From Eq. (7.6.3),

$$v_\theta = r\Omega\frac{z}{h}$$

The shear stress component $\tau_{\theta z}$ is, in turn,

$$\tau_{\theta z} = \mu\frac{\partial v_\theta}{\partial z} = \mu\Delta\Omega\frac{1}{h}$$

The torque T is computed next:

$$T = \int_0^R r\tau_{\theta z}2\pi r^2 dr = \frac{2\pi\mu\Delta\Omega}{h}\int_0^R r^3 dr$$

$$\therefore T = \frac{\pi\mu\Delta\Omega R^4}{2h}$$

Solving for ΔΩ and substituting the pertaining values, we obtain

$$T = \frac{\pi\mu\Delta\Omega R^4}{2h} \rightarrow \Delta\Omega = \frac{2hT}{\pi\mu R^4}$$

$$\therefore \Delta\Omega = \frac{2\times(1.0\times10^{-3})\times50}{\pi\times\left[900\times(30\times10^{-6})\right]\times0.05^4} = \boxed{189{,}000\text{ rad/s}}$$

or, equivalently, 30.1 kHz. Note that the product in square brackets is $\mu = \rho\nu$ (*dynamic viscosity* = *density* × *kinematic viscosity*).

P.8.1 ➡ *Solution*

The functional relationship that describes the pump problem is

$$\Delta p = f(Q, \rho, d, \Omega, g)$$

The dimensional matrix, with *F* (force), *M* (mass), *L* (length), and *T* (time) as primary dimensions, is shown next. Note that g_c is a "dimensional unifier" akin to the one introduced in Section 8.5 of the textbook.

	Δp	Q	ρ	d	Ω	g_c
F	1	0	0	0	0	–1
M	0	0	1	0	0	1
L	–2	3	–3	1	0	1
T	0	–1	0	0	–1	–2

The rank of the dimensional matrix is 4 and the number of variables is 6; accordingly, we shall use 6 − 4 = 2 dimensionless parameters. Let Δp and Q be the dependent variables, and ρ, d, Ω, and g_c be the repeating variables. We begin by building a parameter Π_1 on the basis of Δp:

$$\Pi_1 = \Delta p \rho^a d^b \Omega^c g_c^d$$

$$\therefore \Pi_1 = \left(FL^{-2}\right)\left(ML^{-3}\right)^a \left(L\right)^b \left(T^{-1}\right)^c \left(F^{-1}MLT^{-2}\right)^d$$

$$\therefore \Pi_1 = F^{1-d} M^{a+d} L^{-2-3a+b+d} T^{-c-2d}$$

$$\begin{cases} 1-d=0 \ \ \text{(I)} \\ a+d=0 \ \ \text{(II)} \\ -2-3a+b+d=0 \ \ \text{(III)} \\ -c-2d=0 \ \ \text{(IV)} \end{cases}$$

From (I), it is easy to see that d = 1. From (II), we get a = −1. From (IV), we get $c = -2d = -2$. Lastly, we substitute the available results into (III) to obtain b:

$$-2-3a+b+d=0 \rightarrow -2-3\times(-1)+b+1=0$$

$$\therefore -2+3+b+1=0$$

$$\therefore b=-2$$

Accordingly, the parameter we seek is given by

$$\Pi_1 = \Delta p \rho^{-1} d^{-2} \Omega^{-2} g_c^1 = \frac{\Delta p d^2}{\rho \Omega^2 d^4 / g_c}$$

Next, we produce a parameter Π_2 on the basis of Q:

$$\Pi_2 = Q \rho^a d^b \Omega^c g_c^d$$

$$\therefore \Pi_2 = \left(L^3T^{-1}\right)\left(ML^{-3}\right)^a \left(L\right)^b \left(T^{-1}\right)^c \left(F^{-1}MLT^{-2}\right)^d$$

$$\therefore \Pi_2 = F^{-d} M^{a+d} L^{3-3a+b+d} T^{-1-c-2d}$$

$$\begin{cases} -d=0 \ \ \text{(I)} \\ a+d=0 \ \ \text{(II)} \\ 3-3a+b+d=0 \ \ \text{(III)} \\ -1-c-2d=0 \ \ \text{(IV)} \end{cases}$$

Referring to equation (I), we clearly have d = 0. Likewise, substituting d into (II) yields a = 0. Then, substituting d into (IV) and solving for c,

$$-1-c-2d=0 \rightarrow -1-c-2\times 0=0$$

$$\therefore c=-1$$

We substitute into (III) to obtain the remaining exponent, b:

$$3-3a+b+d=0 \rightarrow 3-3\times 0+b+0=0$$

$$\therefore b=-3$$

Therefore, parameter Π_2 is such that

$$\Pi_2 = Q \rho^0 d^{-3} \Omega^{-1} g_c^0 = \frac{Q}{\Omega d^3}$$

We conclude the analysis by stating the dimensionless relation

$$\boxed{\frac{\Delta p d^2}{\rho \Omega^2 d^4 / g_c} = f\left(\frac{Q}{\Omega d^3}\right)}$$

P.8.2 ➡ ***Solution***

We must first restate the variables with speed as one of the dimensions:

$$\Delta p = \frac{F}{L^2} = \frac{ML}{T^2 L^2} = \frac{M}{T^2 L} = \frac{M}{T^3 \times \frac{L}{T}} = MS^{-1}T^{-3}$$

$$Q = \frac{L^3}{T} = \frac{L^3}{T^3} T^2 = S^3 T^2$$

$$\rho = \frac{M}{L^3} = \frac{M}{\frac{L^3}{T^3} \times T^3} = \frac{M}{\left(\frac{L}{T}\right)^3 \times T^3} = MS^{-3}T^{-3}$$

$$d = L = \frac{L}{T} \times T = ST$$

$$\Omega = T^{-1}$$

The dimensional matrix is shown next.

	Δp	Q	ρ	d	Ω
M	1	0	1	0	0
S	−1	3	−3	1	0
T	−3	2	−3	1	−1

This matrix has rank 3 and the number of variables involved is 5, hence we need to construct 5 − 3 = 2 dimensionless groups. As before, we take Δp and Q as the dependent variables. Parameter Π_1 is constructed as follows.

$$\Pi_1 = \Delta p \rho^a d^b \Omega^c = \left(MS^{-1}T^{-3}\right)\left(MS^{-3}T^{-3}\right)^a \left(ST\right)^b \left(T^{-1}\right)^c$$

$$\therefore \Pi_1 = M^{1+a} S^{-1-3a+b} T^{-3-3a+b-c}$$

$$\begin{cases} 1+a=0 \ \text{(I)} \\ -1-3a+b=0 \ \text{(II)} \\ -3-3a+b-c=0 \ \text{(III)} \end{cases}$$

From (I), we clearly have a = −1. Substituting a in (II) and solving for b, we get

$$-1-3a+b=0 \rightarrow -1-3\times(-1)+b=0$$

$$\therefore b=-2$$

Substituting a and b in (III), we obtain

$$-3-3a+b-c=0 \rightarrow -3-3\times(-2)-4-c=0$$

$$\therefore -3+6-4-c=0$$

$$\therefore c=-1$$

Thus, parameter Π_1 is given by

$$\Pi_1 = \Delta p \rho^{-1} d^{-2} \Omega^{-1} = \frac{\Delta p}{\rho d^2 \Omega}$$

We now turn to parameter Π_2, which is constructed as follows:

$$\Pi_2 = Q\rho^a d^b \Omega^c = \left(S^3T^2\right)\left(MS^{-3}T^{-3}\right)^a \left(ST\right)^b \left(T^{-1}\right)^c$$

$$\therefore \Pi_2 = M^a S^{3-3a+b} T^{2-3a+b-c}$$

$$\begin{cases} a=0 \ \text{(I)} \\ 3-3a+b=0 \ \text{(II)} \\ 2-3a+b-c=0 \ \text{(III)} \end{cases}$$

Equation (I) is trivial. Substituting a into (II) and solving for b,

$$3-3a+b=0 \rightarrow 3-3\times 0+b=0$$

$$\therefore b=-3$$

Lastly, we substitute a and b into (III) to obtain the remaining exponent:

$$2-3a+b-c=0 \rightarrow 2-3\times 0-3-c=0$$

$$\therefore c = -1$$

$$\therefore \Pi_2 = Q\rho^0 d^{-3}\Omega^{-1} = \frac{Q}{\Omega d^3}$$

Gleaning our results, we write

$$\Pi_1 = f(\Pi_2) \rightarrow \boxed{\frac{\Delta p}{\rho d^2 \Omega} = f\left(\frac{Q}{\Omega d^3}\right)}$$

P.8.3 ➡ ***Solution***

One possibility is that the variable containing mass is not actually a variable of the function and should be dropped. A second possibility is that another variable containing mass is missing from the list of variables.

P.8.6 ➡ ***Solution***

Refer to the following dimensional matrix.

	S	*A*	*λ*	*h*	*g*
M	0	0	0	0	0
L	1	1	1	1	1
T	−1	0	0	0	−2

As can be seen, if density were included it would be the only variable with mass as part of its decomposition, and hence could not be nondimensionalized. We can expedite the dimensional analysis procedure with the method of scales. We posit a dimensionless group for wave amplitude *A* by normalizing it with respect to the wave height *h*:

$$\Pi_2 = \frac{A}{h}$$

Likewise, we normalize the wavelength λ with respect to the wave height *h*:

$$\Pi_3 = \frac{\lambda}{h}$$

A reasonable quantity to nondimensionalize velocity in wave problems is the factor $\sqrt{gh}$. Accordingly, we write

$$\Pi_1 = \frac{S}{\sqrt{gh}}$$

Thus, we arrive at the dimensionless form

$$\Pi_1 = f(\Pi_2, \Pi_3)$$

$$\therefore \boxed{\frac{S}{\sqrt{gh}} = f\left(\frac{A}{h}, \frac{\lambda}{h}\right)}$$

P.8.8 ➡ ***Solution***

Refer to the following dimensional matrix.

	P	*d*	*ρ*	*V*	Ω
M	1	0	1	0	0
L	2	1	−3	1	0
T	−3	0	0	−1	−1

The matrix rank is equal to 3; the number of physical parameters is 5; accordingly, the windmill can be modelled with 5 − 3 = 2 dimensionless groups. We take *d, V,* and Ω as the repeating variables, while power and linear velocity are chosen as dependent variables. Denoting the dimensionless group that contains power as Π_1, we proceed to write

$$\Pi_1 = Pd^a\rho^b\Omega^c$$

$$\therefore \Pi_1 = (ML^2T^{-3}) \times (L)^a \times (ML^{-3})^b \times (T^{-1})^c$$

$$\therefore \Pi_1 = M^{1+b}L^{2+a-3b}T^{-3-c}$$

$$\begin{cases} 1+b=0 \ \ \text{(I)} \\ 2+a-3b=0 \ \ \text{(II)} \\ -3-c=0 \ \ \text{(III)} \end{cases}$$

From equations (I) and (III), we find that b = −1 and c = −3, respectively. Substituting b in (II) and solving for the remaining exponent, we get:

$$2+a-3b=0 \rightarrow 2+a-3\times(-1)=0$$
$$\therefore a=-5$$

Thus, dimensionless parameter Π_1 is expressed as

$$\Pi_1 = Pd^{-5}\rho^{-1}\Omega^{-3} = \frac{P}{\rho\Omega^3 d^5}$$

Now, we turn to the parameter that contains velocity:

$$\Pi_2 = Vd^a\rho^b\Omega^c$$
$$\therefore \Pi_2 = \left(LT^{-1}\right)\times(L)^a\times\left(ML^{-3}\right)^b\times\left(T^{-1}\right)^c$$
$$\therefore \Pi_2 = M^b L^{1+a-3b} T^{-1-c}$$
$$\begin{cases} b=0 \ \ \text{(I)} \\ 1+a-3b=0 \ \ \text{(II)} \\ -1-c=0 \ \ \text{(III)} \end{cases}$$

Equation (I) is trivial and gives b = 0. Substituting b in (II) and solving for a gives a = −1. Lastly, solving (III) for c yields c = −1. Therefore, dimensionless parameter Π_2 is given by

$$\Pi_2 = Vd^{-1}\rho^0\Omega^{-1} = \frac{V}{\Omega d}$$

Thus, the dimensionless relationship for the system at hand is

$$\frac{P}{\rho\Omega^3 d^5} = f\left(\frac{V}{\Omega d}\right)$$

For dynamic similarity between full-scale device and model, we must have

$$\left.\left(\frac{V}{\Omega d}\right)\right|_{\text{full-scale}} = \left.\left(\frac{V}{\Omega d}\right)\right|_{\text{model}}$$

$$\therefore \Omega_m = \frac{V_m}{V_{\text{fs}}}\frac{d_{\text{fs}}}{d_m}\Omega_{\text{fs}}$$

$$\therefore \Omega_m = \frac{90}{15}\times\frac{175}{1.75}\times 20 = \boxed{12{,}000\,\text{rpm}}$$

This angular speed corresponds to a power P such that

$$\left.\left(\frac{P}{\rho d^5\Omega^3}\right)\right|_{\text{full-scale}} = \left.\left(\frac{P}{\rho d^5\Omega^3}\right)\right|_{\text{model}}$$

$$\therefore P_m = \left(\frac{d_m}{d_{\text{fs}}}\right)^5\left(\frac{\Omega_m}{\Omega_{\text{fs}}}\right)^3 P_{\text{fs}}$$

$$\therefore P_m = \left(\frac{1.75}{175}\right)^5\times\left(\frac{12{,}000}{20}\right)^3\times 300 = \boxed{6.48\,\text{kW}}$$

The model should be supplied with approximately 6.5 kilowatts of power.

