Quiz EL402
Sampling and Pulse Code Modulation

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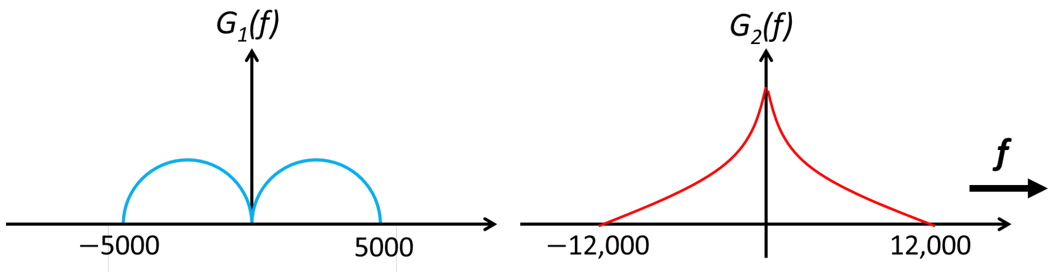
PROBLEMS

Problem 1

The following figure shows Fourier spectra of signals $g_1(t)$ and $g_2(t)$. Determine the Nyquist interval and the sampling rate for signals $g_1(t)$, $g_2(t)$, $g_1^2(t)$, $g_2^m(t)$, and $g_1(t)g_2(t)$.



Problem 2

Determine the Nyquist sampling rate and the Nyquist sampling interval for the following signals:

Problem 2.1: $\mathrm{sinc}\,(2100\pi t)$

Problem 2.2: $5\mathrm{sinc}^2\,(200\pi t)$

Problem 2.3: $\mathrm{sinc}\,(2100\pi t) + 5\mathrm{sinc}^2\,(200\pi t)$

Problem 2.4: $\mathrm{sinc}\,(200\pi t)\,\mathrm{sinc}\,(2100\pi t)$

Problem 3 (Sklar, 2001)

Determine the minimum sampling rate necessary to sample and perfectly reconstruct the signal $x(t) = \sin(6280t)/6280t$.

Problem 4 (Sklar, 2001)

A waveform, $x(t) = 10\cos(1000t + \pi/3) + 20\cos(2000t + \pi/6)$ is to be uniformly sampled for digital transmission.

Problem 4.1: What is the maximum allowable time interval between sample values that will ensure perfect signal reproduction?

Problem 4.2: If we want to reproduce 1 hour of this waveform, how many sample values need to be stored?

Problem 5

A compact disc (CD) records audio signals digitally by using PCM. Assume the audio signal bandwidth to be 15 kHz.

Problem 5.1: What is the Nyquist rate?

Problem 5.2: If the Nyquist samples are quantized into L = 65,536 levels and then binary coded, determine the number of binary digits required to encode a sample.

Problem 5.3: If the audio signal has average power of 0.1 W and peak voltage of 1 V, find the resulting ratio of signal to quantization noise (SQNR) of the uniform quantizer output in part 2.

Problem 5.4: Determine the number of binary digits per second (bit/s) required to encode the audio signal.

Problem 5.5: For practical reasons discussed in Lathi and Ding (2009), signals are sampled at a rate well above the Nyquist rate. Practical CDs use 44,100 samples per second. If L = 65,536, determine the number of bits per second required to encode the signal, and the minimum bandwidth required to transmit the encoded signal.

Problem 6

A television signal (video and audio) has a bandwidth of 4.5 MHz. This signal is sampled, quantized, and binary-coded to obtain a PCM signal.

Problem 6.1: Determine the sampling rate if the signal is to be sampled at a rate 20% above the Nyquist rate.

Problem 6.2: If the samples are quantized into 1024 levels, determine the number of binary pulses required to encode each sample.

Problem 6.3: Determine the binary pulse rate (bits per second) of the binary-coded signal and the minimum bandwidth required to transmit this signal.