P.8.9 ➡ *Solution*

The dimensionless relationship in this case reads

$$P = f(D, N, \rho, \mu)$$

The dimensional matrix is shown next.

	P	D	N	ρ	μ
M	1	0	0	1	1
L	2	1	0	−3	−1
T	−3	0	−1	0	−1

The number of variables is 5 and the rank of the matrix is shown to be 3, hence the number of dimensionless parameters we will be working with is 5 − 3 = 2. We shall take power P and viscosity μ as dependent variables, and D, N, ρ as repeating variables. Let us first obtain parameter Π_1 on the basis of power P:

$$\Pi_1 = PD^a N^b \rho^c$$

$$\therefore \Pi_1 = (ML^2T^{-3})(L)^a (T^{-1})^b (ML^{-3})^c$$

$$\therefore \Pi_1 = M^{1+c} L^{2+a-3c} T^{-3-b}$$

$$\begin{cases} 1+c=0 \ \text{(I)} \\ 2+a-3c=0 \ \text{(II)} \\ -3-b=0 \ \text{(III)} \end{cases}$$

Referring to (I), it is immediately apparent that $c = -1$. From (III), we see that $b = -3$. Substituting c in (II) brings to

$$2+a-3c=0 \rightarrow 2+a-3\times(-1)=0$$
$$\therefore a=-5$$

Substituting the exponents in the definition of Π_1, we have

$$\Pi_1 = PD^{-5}N^{-3}\rho^{-1} = \frac{P}{\rho N^3 D^5}$$

Next, we turn to parameter Π_2, which is based on the dynamic viscosity μ:

$$\Pi_2 = \mu D^a N^b \rho^c$$
$$\therefore \Pi_2 = (ML^{-1}T^{-1})(L)^a (T^{-1})^b (ML^{-3})^c$$
$$\therefore \Pi_2 = M^{1+c} L^{-1+a-3c} T^{-1-b}$$
$$\begin{cases} 1+c=0 \ \text{(I)} \\ -1+a-3c=0 \ \text{(II)} \\ -1-b=0 \ \text{(III)} \end{cases}$$

As in the derivation of the previous parameter, $c = -1$. From (III), we likewise get $b = -1$. Lastly, we substitute c into (II) to obtain

$$-1+a-3c=0 \rightarrow -1+a-3\times(-1)=0$$
$$\therefore -1+a+3=0$$
$$\therefore a=-2$$

so that

$$\Pi_2 = \mu D^{-2}N^{-1}\rho^{-1} = \frac{\mu}{\rho N D^2}$$

Notice that this is essentially a reciprocal Reynolds number for rotational flow. For similarity to hold, Π_2 must be the same for the two scenarios. Denoting conditions in water and air with subscripts 'w' and 'a', respectively, we can equate parameters Π_2 and solve for the speed N_a,

$$\left(\frac{\mu}{\rho N D^2}\right)_w = \left(\frac{\mu}{\rho N D^2}\right)_a \rightarrow N_a = \frac{\mu_a}{\mu_w}\left(\frac{D_w}{D_a}\right)^2 \frac{\rho_w}{\rho_a}$$

$$\therefore N_a = \frac{1.86\times10^{-5}}{1.01\times10^{-3}}\times\left(\frac{225}{675}\right)^2\times\frac{1000}{1.2}\times23 = \boxed{39.2\,\text{rps}}$$

Likewise, we can equate power coefficients Π_1 and solve for power:

$$\left(\frac{P}{\rho N^3 D^5}\right)_w = \left(\frac{P}{\rho N^3 D^5}\right)_a \to P_a = \frac{\rho_a}{\rho_w}\left(\frac{N_a}{N_w}\right)^3\left(\frac{D_a}{D_w}\right)^5 P_w$$

$$\therefore P_a = \frac{1.2}{1000}\times\left(\frac{39.2}{23}\right)^3\times\left(\frac{675}{225}\right)^5\times159 = \boxed{230\,\text{N}\cdot\text{m/s}}$$

P.8.11 ➡ ***Solution***

The flow must be fully developed. The pressure drop is assumed to be constant and equal to the ratio of pressure at the initial point, p_1, and pressure at the final point, p_2, divided by the length L that separates the two points; in mathematical terms, $\Delta p = (p_2 - p_1)/L$. Since we are including the roughness height, the pipe wall needs not be smooth. The dimensional matrix is shown next.

	$\Delta p/L$	ρ	V	μ	ε	d
M	1	1	0	1	0	0
L	−2	−3	1	−1	1	1
T	−2	0	−1	−1	0	0

The number of variables is 6 and the rank of the dimensional matrix is 3, hence we shall work with 6 − 3 = 3 dimensionless groups. Let us take $\Delta p/L$, μ, and ε as the dependent variables. We first formulate a parameter Π_1 on the basis of pressure drop:

$$\Pi_1 = \frac{\Delta p}{L}\rho^a V^b d^c$$

$$\therefore \Pi_1 = \left(ML^{-2}T^{-2}\right)\left(ML^{-3}\right)^a\left(LT^{-1}\right)^b\left(L\right)^c$$

$$\therefore \Pi_1 = M^{1+a}L^{-2-3a+b+c}T^{-2-b}$$

$$\begin{cases} 1+a=0 \ \text{(I)} \\ -2-3a+b+c=0 \ \text{(II)} \\ -2-b=0 \ \text{(III)} \end{cases}$$

From (I) and (III), it is easy to see that a = −1 and b = −2, respectively. It remains to substitute these into (II) and establish the value of c:

$$-2-3a+b+c=0 \to -2-3\times(-1)-2+c=0$$

$$\therefore c=1$$

Thus, parameter Π_1 is given by

$$\Pi_1 = \frac{\Delta p}{L}\rho^{-1}V^{-2}d^1 = \frac{\Delta p/L}{\rho V^2/d}$$

We now turn to parameter Π_2, which is based on the dynamic viscosity μ:

$$\Pi_2 = \mu\rho^a V^b d^c$$

$$\therefore \Pi_2 = \left(ML^{-1}T^{-1}\right)\left(ML^{-3}\right)^a\left(LT^{-1}\right)^b\left(L\right)^c$$

$$\therefore \Pi_2 = M^{1+a}L^{-1-3a+b+c}T^{-1-b}$$

$$\begin{cases} 1+a=0 \ \text{(I)} \\ -1-3a+b+c=0 \ \text{(II)} \\ -1-b=0 \ \text{(III)} \end{cases}$$

From (I) and (III), it is easy to see that $a = b = -1$. Substituting in (II), it follows that

$$-1-3a+b+c=0 \to -1-3(-1)-1+c=0$$

$$\therefore c=-1$$

Thus, parameter Π_2 is found to be

$$\Pi_2 = \mu\rho^{-1}V^{-1}d^{-1} = \frac{\mu}{\rho V d}$$

which is essentially the reciprocal Reynolds number with tube diameter as the reference length. We now turn to parameter Π_3, which is based on the roughness height ε:

$$\Pi_3 = \varepsilon\rho^a V^b d^c$$

$$\therefore \Pi_3 = (L)(ML^{-3})^a (LT^{-1})^b (L)^c$$

$$\therefore \Pi_3 = M^a L^{1-3a+b+c} T^b$$

We needn't even write the system of equations to notice that a = b = 0. Substituting in the exponent for L brings to

$$1-3a+b+c = 0 \rightarrow 1-3\times 0+0+c$$
$$\therefore c = -1$$

so that

$$\Pi_3 = \varepsilon\rho^0 V^0 d^{-1} = \frac{\varepsilon}{d}$$

This is essentially a nondimensionalized roughness height. Finally, the dimensionless relationship we seek is

$$\Pi_1 = f(\Pi_2, \Pi_3) \rightarrow \boxed{\frac{\Delta p/L}{\rho V^2/d} = f\left(\frac{\mu}{\rho V d}, \frac{\varepsilon}{d}\right)}$$

P.8.13 ➡ *Solution*

This one is a little less boring than previous ones because it involves a fourth primary dimension, namely, the temperature θ.The relationship we aim for can be written as

$$p_2 = f(p_1, V_1, T_1, R, c_v)$$

Where R is the universal gas constant and c_v is specific heat capacity at constant volume. The corresponding dimensional matrix is shown next.

	p_2	p_1	V_1	T_1	R	c_v
M	1	1	0	0	0	0
L	−1	−1	1	0	2	2
T	−2	−2	−1	0	−2	−2
θ	0	0	0	1	−1	−1

This formulation can be simplified if we seek the pressure *ratio* p_2/p_1 instead of the individual pressures themselves; the updated functional form is then

$$\frac{p_2}{p_1} = f(V_1, T_1, R, c_v)$$

	p_2/p_1	V_1	T_1	R	c_v
M	0	0	0	0	0
L	0	1	0	2	2
T	0	−1	0	−2	−2
θ	0	0	1	−1	−1

Even greater simplification can be achieved if we replace temperature T_1 with the product RT_1. Likewise, we may nondimensionalize specific heat by dividing it by the gas constant. What's more, we can exclude mass from our solution because none of the variables in the dimensional matrix above include M. The updated functional form then becomes

$$\frac{p_2}{p_1} = f\left(V_1, RT_1, \frac{c_v}{R}\right)$$

	p_2/p_1	V_1	RT_1	c_v/R
L	0	1	2	0
T	0	−1	−2	0
θ	0	0	−1	0

Thus, the following dimensionless groups become apparent. We first have the pressure ratio

$$\Pi_1 = \frac{p_2}{p_1}$$

We also have the ratio of constant-volume specific heat to gas constant,

$$\Pi_2 = \frac{c_v}{R}$$

But, recalling that $c_p = c_v + R$, where c_p is the constant-pressure specific heat capacity, and $\gamma = c_p/c_v$, we may restate Π_2 as

$$\Pi_2 = \frac{1}{\gamma - 1}$$

which is likewise dimensionless. The third and final dimensionless group is

$$\Pi_3 = \frac{V_1^2}{RT_1}$$

which, including the specific-heat ratio γ in the denominator (which is reasonable because γ is itself dimensionless), becomes

$$\Pi_3 \sim \frac{V_1^2}{\gamma RT_1}$$

This, in turn, is the square of the ratio of flow speed to the speed of sound; in other words, the expression above is the squared Mach number:

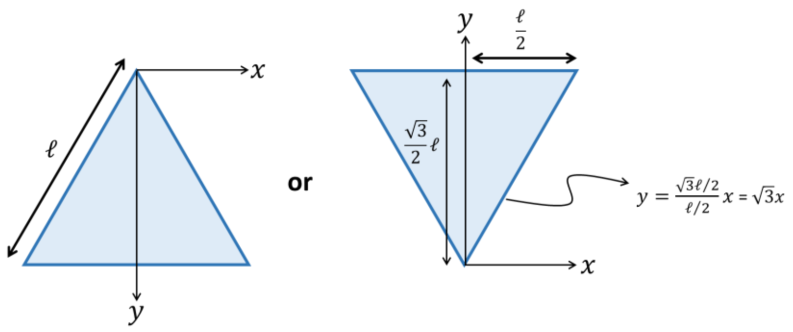
$$\Pi_3 \sim \left(\frac{V_1}{\sqrt{\gamma RT_1}}\right)^2 = \mathrm{Ma}^2$$

In summary:

$$\Pi_1 = f(\Pi_2, \Pi_3) \rightarrow \boxed{\frac{p_2}{p_1} = f\left(\frac{c_v}{R}, \mathrm{Ma}\right)}$$

P.11.1 ➡ *Solution*

The triangular cross-section and an alternative view are shown below.



We were asked to check that the velocity *w(x,y)* can be described by the expression

$$w(x,y) = \frac{-dp/dz}{2\sqrt{3}\mu\ell}\left(y - \frac{\sqrt{3}}{2}\ell\right)(3x^2 - y^2)$$

For the moment, let us employ the simplified notation

$$\phi = \frac{-dp/dz}{2\sqrt{3}\mu\ell}$$

so *w(x,y)* can be rewritten as

$$w(x,y)=\phi\left(y-\frac{\sqrt{3}}{2}\ell\right)\left(3x^2-y^2\right)$$

In order to ascertain the velocity profile in question, we may check whether it satisfies the Navier-Stokes equations. One form that would be pertinent for steady flow in a duct is:

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{1}{\mu}\frac{dp}{dz} \quad \text{(I)}$$

Obtaining the second-order partials of *w(x,y)* is a simple calculus exercise; let's speed things up with Mathematica:

```
In[1003]:= w = ϕ * (y - √3/2 * L) * (3 x^2 - y^2);

In[1006]:= partialwx2 = Simplify[D[w, {x, 2}]]
Out[1006]= -3 √3 L ϕ + 6 y ϕ

In[1007]:= partialwy2 = Simplify[D[w, {y, 2}]]
Out[1007]= (√3 L - 6 y) ϕ

In[1010]:= Simplify[partialwx2 + partialwy2]
Out[1010]= -2 √3 L ϕ
```

That is, the software says that

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=-2\sqrt{3}\ell\phi$$

or, replacing ϕ,

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=-2\sqrt{3}\ell\times\left(-\frac{dp/dz}{2\sqrt{3}\mu\ell}\right)=\frac{1}{\mu}\frac{dp}{dz}$$

which is identical to the NS equation (I); this indicates that the velocity profile for the equilateral-triangle cross-section is indeed valid. To complete our solution, we integrate over the cross-section to obtain the flow rate:

$$Q=\int_A w\,dA=2\int_0^{\sqrt{3}\ell/2}\int_0^{x=y/\sqrt{3}} w\,dx\,dy$$

```
In[1015]:= flowRate = 2 * Integrate[Integrate[ϕ * (y - √3/2 * L) * (3 x^2 - y^2), {x, 0, y/√3}, Assumptions → y > 0],
              {y, 0, √3 * L/2}, Assumptions → L > 0]
Out[1015]= 3 L^5 ϕ / 160
```

The result afforded by the program is

$$Q=\frac{3\ell^5\phi}{160}$$

Replacing ϕ as before, we obtain:

$$Q=\frac{3\ell^5}{160}\times\left(-\frac{dp/dz}{2\sqrt{3}\mu\ell}\right)=\boxed{\frac{\sqrt{3}}{320}\frac{\ell^4}{\mu}\left(-\frac{dp}{dz}\right)}$$

Interestingly, the flow rate in an equilateral-triangle cross-section is proportional to the fourth power of the triangle's edge, much like Poiseuille flow in a cylindrical tube is proportional to the fourth power of the radius. Panton notes that similar flow rate-pressure relationships occur for all cross-sectional geometries.

P.11.2 ➡ *Solution*

The wave motion oscillates at a frequency Ω such that

$$\Omega = \frac{2\pi}{T} = \frac{2\pi}{5.0} = 1.26\,\text{s}^{-1}$$

so that, taking $\nu \approx 1.14\times10^{-6}$ m^2/s as the kinematic viscosity of water,

$$\delta = 4.5\left(\frac{2\nu}{\Omega}\right)^{1/2} = 4.5\times\left[\frac{2\times\left(1.14\times10^{-6}\right)}{1.26}\right]^{1/2} = 6.05\times10^{-3}\,\text{m}$$

$$\therefore \boxed{\delta = 6.05\,\text{mm}}$$

P.11.12 ➡ *Solution*

The Oseen vortex has a velocity field given by

$$v_\theta = \frac{\Gamma}{2\pi r}\left[1-\exp\left(-\frac{r^2}{4\upsilon t}\right)\right]$$

Differentiating with respect to *r*, we get

```
FullSimplify[D[Γ/(2*π*r) * (1 - Exp[-r^2/(4*υ*t)]), {r, 1}]]
```

$$\frac{\Gamma\left(-2+e^{-\frac{r^2}{4 t \upsilon}}\left(2+\frac{r^2}{t\upsilon}\right)\right)}{4\pi r^2}$$

That is,

$$\frac{\partial v_\theta}{\partial r} = \frac{\Gamma}{4\pi r^2}\left[-2+\exp\left(-\frac{r^2}{4\upsilon t}\right)\left(2+\frac{r^2}{\upsilon t}\right)\right]$$

Setting the term in square brackets to zero gives

$$-2+\exp\left(-\frac{r^2}{4\upsilon t}\right)\left(2+\frac{r^2}{\upsilon t}\right)=0$$

$$\therefore \exp\left(-\frac{r^2}{4\upsilon t}\right)\left(2+\frac{r^2}{\upsilon t}\right)=2$$

$$\therefore \exp\left(-\frac{r^2}{4\upsilon t}\right)\left(1+\frac{r^2}{2\upsilon t}\right)=1$$

$$\therefore \exp\left(-\frac{r^2}{4\upsilon t}\right)\left(1+\frac{2r^2}{4\upsilon t}\right)=1$$

Setting the term in red to a new variable ξ, we write

$$\exp(-\xi)(1+2\xi)=1$$

The relationship above is a transcendental equation in ξ. One immediately apparent solution is ξ = 0, which is trivial. To look for a different solution, we solve the equation numerically with Mathematica's *FindRoot* command:

```
In[1147]:=
FindRoot[Exp[-ξ] * (1 + 2 * ξ) - 1, {ξ, 1}]
Out[1147]=
{ξ → 1.25643}
```

That is, $\xi \approx 1.26$. Substituting in the definition of ξ, we obtain

$$\xi = \frac{r_0^2}{4\upsilon t} = 1.26 \rightarrow r = \sqrt{4\xi\upsilon t}$$

$$\therefore r = \sqrt{4 \times 1.26 \times \upsilon t}$$

$$\therefore \boxed{\tilde{r} = 2.25\sqrt{\upsilon t}}$$

To obtain the corresponding velocity, we substitute $\tilde{r}$ into the expression for v_θ, namely

In[1150]:=

```
FullSimplify[(Γ/(2*π*r) * (1 - Exp[-r^2/(4*υ*t)])) /. r → 2.25 Sqrt[υ*t]]
```

Out[1150]=

$$\frac{0.0507837\,\Gamma}{\sqrt{t\,\upsilon}}$$

That is,

$$\boxed{v_{\theta,\max} \approx 0.0508\frac{\Gamma}{\sqrt{\upsilon t}}}$$

In summary, the maximum circumferential velocity of the Oseen vortex decreases with the square root of time, while the radial position at which this velocity is attained increases with the square root of time.

P.11.14 ➡ *Solution*

Noting that the Reynolds number is 1000; we can estimate the kinematic viscosity of the air surrounding the cylinder:

$$\mathrm{Re} = \frac{VD}{\upsilon} \rightarrow \upsilon = \frac{VD}{\mathrm{Re}}$$

$$\therefore \upsilon = \frac{15 \times 10}{1000} = 0.15\,\mathrm{cm^2/s}$$

The constant a is related to other parameters by Eq. (11.9.4),

$$a = \frac{\alpha U_\infty}{r_0}$$

where α is a constant that depends on the shape of the cylinder; for a circular cylinder, we have α = 2. Thus,

$$a = \frac{2 \times 15}{5} = 6.0\,\mathrm{s^{-1}}$$

Now, noting that variable η is given by one of Eqs. (11.9.14), we rearrange as follows:

$$\eta = \frac{y}{\sqrt{\upsilon/a}} = \frac{y}{\sqrt{\frac{\upsilon r_0}{2U}}} = \frac{y}{r_0\sqrt{\frac{\upsilon}{2Ur_0}}} = \frac{y}{r_0\sqrt{\mathrm{Re}^{-1}}}$$

$$\therefore \eta = \frac{y}{r_0}\sqrt{\mathrm{Re}}$$

Since we're interested in conditions 0.15 cm away from the surface, we substitute above to obtain

$$\eta = \frac{0.15}{5} \times \sqrt{1000} = 0.949$$

Entering this value of η into the chart in Fig. 11.10, we read $F(0.949) \approx 0.40$ and $F'(0.949) \approx 0.75$. Solving Eqs. (11.9.14) for velocity components u and v brings to

$$F = -\frac{v}{\sqrt{\upsilon a}} \rightarrow v = -F\sqrt{\upsilon a}$$

$$v = -0.40 \times \sqrt{0.15 \times 6.0} = \boxed{-0.379\,\mathrm{cm/s}}$$

$$F' = \frac{u}{ax} \rightarrow u = F'ax$$

$$\therefore u = 0.75 \times 6.0 \times 0.5 = \boxed{2.25\,\mathrm{cm/s}}$$

Lastly, we appeal to the shear stress-strain rate relation and write

$$|\tau| = \mu\frac{\partial u}{\partial y} = \mu\frac{\partial}{\partial \eta}\left(ax\times F'\right)\frac{\partial \eta}{\partial y} = \mu axF''\times\frac{1}{r_0}\sqrt{\text{Re}}$$

At the wall, η = 0. Further, $F''(\eta = 0) = 1.2$. Then, the only missing quantity in the relation above is the dynamic viscosity of air, which we may take as 1.81×10⁻⁵ Pa·s. Thus,

$$|\tau| = \left(1.81\times10^{-5}\right)\times6.0\times0.005\times1.2\times\frac{1}{0.05}\times\sqrt{1000} = \boxed{4.21\times10^{-4}\text{ N/m}^2}$$