Problem 7

In a binary PCM system, if the quantizing noise is not to exceed $\pm P$ percent of the peak-to-peak analog level, show that the number of bits in each PCM word needs to be

$$n \geq \log_2(10)\left[\log_{10}\left(\frac{50}{P}\right)\right] = 3.32\log_{10}\left(\frac{50}{P}\right)$$

Problem 8

The information in an analog voltage waveform is to be transmitted over a PCM system with a $\pm 0.2\%$ accuracy (full scale). The analog waveform has an absolute bandwidth of 200 Hz and an amplitude range of −5 to +5 V.

Problem 8.1: Determine the minimum sampling rate needed.

Problem 8.2: Determine the number of bits needed in each PCM word.

Problem 8.3: Determine the minimum bit rate required in the PCM signal.

Problem 9

A 700-Mbyte CD is used to store PCM data. Suppose that a voice-frequency (VF) signal is sampled at 8 ksamples/s and the encoded PCM is to have an average SNR of at least 30 dB. How many hours of VF conversation (i.e., PCM data) can be stored in the disk?

Problem 10

The output SNR of a 13-bit PCM was found to be insufficient at 30 dB. To achieve the desired SNR of 42 dB, it was decided to increase the number of quantization levels L. Find the fractional increase in the transmission bandwidth required for this increase in L.

Problem 11

For a PCM signal, determine the number of quantization levels L if the compression parameter μ = 100 and the minimum SNR required is 45 dB. Determine the output SNR with this value of L. Remember that L must be a power of 2, that is, $L = 2^n$ for a binary PCM.

Problem 12

A signal band-limited to 1 MHz is sampled at a rate 25% higher than the Nyquist rate and quantized into 256 levels using a μ-law quantizer with μ = 255.

Problem 12.1: Determine the signal-to-quantization noise ratio.

Problem 12.2: The SNR (the received signal quality) you found in the previous part was unsatisfactory. It must be increased at least by 10 dB. Would you be able to obtain the desired SNR without increasing the transmission bandwidth if a sampling rate 20% above the Nyquist rate were found to be adequate? If so, explain how. What is the maximum SNR that can be realized in this way?

SOLUTIONS

P.1 ➡ *Solution*

The Nyquist interval is $T_s = 1/2B$, where $2B$ is the sampling rate, which we also aim to find, and B is the bandwidth of the signal. By inspection, the bandwidth of $g_1(t)$ is 5 kHz, which means that the sampling rate is 2 × 5 = 10 kHz. In turn, the Nyquist interval is T_s = 1/10,000 = 0.1 ms.

Likewise, signal $g_2(t)$ has bandwidth equal to 12 kHz and sampling rate 2 × 12 = 24 kHz. The Nyquist interval is T_s = 1/24,000 = 4.17×10^{-5} s or 41.7 μs.

To establish the bandwidth of $g_1^2(t)$, first note that the product of two signals, in this case $g_1(t)$ with itself, has a frequency spectrum given by the convolution of the two signals. Also, the bandwidth of the convolution is given by the sum of the bandwidth of the individual signals. In the present case, signal $g_1(t)$ has bandwidth 5 kHz, so $g_1^2(t)$ will have a bandwidth equal to 5 + 5 = 10 kHz. The sampling rate of $g_1^2(t)$ is then 2 × 10 = 20 kHz. The sampling interval is 1/20,000 = 50 μs.

As for $g_2^m(t)$, note first that the signal $g_2(t)$ taken to the m power can be restated as

$$g_2^m(t) = g_2(t)\left[g_2^{m-1}(t)\right] = g_2(t)\left\{g_2(t)\left[g_2^{m-2}(t)\right]\right\} = \underbrace{g_2(t)g_2(t)\times\ldots\times g_2(t)}_{=m\,\text{times}}$$

so that, equivalently,

$$g_2^m(t) = \underbrace{G_2(f)*G_2(f)*\ldots*G_2(f)}_{=m\,\text{times}}$$

It follows that $g_2^m(t)$ will have a bandwidth given by m times 12 kHz, that is, B = 12,000m Hz, and a corresponding sampling rate 2 × 12,000m = 24,000m Hz. The sampling interval is 1/(24,000m) = $4.17\times10^{-5}m^{-1}$ sec. For m = 10, for example, we'd obtain a sampling rate 24,000 × 10 = 240 kHz and a Nyquist interval $4.17\times10^{-5} \div 10 = 4.17$ μs.