P.11.16 ➡ *Solution*

The differential equation at hand is

$$f'' + \left(\frac{\eta}{2} - \frac{1}{\eta}\right)f' = 0$$

Separating variables,

$$f'' + \left(\frac{\eta}{2} - \frac{1}{\eta}\right)f' = 0 \rightarrow \frac{df'}{f''} = \left(-\frac{\eta}{2} + \frac{1}{\eta}\right)d\eta$$

$$\therefore \frac{df'}{f''} = -\frac{\eta}{2}d\eta + \frac{1}{\eta}d\eta$$

$$\therefore \ln f' = -\frac{\eta^2}{4} + \ln\eta + \ln C_1$$

Grouping the logarithms:

$$\ln f' - \ln\eta - \ln C_1 = -\frac{\eta^2}{4}$$

$$\ln\left(\frac{f'\eta^{-1}}{C_1}\right) = -\frac{\eta^2}{4}$$

Exponentiating:

$$\exp\left[\ln\left(\frac{f'\eta^{-1}}{C_1}\right)\right] = \exp\left(-\frac{\eta^2}{4}\right)$$

$$\therefore \frac{f'\eta^{-1}}{C_1} = \exp\left(-\frac{\eta^2}{4}\right)$$

Isolating f':

$$\frac{f'\eta^{-1}}{C_1} = \exp\left(-\frac{\eta^2}{4}\right)$$

$$\therefore f' = C_1\eta\exp\left(-\frac{\eta^2}{4}\right)$$

$$\therefore \frac{df}{d\eta} = C_1\eta\exp\left(-\frac{\eta^2}{4}\right)$$

$$\therefore df = C_1\eta\exp\left(-\frac{\eta^2}{4}\right)d\eta$$

Using integration by parts, the expression on the right-hand side can be shown to yield

$$\int\eta\exp\left(-\frac{\eta^2}{4}\right)d\eta = -2\exp\left(-\frac{\eta^2}{4}\right)$$

so that

$$df = C_1\eta\exp\left(-\frac{\eta^2}{4}\right)d\eta \rightarrow \int df = \int C_1\eta\exp\left(-\frac{\eta^2}{4}\right)d\eta$$

$$\therefore f = -2C_1\exp\left(-\frac{\eta^2}{4}\right) + C_2$$

Constants C_1 and C_2 can be evaluated from the boundary conditions $f(0) = 0$ and $f(\infty) = 1$. Applying the first BC, we get

$$f(\eta=0)=-2C_1\exp\left(-\frac{0^2}{4}\right)+C_2=0$$

$$\therefore -2C_1+C_2=0$$

$$\therefore C_2=2C_1 \;\; \text{(I)}$$

Applying the second BC, we get

$$f(\eta=\infty)=-2C_1\underbrace{\exp\left(-\frac{\infty^2}{4}\right)}_{\to 0}+C_2=1$$

$$\therefore 0+C_2=1$$

$$\therefore C_2=1$$

Substituting C_2 in (I), we obtain C_1 = 1/2. Therefore, the solution $f(\eta)$ is found to be

$$f=-2C_1\exp\left(-\frac{\eta^2}{4}\right)+C_2=-2\times\frac{1}{2}\times\exp\left(-\frac{\eta^2}{4}\right)+1$$

$$\therefore \boxed{f(\eta)=1-\exp\left(-\frac{\eta^2}{4}\right)}$$

which is identical to Eq. (11.8.10).

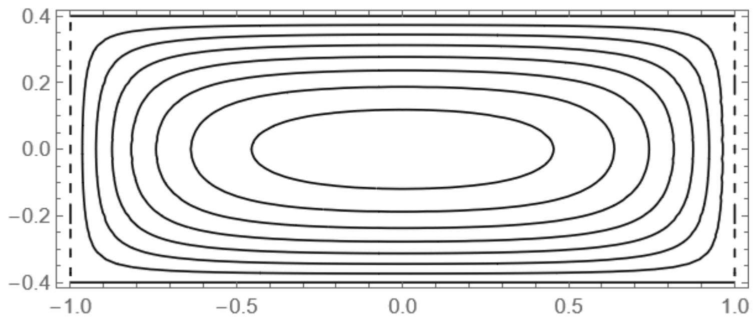
P.11.21 ➡ ***Solution***

The contours can be plotted with the following custom Mathematica function, *rectDuct*, which is basically a *ContourPlot* call with some adjustments:

```
rectDuct[a_,n_]:=ContourPlot[(a^2-y^2)/2+(2/a)*Sum[With[{α=(2k-1)
π/(2a)},(-1)^k/α^3 Cos[α y] Cosh[α
x]/Cosh[α]],{k,1,n}]//Evaluate,{x,-1,1},{y,-a,a},AspectRatio-
>Automatic,ContourShading->None];
```

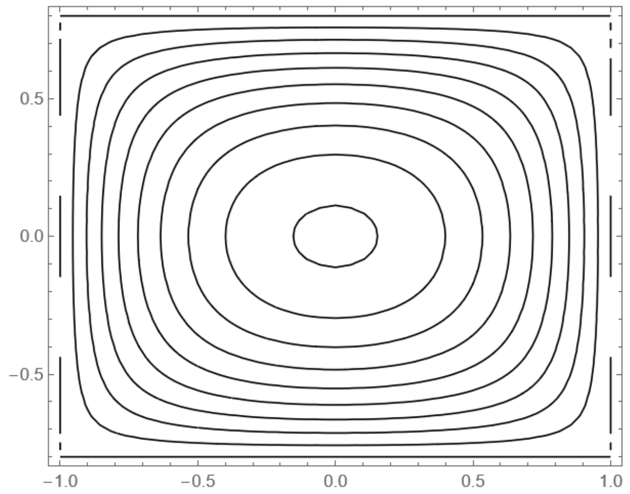
Entering *rectDuct*[0.4, 15], for example, plots contours for a rectangular duct with aspect ratio a = 0.4 and 15 terms in the sum that appears in the velocity distribution formula:

```
rectDuct[0.4,15]
```



We could just as well plot contours for a section with aspect ratio equal to, say, 0.8, and 25 terms in the summation (although using only about 5 or 6 terms is quite adequate for most purposes):

```
rectDuct[0.8,25]
```



P.12.1 ➡ *Solution*

Appealing to the definition of streamfunction for a two-dimensional flow, we may write

$$\begin{cases} \dfrac{\partial \psi}{\partial y} = u = ax \;\; \text{(I)} \\ -\dfrac{\partial \psi}{\partial x} = v = -ay \;\; \text{(II)} \end{cases}$$

Separating variables and integrating, we obtain

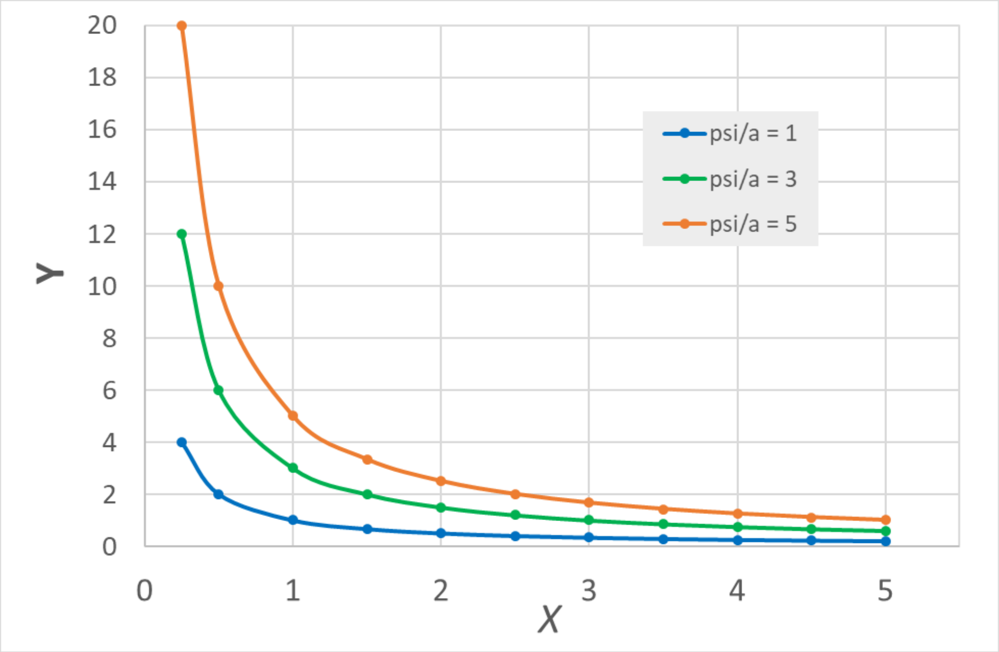
$$\begin{cases} \psi = axy + f(x) \;\; \text{(from I)} \\ \psi = axy + g(y) \;\; \text{(from II)} \end{cases}$$

Comparing the two relations, $f(x) = g(y)$ = constant; let this constant be denoted as ψ_0, so that

$$\psi(x,y) = axy + \psi_0$$

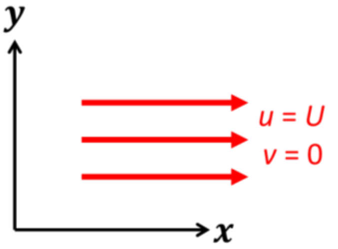
Take $\psi_0 = 0$, so that $\psi(x,y) = axy$. Normalizing with respect to a gives $\psi(x,y)/a = xy$. As asked in the problem statement, we set $[\psi(x,y)/a]$ to incremental values – say, 1, 3, and 5 – and sketch the corresponding streamlines. As the reader will surely recall, functions of the form $y = C/x$, where C is a constant, are actually hyperbolae bounded by the first and third quadrants of the Cartesian plane. The data are tabulated and plotted below.

$\psi(x,y)/a=1$		$\psi(x,y)/a=3$		$\psi(x,y)/a=5$	
x	y	x	y	x	y
0.25	4.00	0.25	12.00	0.25	20.00
0.5	2.00	0.5	6.00	0.5	10.00
1	1.00	1	3.00	1	5.00
1.5	0.67	1.5	2.00	1.5	3.33
2	0.50	2	1.50	2	2.50
2.5	0.40	2.5	1.20	2.5	2.00
3	0.33	3	1.00	3	1.67
3.5	0.29	3.5	0.86	3.5	1.43
4	0.25	4	0.75	4	1.25
4.5	0.22	4.5	0.67	4.5	1.11
5	0.20	5	0.60	5	1.00



P.12.4 ➡ *Solution*

Consider first the system in Cartesian coordinates.



Resorting to the relationship between the horizontal velocity component and the streamfunction, we have:

$$u = U = \frac{\partial \psi}{\partial y} \rightarrow \int \partial \psi = \int U \partial y$$
$$\therefore \psi = Uy + f(x)$$

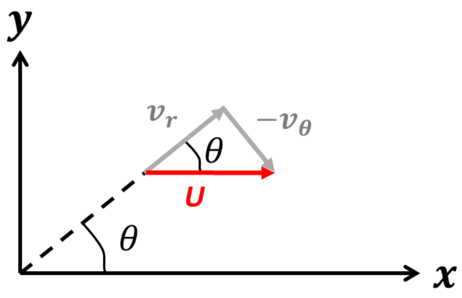
Likewise, for the vertical component:

$$v = 0 = \frac{\partial \psi}{\partial x} \rightarrow \psi = f(x) = \psi_0$$

where ψ_0 is arbitrary. Thus, the streamfunction we aim for is

$$\underline{\psi(x,y) = Uy + \psi_0}$$

Consider now the same system in a cylindrical coordinate frame, as illustrated below.



The velocity components may be described as

$$v_r = U\cos\theta = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$
$$v_\theta = -U\sin\theta = -\frac{\partial \psi}{\partial r}$$

Integrating:

$$\psi = \int \frac{\partial \psi}{\partial \theta} d\theta + f(r)$$

$$\therefore \psi = Ur\int \cos\theta d\theta + f(r)$$

$$\therefore \psi = Ur\sin(\theta) + f(r) \quad \text{(I)}$$

Proceeding similarly with $\partial\psi/\partial r$:

$$\psi = \int \frac{\partial \psi}{\partial r} dr + g(\theta) = U\sin\theta \int dr + g(\theta)$$
$$\therefore \psi = Ur\sin\theta + g(\theta) \quad \text{(II)}$$

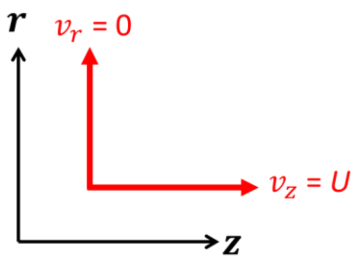
Comparing (I) and (II),

$$f(r) = g(\theta) = \psi_0$$

where ψ_0 is arbitrary. Accordingly, the streamfunction is

$$\underline{\psi(r,\theta) = Ur\sin(\theta) + \psi_0}$$

We now consider the same system in axisymmetric coordinates.



For a uniform stream moving from left to right, v_r = 0 and v_z = U. Integrating v_z brings to

$$v_z = U = \frac{1}{r}\frac{\partial \psi}{\partial r} \rightarrow \int \partial \psi = \int U r \partial r$$

$$\therefore \psi = \frac{1}{2} r^2 U + \psi_0$$

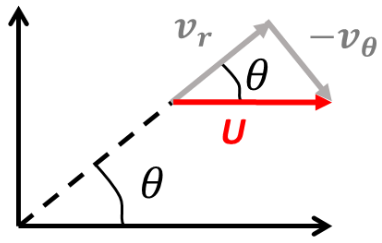
Likewise, we integrate v_r to obtain

$$v_r = 0 = -\frac{1}{r}\frac{\partial \psi}{\partial z} \rightarrow f(z) = \psi_0$$

where ψ_0 is arbitrary. Combining the two previous results, the streamfunction is found to be

$$\underline{\psi(r,z) = \frac{1}{2} r^2 U + \psi_0}$$

Finally, we turn to spherical coordinates.



The velocity components are given by the following expressions:

$$v_r = U\cos\theta = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}$$

$$v_\theta = -U\sin\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \theta}$$

The streamfunction follows as

$$\psi = \int_{(0,0)}^{(r,0)} \frac{\partial \psi}{\partial r} dr + \int_{(r,0)}^{(r,\theta)} \frac{\partial \psi}{\partial \theta} d\theta + \psi_0$$

$$\therefore \psi = \int_0^r U r \sin^2 \theta dr + \int_0^\theta U r^2 \cos\theta \sin\theta d\theta + \psi_0$$

The first integral on the right-hand side should yield zero because in it θ is fixed at 0 and $sin^2 0$ = 0. Noting that $cos\theta sin\theta = sin(2\theta)/2$ and $\int_0^\theta \sin(2\theta)\, d\theta = \sin^2 \theta$, we proceed to evaluate the second integral:

$$\psi = \underbrace{\int_0^r U r \sin^2 \theta dr}_{=0} + \int_0^\theta U r^2 \cos\theta \sin\theta d\theta + \psi_0$$

$$\therefore \psi = \int_0^\theta U r^2 \cos\theta \sin\theta d\theta + \psi_0$$

$$\therefore \psi = \frac{1}{2}\int_0^\theta U r^2 \sin(2\theta) d\theta + \psi_0$$

$$\therefore \underline{\psi = \frac{1}{2} r^2 U \sin^2 \theta + \psi_0}$$

P.12.13 ➡ *Solution*

Applying the usual streamfunction relations, we obtain

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r} \times \left[r\left(1 - \frac{r_0^2}{r^2}\right) U\cos\theta \right]$$

$$\therefore v_r = U\cos\theta \left(1 - \frac{r_0^2}{r^2}\right)$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U\sin\theta \left(1 + \frac{r_0^2}{r^2}\right)$$

As the reader may note, this is actually the ideal solution, hence the Bernoulli equation could be used. However, here we opt for the "long" route. Integrating along the stagnation streamline, we write

$$p_0 - p_\infty = \left(\int_\infty^{r_0} \frac{\partial p}{\partial r} dr \right) \Bigg|_{\theta=\pi} \quad \text{(I)}$$

with $v_r = -U(1 - r_0^2/r^2)$ and $v_\theta = 0$. In order to obtain the pressure distribution, we first appeal to the momentum equation

$$-\frac{\partial p}{\partial r} = \rho v_r \frac{\partial v_r}{\partial r} - \mu \left\{ \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r) \right] + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \right\}$$

Note that the term that multiplies the dynamic viscosity μ should yield zero, because the flow is inviscid. We are left with the term in blue, which becomes

$$-\frac{\partial p}{\partial r} = \rho v_r \frac{\partial v_r}{\partial r} = \frac{2\rho U^2 r_0^2 \left(r^2 - r_0^2 \right)}{r^5}$$

Mathematica can be used to verify our calculations in this particular passage:

```
Simplify[ρ*U*(1 - r0^2/r^2)*D[U*(1 - r0^2/r^2), {r, 1}]]
```

$$\frac{2\, r0^2 \left(r^2 - r0^2\right) U^2 \rho}{r^5}$$

Then, we substitute $\partial p/\partial r$ in equation (I) and carry out the integration:

$$p_0 - p_\infty = 2\rho U^2 \int_\infty^{r_0} \left(\frac{r_0^4}{r^5} - \frac{r_0^2}{r^3} \right) dr = 2\rho U^2 \left(\frac{r_0^4}{-4} r^{-4} - \frac{r_0^2}{-2} r^{-2} \right) \Bigg|_\infty^{r_0}$$

$$\therefore p_0 - p_\infty = 2\rho U^2 \left(\frac{1}{2} - \frac{1}{4} \right) = \boxed{\frac{\rho U^2}{2}}$$

This is the so-called *dynamic pressure* that appears in the Bernoulli equation and is often used in dimensionless coefficients of aerodynamics and naval architecture.

Now, on the *surface* of the cylinder, we have

$$p(\theta, r_0) - p_0 = \left(\int_\pi^\theta \frac{\partial p}{\partial \theta} d\theta \right) \Bigg|_{r=r_0}$$

But, from the momentum equation,

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = \rho \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \mu \times 0$$

$$\therefore \frac{1}{r}\frac{\partial p}{\partial \theta} = -\rho \frac{1}{r}\left[-U \sin\theta \left(1 + \frac{r_0^2}{r^2} \right) \right] \times \left[-U \cos\theta \left(1 + \frac{r_0^2}{r^2} \right) \right]$$

$$\therefore \frac{1}{r}\frac{\partial p}{\partial \theta} = -4 \frac{\rho U^2}{r} \sin\theta \cos\theta$$

$$\therefore \frac{1}{r}\frac{\partial p}{\partial \theta} = -2 \frac{\rho U^2}{r} \sin 2\theta$$

where we have used the trig relationship $\sin(2\theta)/2 = \sin\theta\cos\theta$. Finally,

$$p - p_0 = \int_\pi^\theta \left(-2\rho U^2 \sin 2\theta \right) d\theta = \boxed{-2\rho U^2 \sin^2 \theta}$$

P.13.1 ➡ ***Solution***

For the disk in question, the velocity components are $v_\theta = r\Omega$ and $v_r = v_z = 0$. It follows that vorticity components ω_r and ω_θ are both zero, whereas the remaining component, ω_z, is calculated to be

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) = \frac{1}{r}\frac{\partial}{\partial r}(r \times r\Omega) = \frac{1}{r}\frac{\partial}{\partial r}(r^2\Omega) = \frac{1}{r}\times 2r\Omega$$

$$\therefore \underline{\omega_z = 2\Omega}$$

Note that there is no r term in the result above; accordingly, the vorticity of the particles at any distance from the center of the disk is constant and equal to 2Ω.

P.13.2 ➡ ***Solution***

The vorticity components ω_r and ω_θ are easily shown to equal zero:

$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} = 0$$

$$\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0$$

It remains to compute the z-component ω_z:

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

$$\therefore \omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - 0$$

$$\therefore \omega_z = \frac{1}{r}\frac{\partial}{\partial r}\left\{r \times \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{ar^2}{2v}\right)\right]\right\}$$

$$\therefore \omega_z = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\Gamma}{2\pi} - \frac{\Gamma}{2\pi}\exp\left(-\frac{ar^2}{2v}\right)\right]$$

$$\therefore \omega_z = \frac{1}{r}\left\{\underbrace{\frac{\partial}{\partial r}\left(\frac{\Gamma}{2\pi}\right)}_{=0} - \frac{\partial}{\partial r}\left[\frac{\Gamma}{2\pi}\exp\left(-\frac{ar^2}{2v}\right)\right]\right\}$$

$$\therefore \boxed{\omega_z = \frac{a\Gamma}{2\pi v}\exp\left(-\frac{ar^2}{2v}\right)}$$

P.13.3 ➡ ***Solution***

The von Kármán viscous pump has velocity components given by

$$\begin{cases} v_\theta = r\Omega G(z) \\ v_r = r\Omega F(z) \\ v_z = \sqrt{\upsilon\Omega}H(z) \end{cases}$$

Accordingly, we write the vorticity components

$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} = \frac{1}{r}\times 0 - r\Omega G'(z)$$

$$\therefore \omega_r = -r\Omega G'(z)$$

$$\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = r\Omega F'(z) - 0$$

$$\therefore \omega_\theta = r\Omega F'(z)$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial z}\left[r \times (r\Omega G(z))\right] - \frac{1}{r}\times 0 = 2r\Omega G(z)$$

$$\therefore \omega_z = 2\Omega G(z)$$

At the wall, we have z = 0 and the vorticity components become

$$\omega_r = -r\Omega G'(0)$$

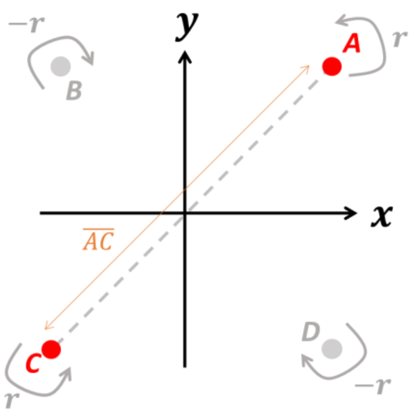
$$\omega_\theta = r\Omega F'(0)$$

$$\omega_z = 2\Omega \underbrace{G(0)}_{=1} = 2\Omega$$

The z-component of vorticity is found to have the same value as in Problem 13.1. However, the other two components, which are both zero in that problem, are not nil in the wall of a von Kármán pump.

P.13.6 ➡ ***Solution***

The system at hand is illustrated below.



Let V_{AC} denote the velocity of A induced by C. With reference to the figure, we may write

$$V_{\overline{AC}} = \frac{\Gamma}{2\pi \times \overline{AC}}$$

$$\therefore V_{\overline{AC},x} = \frac{\Gamma}{2\pi \times \overline{AC}} \times \frac{(-2y)}{\overline{AC}} = -\frac{\Gamma 2y}{2\pi \times \left(\overline{AC}\right)^2}$$

$$\therefore V_{\overline{AC},y} = \frac{\Gamma 2x}{2\pi \times \left(\overline{AC}\right)^2}$$

where $\overline{AC} = 4x^2 + 4y^2$. In turn, source B imparts an induced velocity with y-component only, and is given by

$$V_{\overline{AB}} = V_{\overline{AB}y} = -\frac{\Gamma}{2\pi \times 2x}$$

In a similar manner, source D imparts an induced velocity with x-component only, namely

$$V_{\overline{AD}} = V_{\overline{AD}x} = \frac{\Gamma}{2\pi \times 2y}$$

Adding velocities in the x-direction:

$$V_{A_x} = V_{\overline{AC}x} + V_{\overline{AB}x} = \frac{\Gamma}{2\pi}\left[\frac{1}{2y} - \frac{2y}{4x^2+4y^2}\right] = \frac{\Gamma}{2\pi}\left[\frac{4x^2 + 4y^2 - 4y^2}{2y\left(4x^2+4y^2\right)}\right]$$

$$\therefore V_{A_x} = \frac{\Gamma}{4\pi y} \times \frac{4x^2}{\left(4x^2+4y^2\right)} = \frac{\Gamma}{4\pi y}\frac{x^2}{\left(x^2+y^2\right)}$$

Similarly, we add velocities in the y-direction to obtain

$$V_{A_y} = V_{\overline{AC}y} + V_{\overline{AB}y} = \frac{\Gamma}{2\pi}\left[\frac{2x}{4x^2+4y^2} - \frac{1}{2x}\right] = \frac{\Gamma}{2\pi}\left[\frac{-y^2}{x\left(x^2+y^2\right)}\right]$$

Thus, the motion of vortex A is described by

$$\frac{dy}{dx} = \frac{V_{A,y}}{V_{A,x}} = \frac{-y^2/x}{x^2/y} \rightarrow \frac{dy}{y^3} = -\frac{dx}{x^3}$$

Integrating and rearranging, we ultimately obtain

$$\boxed{\frac{h^2}{x^2} + \frac{h^2}{y^2} = 1}$$

where h is the asymptote of the cross curve.

Now, to establish the corresponding streamfunction ψ, we write the differential form

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -vdx + udy$$

which becomes

$$d\psi = \frac{\Gamma}{4\pi}\left(\frac{1}{x} - \frac{x}{x^2+y^2}\right)dx + \frac{\Gamma}{4\pi}\left(\frac{1}{y} - \frac{y}{x^2+y^2}\right)dy$$

$$\therefore d\psi = \frac{\Gamma}{4\pi}\left(\frac{dx}{x} + \frac{x}{x^2+y^2}dx + \frac{dy}{y} - \frac{y}{x^2+y^2}dy\right)$$

$$\therefore d\psi = \frac{\Gamma}{4\pi}\left[\frac{dx}{x} + \frac{dy}{y} - \frac{1}{2}\frac{d\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right]$$

Then, we integrate to obtain

$$\int d\psi = \frac{\Gamma}{4\pi}\left[\int\frac{dx}{x} + \int\frac{dy}{y} - \frac{1}{2}\int\frac{d\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right]$$

$$\therefore \psi = \frac{\Gamma}{4\pi}\left[\ln x + \ln y - \frac{1}{2}\ln\left(x^2+y^2\right)\right]$$

Multiplying through by $8\pi/\Gamma$,

$$\frac{8\pi}{\Gamma}\psi = \left[2\ln x + 2\ln y - 2\times\frac{1}{2}\ln\left(x^2+y^2\right)\right]$$

$$\therefore \frac{8\pi}{\Gamma}\psi = \left[\ln x^2 + \ln y^2 - \ln\left(x^2+y^2\right)\right]$$

$$\therefore \frac{8\pi}{\Gamma}\psi = \left[\ln x^2y^2 - \ln\left(x^2+y^2\right)\right]$$

$$\therefore \boxed{\frac{8\pi}{\Gamma}\psi = \ln\frac{x^2y^2}{x^2+y^2}}$$

P.13.12 ➡ *Solution*

We are to verify if the inviscid Laplace equation,

$$\nabla^2\psi = -\omega \quad \text{(I)}$$

is valid for the flow in question. Differentiating ψ twice with respect to x and y, respectively, we get

```
In[1084]:= dx = Simplify[D[Log[c * Cosh[y] + Sqrt[c^2 - 1 * Cos[x]], {x, 2}]]
```

$$\text{Out[1084]=} \; -\frac{-1 + c^2 + c\sqrt{-1+c^2}\,\text{Cos}[x]\,\text{Cosh}[y]}{\left(\sqrt{-1+c^2}\,\text{Cos}[x] + c\,\text{Cosh}[y]\right)^2}$$

```
In[1085]:= dy = Simplify[D[Log[c * Cosh[y] + Sqrt[c^2 - 1 * Cos[x]], {y, 2}]]
```

$$\text{Out[1085]=} \; \frac{c\left(c + \sqrt{-1+c^2}\,\text{Cos}[x]\,\text{Cosh}[y]\right)}{\left(\sqrt{-1+c^2}\,\text{Cos}[x] + c\,\text{Cosh}[y]\right)^2}$$

```
In[1086]:= Simplify[dx + dy]
```

$$\text{Out[1086]=} \; \frac{1}{\left(\sqrt{-1+c^2}\,\text{Cos}[x] + c\,\text{Cosh}[y]\right)^2}$$

Note that $\psi_{xx} + \psi_{yy}$ is such that

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \left[\sqrt{C^2-1}\cos(x) + C\cosh(y)\right]^{-2}$$

Compare this with ω_z = $exp(-2\psi)$, that is,

```
In[1087]:= Simplify[Exp[-2 * Log[c * Cosh[y] + Sqrt[c^2 - 1] * Cos[x]]]]
```

$$\text{Out[1087]=} \frac{1}{\left(\sqrt{-1+c^2}\,\text{Cos}[x] + c\,\text{Cosh}[y]\right)^2}$$

but this is identical to the sum $\psi_{xx} + \psi_{yy}$ we've obtained above. Accordingly, the two sides of expression (I) are equal and the inviscid Laplace equation is satisfied.

Finding the velocity components *u* and *v* is straightforward:

```
In[1101]:= ψ[x_, y_] := Log[c * Cosh[y] + Sqrt[c^2 - 1] * Cos[x]];

In[1090]:= Clear[u]

In[1102]:= u = D[ψ[x, y], {y, 1}]
```

$$\text{Out[1102]=} \frac{c\,\text{Sinh}[y]}{\sqrt{-1+c^2}\,\text{Cos}[x] + c\,\text{Cosh}[y]}$$

```
In[1103]:= Clear[v]

In[1104]:= v = -D[ψ[x, y], {x, 1}]
```

$$\text{Out[1104]=} \frac{\sqrt{-1+c^2}\,\text{Sin}[x]}{\sqrt{-1+c^2}\,\text{Cos}[x] + c\,\text{Cosh}[y]}$$

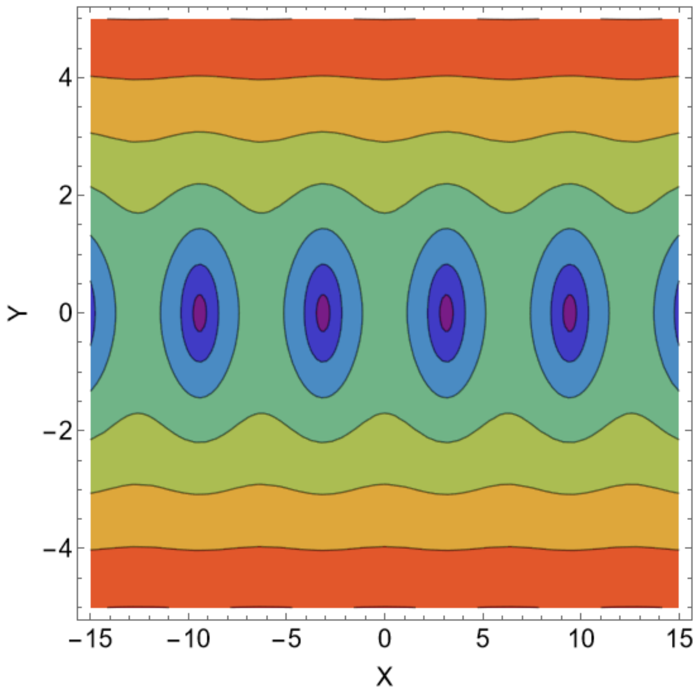
As shown, the software returns

$$u(x,y) = \frac{C\sinh(y)}{\sqrt{C^2-1}\cos(x) + C\cosh(y)}$$

$$v(x,y) = \frac{\sqrt{C^2-1}\sin(x)}{\sqrt{C^2-1}\cos(x) + C\cosh(y)}$$

It remains to visualize some streamlines for C = 2.0. We may use Mathematica's *StreamPlot* or *ContourPlot* commands; the latter is more straightforward to use in the case at hand (even though the problem statement asked for streamlines, not contours):

```
ContourPlot[Log[2 * Cosh[y] + Sqrt[2^2 - 1] * Cos[x]], {x, -15, 15}, {y, -5, 5}, ColorFunction → "Rainbow"]
```



P.13.13 ➡ *Solution*

We saw in Prob. 13.12 that

$$u(x,y)=\frac{C\sinh(y)}{\sqrt{C^2-1}\cos(x)+C\cosh(y)}$$

Setting C = 1 brings to

$$u(x,y)=\frac{1.0\times\sinh(y)}{\underbrace{\sqrt{1.0^2-1}}_{=0}\times\cos(x)+1.0\times\cosh(y)}=\frac{1.0\times\sinh(y)}{1.0\times\cosh(y)}=\boxed{\tanh(y)}$$

as we intended to show.

P.13.15 ➡ *Solution*

Note first that η can be stated as

$$\eta^2=\frac{ar^2}{2\upsilon}\rightarrow\eta=r\sqrt{\frac{a}{2\upsilon}}$$

Noting that a has dimensions of reciprocal time, we can scale length variables as ~ $\sqrt{a/v}$ and velocity variables as ~ $\sqrt{va}$. Accordingly, we proceed to nondimensionalize v_r:

$$v_r=-ar+\frac{6\upsilon}{r}\left[1-\exp\left(-\eta^2\right)\right]=-ar\times\sqrt{\frac{a}{2\upsilon}}\times\sqrt{\frac{2\upsilon}{a}}+\frac{6\upsilon}{r\sqrt{\frac{2a}{\upsilon}}}\sqrt{\frac{2a}{\upsilon}}\left[1-\exp\left(-\eta^2\right)\right]$$

$$\therefore v_r=-\sqrt{2\upsilon a}\eta+6\sqrt{2\upsilon a}\frac{1}{\eta}\left[1-\exp\left(-\eta^2\right)\right]$$

Lastly, we define $v_r^\star = v_r/\sqrt{2va}$, so that

$$v_r^*=\frac{v_r}{\sqrt{2\upsilon a}}=-\eta+\frac{6}{\eta}\left[1-\exp\left(-\eta^2\right)\right]\ \text{(I)}$$

Now, let $z^\star = \xi = z\sqrt{a/2v}$ and $v_z^\star = v_z/\sqrt{2va}$. With these substitutions, we can easily nondimensionalize velocity component v_z:

$$v_z=2az\times\sqrt{\frac{a}{2\upsilon}}\times\sqrt{\frac{2\upsilon}{a}}\times\left[1-3\exp\left(-\eta^2\right)\right]$$

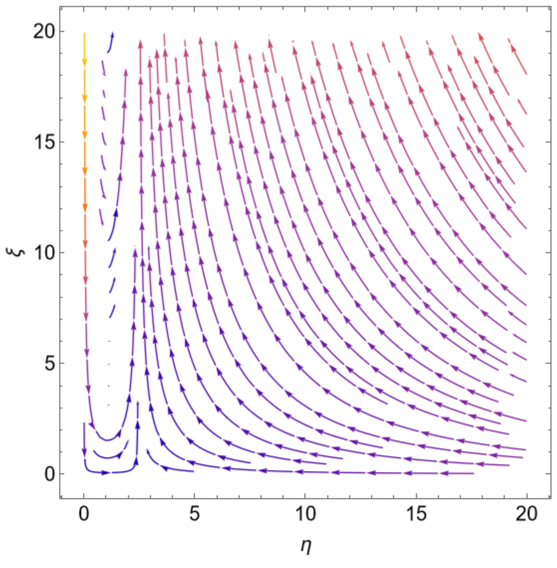
$$\therefore v_z=2\sqrt{2\upsilon a}\xi\times\left[1-3\exp\left(-\eta^2\right)\right]$$

$$\therefore\frac{v_z}{\sqrt{2\upsilon a}}=2\xi\left[1-3\exp\left(-\eta^2\right)\right]$$

$$\therefore v_z^*=2\xi\left[1-3\exp\left(-\eta^2\right)\right]\ \text{(II)}$$

Equations (I) and (II) are ready to be plotted in the r-z plane – or, in view of our dimensionless variables, the η-ξ plane. One way to go is to use Mathematica's *StreamPlot* command:

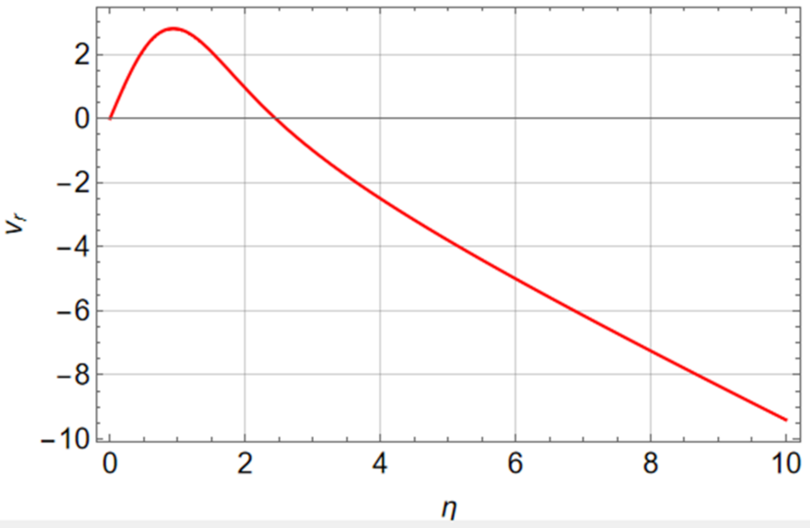
$$\text{StreamPlot}[\{-\eta+\frac{6}{\eta}*(1-\text{Exp}[-\eta^2]),2*\xi*(1-3*\text{Exp}[-\eta^2])\},\{\eta,0,20\},\{\xi,0,20\}]$$



Next, we plot the velocity profiles. Equation (I) can be readily plotted as a function of η with Mathematica's *Plot* command:

```
In[1022]:= vr[η_] := -η + 6/η * (1 - Exp[-η^2]);

In[1026]:= Plot[vr[η], {η, 0, 10}, Frame → True, GridLines → Automatic, PlotStyle → Red]
```

Out[1027]=



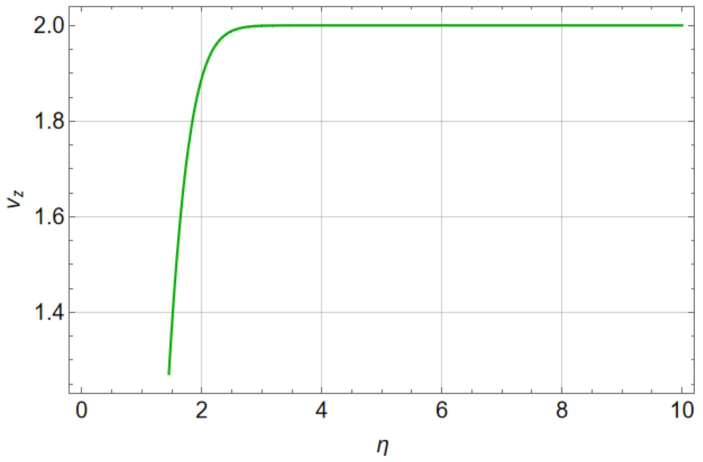
Next, in order to plot v_z, we first divide (II) through by ξ,

$$v_z^* = 2\xi\left[1-3\exp\left(-\eta^2\right)\right] \rightarrow \frac{v_z^*}{\xi} = 2\left[1-3\exp\left(-\eta^2\right)\right]$$

so that, defining a Mathematica function and applying *Plot* as before:

```
In[1028]:= vz[η_] := 2 * (1 - 3 * Exp[-η^2]);

In[1030]:= Plot[vz[η], {η, 0, 10}, Frame → True, GridLines → Automatic, PlotStyle → Darker[Green]]
```

Out[1031]=



It remains to plot velocity component v_θ,

$$v_\theta = \frac{\kappa}{r}\frac{H\left(\eta^2\right)}{H\left(\infty\right)} \;\; ; \;\; \text{where } \eta^2 \equiv \frac{ar^2}{2\upsilon}$$

Since there is a *r* in the denominator and $\eta \propto r$, we may recast the equation as:

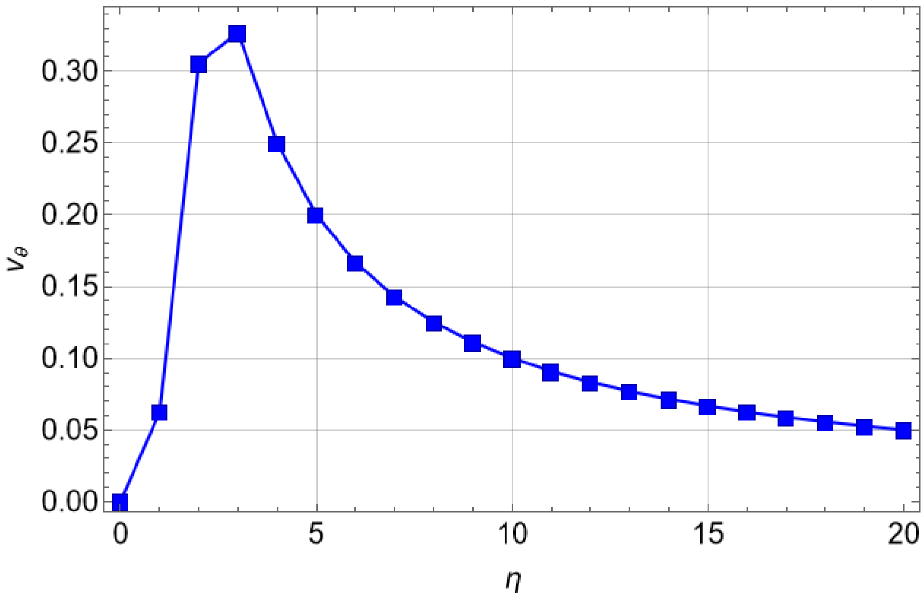
$$\frac{v_\theta}{\kappa} = \frac{1}{r}\frac{H\left(\eta^2\right)}{H\left(\infty\right)} \propto \frac{1}{\eta}\frac{H\left(\eta^2\right)}{H\left(\infty\right)}$$

where

$$H\left(\eta^2\right) \equiv \int_0^{\eta^2} \exp[-t + 3\int_0^t \left(1-e^{-s}\right)s^{-1}ds]dt$$

The Mathematica code we need is reproduced below. We first prepare a function $v\theta$ to evaluate function $H(\eta^2)$ shown above. Then, we prepare a dataset *sullivanSwirl* to establish values for $(1/\eta)(H(\eta^2)/H(\infty)$ in the range $\eta \in (1;20)$. We then prepend a 0 to the beginning of the ensuing dataset (we did not do this in the *Table* command so as to avoid a division by zero). Then, we transpose the dataset to create the array *sullivanData*. Lastly, we plot this dataset with *ListPlot*, as shown in continuation.

```
In[1056]:= vθ[η_] := NIntegrate[Exp[-t + 3 * Integrate[(1 - Exp[-s]) * s^-1, {s, 0, t}, Assumptions → t > 0]], {t, 0, η^2}];

In[1066]:= sullivanSwirl = Table[vθ[η]/(37.905 * η), {η, Range[1, 20]}]

In[1067]:= PrependTo[sullivanSwirl, 0]

In[1069]:= sullivanData = Transpose@{Range[0, 20], sullivanSwirl}

Out[1069]= {{0, 0}, {1, 0.0624799}, {2, 0.305509}, {3, 0.327}, {4, 0.249975}, {5, 0.199997},
  {6, 0.166664}, {7, 0.142855}, {8, 0.124998}, {9, 0.111109}, {10, 0.0999983},
  {11, 0.0909075}, {12, 0.0833319}, {13, 0.0769218}, {14, 0.0714273}, {15, 0.0666655},
  {16, 0.0624989}, {17, 0.0588225}, {18, 0.0555546}, {19, 0.0526307}, {20, 0.0499991}}

In[1079]:= ListPlot[sullivanData, PlotStyle → Blue, Frame → True, GridLines → Automatic, Joined → True,
  PlotMarkers → {Blue, Medium}]
```



Note that the list plot is a bit rough because we used the coarse interval {0,1,2,3,...}. Using more closely spaced values should improve the plot, especially in the interval $\eta \in (0;5)$.

P.18.2 ➡ ***Solution***

Expanding function *F*, we obtain

$$F = Uz^2 = U \times (x+iy)^2 = U \times (x^2 + 2ixy - y^2)$$

$$\therefore F = U(x^2 - y^2) + 2Uxyi$$

so that

$$\phi = (x^2 - y^2)U$$

$$\psi = 2xyU$$

Velocity component *u* can be obtained from ϕ or ψ:

$$u = \frac{\partial \phi}{\partial x} = 2Ux \;\; ; \;\; u = \frac{\partial \psi}{\partial y} = 2Ux$$

Likewise, *v* is such that

$$u = \frac{\partial \phi}{\partial y} = -2Uy \;\; ; \;\; u = -\frac{\partial \psi}{\partial x} = -2Uy$$

Lastly, we determine the complex velocity *w*:

$$w = \frac{dF}{dz} = \frac{d}{dz}(Uz^2) = 2Uz = 2U(x+iy) = 2Ux + i2Uy$$

$$\therefore \underline{w = u - iv}$$

P.18.4 ➡ ***Solution***

As mentioned in the textbook, the streamlines of a doublet are supposedly represented by the circle equation

$$x^2 + \left(y + \frac{\mu}{2\pi\psi}\right)^2 = \left(\frac{\mu}{2\pi\psi}\right)^2$$

The complex potential for a doublet is $F = \mu/\pi z$, which can be manipulated to yield

$$F = \frac{\mu}{\pi z} \times \frac{\tilde{z}}{\tilde{z}} = \frac{\mu}{\pi\left(x^2+y^2\right)}(x-iy)$$

But $F = \phi + i\psi$, with the result that

$$\phi = \frac{\mu x}{\pi\left(x^2+y^2\right)} \text{ and } \psi = -\frac{\mu y}{\pi\left(x^2+y^2\right)}$$

or

$$x^2+y^2 = -\frac{\mu y}{\pi\psi} \text{ (I)}$$

Expanding Eq. 18.5.3, we may write

$$x^2 + \left(y+\frac{\mu}{2\pi\psi}\right)^2 = x^2+y^2+\frac{\mu}{\pi\psi}+\left(\frac{\mu}{2\pi\psi}\right)^2 = \left(\frac{\mu}{2\pi\psi}\right)^2$$

$$\therefore x^2+y^2+\frac{\mu}{\pi\psi} = 0$$

$$\therefore x^2+y^2 = -\frac{\mu}{\pi\psi}$$

Notice that this is identical to (I); accordingly, the relationship has been verified.

To find the velocity components, we differentiate the doublet potential and manipulate:

$$F = \frac{\mu}{\pi}z^{-1} \rightarrow w = -\frac{\mu}{\pi}z^{-2} = -\frac{\mu}{\pi}\times\left(r^{-2}e^{-i2\theta}\right)$$

$$\therefore w = -\frac{\mu}{\pi r^2}\times\left(e^{-i2\theta}\right) = -\frac{\mu}{\pi r^2}\left(e^{-i\theta}\right)e^{-i\theta}$$

$$\therefore w = -\frac{\mu}{\pi r^2}(\cos\theta - i\sin\theta)e^{-i\theta}$$

With reference to Eq. 18.1.13, we conclude that

$$\boxed{v_r = \frac{\mu}{\pi r^2}\cos\theta \text{ ; } v_\theta = -\frac{\mu}{\pi r^2}\sin\theta}$$

P.18.6 ➡ *Solution*

The line source and vortex can have their potential functions added to yield

$$F = \frac{m}{2\pi}\ln z - i\frac{\Gamma}{2\pi}\ln z$$

$$\therefore F = (m-i\Gamma)\frac{1}{2\pi}\ln z \text{ (I)}$$

Differentiating with respect to z,

$$\frac{dF}{dz} = W = (m-i\Gamma)\frac{1}{2\pi z}$$

There corresponds a $\overline{W}$ to the W specified above such that

$$\overline{W} = (m+i\Gamma)\frac{1}{2\pi z}$$

Multiplying W by $\overline{W}$ brings to

$$W\overline{W} = (m-i\Gamma)(m+i\Gamma)\frac{1}{4\pi^2 z\times\overline{z}}$$

$$\therefore W\overline{W} = \left(m^2-i^2\Gamma^2\right)\frac{1}{4\pi^2 z\times\overline{z}}$$

$$\therefore W\overline{W} = \left(m^2+\Gamma^2\right)\frac{1}{4\pi^2 r^2} \text{ (II)}$$

Returning to (I),

$$F=(m-i\Gamma)\frac{1}{2\pi}\ln(z)=(m-i\Gamma)\frac{1}{2\pi}\ln\left(re^{i\theta}\right)$$

$$\therefore F=(m-i\Gamma)\frac{1}{2\pi}\left[\ln(r)+\ln\left(e^{i\theta}\right)\right]$$

$$\therefore F=\frac{1}{2\pi}(m-i\Gamma)\left[\ln(r)+i\theta\right]$$

$$\therefore F=\frac{1}{2\pi}\left[m\ln(r)+m\theta i-\Gamma\ln(r)i+\Gamma\theta\right]$$

$$\therefore F=\frac{1}{2\pi}\left\{\underbrace{m\ln(r)+\Gamma\theta}_{=\phi}+\left[\underbrace{m\theta-\Gamma\ln(r)}_{=\psi}\right]i\right\}$$

$$\therefore F=\frac{1}{2\pi}(\phi+i\psi)$$

The imaginary part of *F* yields the streamfunction ψ,

$$\psi=\frac{m}{2\pi}\theta-\frac{\Gamma}{2\pi}\ln(r)$$

Solving for a given ψ_0 gives the streamline equation $r(\theta)$:

$$\underline{r(\theta)=\exp\left[\frac{2\pi}{\Gamma}\left(\psi_0-\frac{m}{2\pi}\theta\right)\right]}$$

Using the Bernoulli equation and (II), we can establish the pressure distribution:

$$\frac{p}{\rho}+\frac{1}{2}W\bar{W}=\text{const.}$$

$$\therefore \frac{p}{\rho}+\frac{m^2+\Gamma^2}{4\pi^2r^2}=\text{const.}=\frac{p_\infty}{\rho}$$

$$\therefore \underline{\frac{p_\infty-p}{\rho}=\frac{m^2+\Gamma^2}{4\pi^2r^2}}$$

P.18.7 ➡ ***Solution***

The system is described by the potential function

$$F=\frac{m}{2\pi}\ln(z-ih)+\frac{m}{2\pi}\ln(z+ih)$$

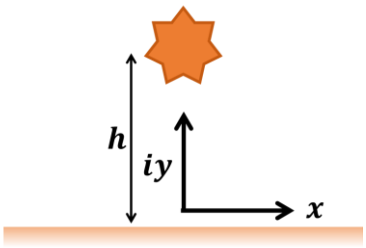
which can be derived to yield

$$W=\frac{dF}{dz}=\frac{m}{2\pi}\left(\frac{1}{z-ih}\right)+\frac{m}{2\pi}\left(\frac{1}{z+ih}\right)$$

$$\therefore W=\frac{m}{2\pi}\left[\left(\frac{1}{z-ih}\right)+\left(\frac{1}{z+ih}\right)\right]$$

$$\therefore W=\frac{m}{2\pi}\left[\frac{z+\cancel{ih}+z-\cancel{ih}}{(z-ih)(z+ih)}\right]$$

$$\therefore W=\frac{m}{2\pi}\left(\frac{2z}{z^2+h^2}\right)$$



Noting that $z = x$ on the wall, we appeal to the Bernoulli equation and write

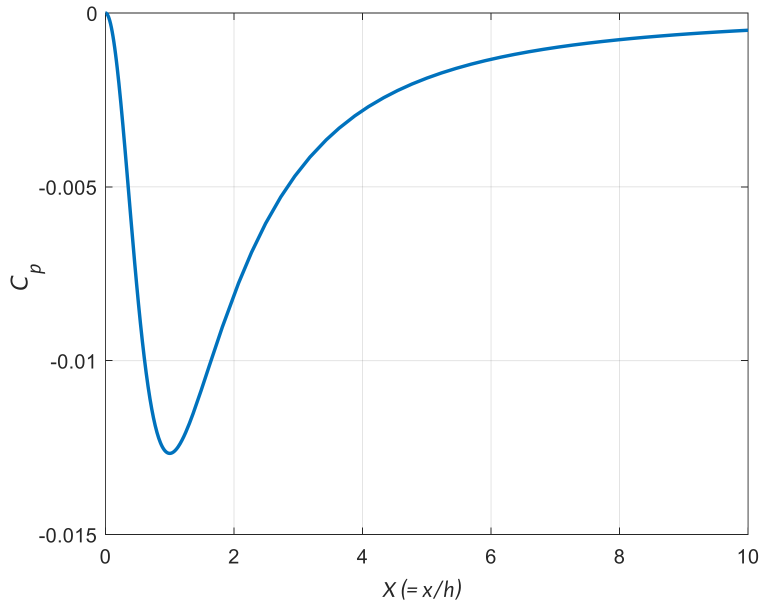
$$\frac{p-p_0}{\rho}=\frac{1}{2}W\bullet\overline{W}=\frac{m^2/h^2}{2\pi^2}\frac{(x/h)^2}{\left[(x/h)^2+1\right]^2}$$

At this point, we define a pressure coefficient C_p such that

$$C_p=\frac{p-p_0}{\rho(m/h)^2}=\frac{1}{2\pi^2}\frac{X^2}{\left(X^2+1\right)^2}$$

where $X = x/h$. We proceed to plot C_p with the following MATLAB code:

```
Cp =@(X) -(1/(2*pi^2))*(X^2/(X^2+1)^2);
fplot(Cp,'LineWidth',2.0)
xlim([0 10]);
ylim([-0.015 0]);
grid on
```



P.18.10 ➡ ***Solution***

We first write the semi-axes ratio

$$\frac{a}{b}=\zeta=\frac{R_0+c^2/R_0}{R_0-c^2/R_0}=\frac{R_0^2+c^2}{R_0^2-c^2}=5$$

Rearranging,

$$\frac{R_0^2+c^2}{R_0^2-c^2}=5\rightarrow R_0^2+c^2=5R_0^2-5c^2$$

$$\therefore R_0=\frac{\sqrt{6}c}{2}$$

or, equivalently, $c = 2R_0/\sqrt{6}$. Now, for flow over a cylinder, we may write the potential

$$F(\zeta)=U\zeta+U\frac{R_0^2}{\zeta}$$

Further, coordinate z is related to the transformed coordinate ζ by the simple expression

$$z=\zeta+\frac{c^2}{\zeta}$$

Differentiating F with respect to ζ,

$$F=U\zeta+U\frac{R_0^2}{\zeta}\rightarrow\frac{dF}{d\zeta}=U-U\left(\frac{R_0}{\zeta}\right)^2$$

Similarly, we differentiate z with respect to ζ,

$$z=\zeta+\frac{c^2}{\zeta}\rightarrow\frac{dz}{d\zeta}=1-\frac{c^2}{\zeta^2}$$

Combining the two previous results, we obtain the potential velocity W:

$$W=\frac{dF}{dz}=\frac{\left(dF/d\zeta\right)}{\left(dz/d\zeta\right)}=\frac{U\left[1-\left(R_0/\zeta\right)^2\right]}{\left(\zeta^2-c^2\right)/\zeta^2}$$

Simplifying:

$$\boxed{W=U\left(\frac{\zeta^2-R_0^2}{\zeta^2-c^2}\right)}$$

with $\zeta=\zeta(z)$.

P.18.11 ➡ *Solution*

Knowing that

$$z=\zeta+\frac{c^2}{\zeta}$$

we apply differentials to obtain

$$dz=d\zeta+c^2\zeta^{-2}d\zeta$$

$$\therefore dz=\left(1-\frac{c^2}{\zeta^2}\right)d\zeta$$

Similarly, for the conjugate:

$$d\bar{z}=\left(1-\frac{c^2}{\bar{\zeta}^2}\right)d\bar{\zeta}$$

It follows that

$$dz\times d\bar{z}=\left(\frac{\zeta^2-c^2}{\zeta^2}\right)\left(\frac{\bar{\zeta}^2-c^2}{\bar{\zeta}^2}\right)d\zeta\, d\bar{\zeta}$$

$$\therefore dz\times d\bar{z}=\frac{\left(\zeta\bar{\zeta}\right)^2-c^2\zeta^2-c^2\bar{\zeta}^2+c^4}{\left(\zeta\bar{\zeta}\right)^2}d\zeta\, d\bar{\zeta}\quad \text{(I)}$$

Further,

$$\bar{\zeta}^2+\zeta^2=R_0^2e^{-i2\theta}+R_0^2e^{i2\theta}=R_0^2\times 2\cos\left(2\theta\right)$$

Also, $d\zeta=R_0ie^{i\theta}d\theta$ and $d\bar{\zeta}=-iR_0e^{-i\theta}d\theta$; the product of these two differentials is $d\zeta d\bar{\zeta}=R_0^2(d\theta)^2$. It follows that (I) can be restated as

$$dz\times d\bar{z}=\left(d\zeta\right)^2=\frac{R_0^4-2c^2R_0^2\cos\left(2\theta\right)+c^4}{R_0^4}R_0^2\left(d\theta\right)^2$$

$$\therefore \frac{\left(d\zeta\right)^2}{R_0^2}=\left[\frac{R_0^4}{R_0^4}-\frac{2c^2R_0^2\cos\left(2\theta\right)}{R_0^4}+\frac{c^4}{R_0^4}\right]\left(d\theta\right)^2$$

$$\therefore \frac{d\zeta}{R_0}=\left[1-2\left(\frac{c}{R_0}\right)^2\cos\left(2\theta\right)+\left(\frac{c}{R_0}\right)^4\right]^{\frac{1}{2}}d\theta$$

Integrating on both sides,

$$\frac{\zeta}{R_0}=\int_{\pi}^{\pi+t}\left[1-2\left(\frac{c}{R_0}\right)^2\cos\left(2\theta\right)+\left(\frac{c}{R_0}\right)^4\right]^{\frac{1}{2}}d\theta\quad \text{(II)}$$

The expression on the right-hand side is an elliptic integral and can be evaluated with Mathematica. Now, from Prob. 18.10 we had

$$W=\frac{\zeta^2-1}{\zeta^2-c^2}\ ;\ \bar{W}=\frac{\bar{\zeta}^2-1}{\bar{\zeta}^2-c^2}$$

so that

$$q^2 = W\bar{W} = \frac{(\zeta\bar{\zeta})^2 - \bar{\zeta}^2 - \zeta^2 + 1}{(\zeta\bar{\zeta})^2 - c^2\bar{\zeta}^2 - c^2\zeta^2 + c^4}$$

$$\therefore q^2 = \frac{1 - 2\cos(2\theta) + 1}{1 - 2c^2\cos(2\theta) + c^4} = \frac{2[1 - \cos(2\theta)]}{1 + c^4 - 2c^2\cos(2\theta)} \quad \text{(III)}$$

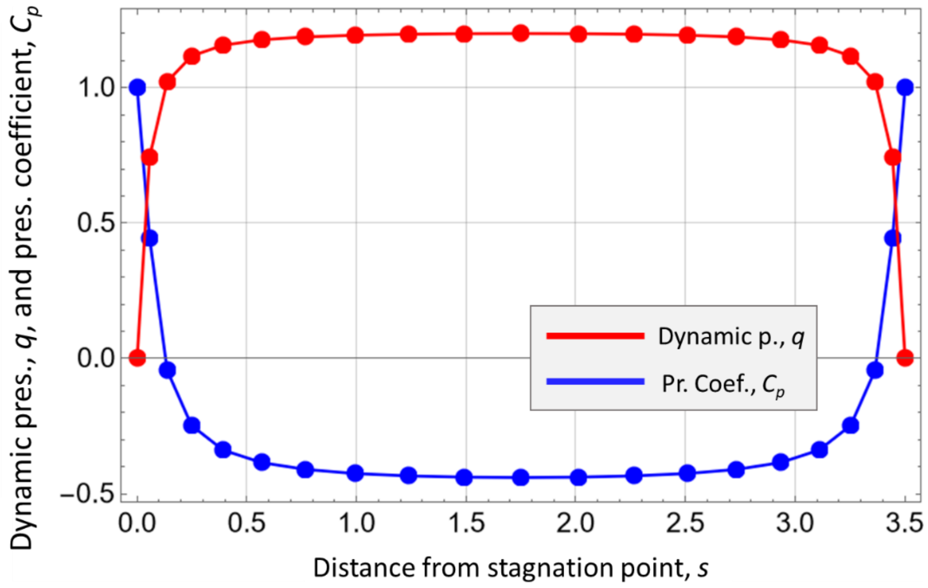
The pressure coefficient C_p is given by

$$C_p = 1 - q^2 \quad \text{(IV)}$$

The dynamic pressure q is, of course, the square root of (III).

$$q = \sqrt{\frac{1 - 2\cos(2\theta) + 1}{1 - 2c^2\cos(2\theta) + c^4}} \quad \text{(V)}$$

We proceed to plot pressure coefficient (equation (IV)) versus distance from stagnation point (eq. (II)); we also prepare a plot of dynamic pressure (equation (V)) versus s.



REFERENCE

- PANTON, R.L. (2013). *Incompressible Flow*. 4th edition. Hoboken: John Wiley and Sons.