The product of $g_1(t)$, which has bandwidth of 5 kHz, and $g_2(t)$, which has bandwidth of 12 kHz, will surely have a bandwidth of 5 + 12 = 17 kHz, giving a sampling rate of 2 × 17 = 34 kHz and a Nyquist interval 1/34,000 = 2.94×10^{-5} s, or 29.4 μs.

P.2 ➡ *Solution*

Problem 2.1: Applying Fourier transforms brings to

$$\text{sinc}(2100\pi t) \Leftrightarrow \frac{1}{2100}\Pi\left(\frac{f}{2100}\right)$$

where Π denotes a rectangular function. The bandwidth of the transformed signal is 1050 Hz, and the Nyquist rate follows as $f_s = 2B$ = 2 × 1050 = 2100 Hz. The Nyquist interval is T_s = 1/2100 = 4.76×10^{-4} s = 476 μs.

Problem 2.2: Applying Fourier transforms gives

$$5\,\text{sinc}^2(200\pi t) \Leftrightarrow \frac{5}{200}\Delta\left(\frac{f}{400}\right)$$

where Δ denotes a triangular function. The bandwidth of the transformed signal is 200 Hz, and the Nyquist rate follows as $f_s = 2B$ = 2 × 200 = 400 Hz. The Nyquist interval is T_s = 1/400 = 2.5 ms.

Problem 2.3: The signal consists of the sum of the two foregoing signals. The same individual Fourier transforms apply, giving

$$\text{sinc}(2100\pi t) + 5\,\text{sinc}^2(200\pi t) \Leftrightarrow \frac{1}{2100}\Pi\left(\frac{f}{2100}\right) + \frac{5}{200}\Delta\left(\frac{f}{400}\right)$$

The bandwidth of the first signal was determined to be 1050 Hz, while that of the second was established as 200 Hz; the bandwidth of a signal made up of the sum of the individual signals will be the larger value, namely 1050 Hz. The Nyquist rate is then f_s = 2 × 1050 = 2100 Hz. The Nyquist interval is T_s = 1/2100 = 476 μs.

Problem 2.4: Taking the product of two signals means obtaining the convolution of their spectra. The width property of convolution states that the bandwidth of the convolution of two spectra is the sum of the individual bandwidths. The individual Fourier transforms are, in the present case,

$$\text{sinc}(200\pi t) \Leftrightarrow \frac{1}{200}\Pi\left(\frac{f}{200}\right)$$
$$\text{sinc}(2100\pi t) \Leftrightarrow \frac{1}{2100}\Pi\left(\frac{f}{2100}\right)$$

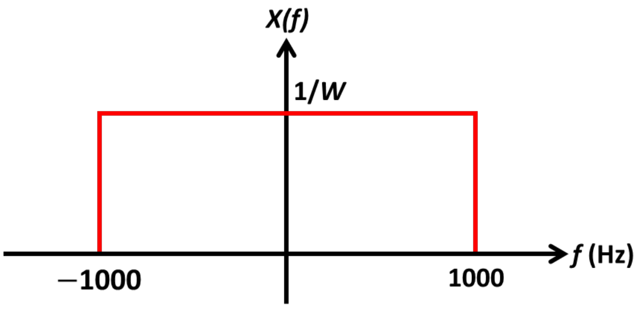
The bandwidth of the upper signal is 100 Hz, while that of the lower one is 1050 Hz. Accordingly, the product of the two signals will have bandwidth 100 + 1050 = 1150 Hz, giving a Nyquist rate f_s = 2 × 1150 = 2300 Hz. The Nyquist interval is T_s = 1/2300 = 4.35×10^{-4} s = 435 μs.

P.3 ➡ *Solution*

The signal at hand has the general form

$$x(t) = \frac{\sin(6280t)}{6280t} = \frac{\sin(Wt/2)}{Wt/2}$$

which means that $W/2 = 2\pi f$ = 6280 rad, or f = 1000 Hz. The frequency spectrum of the signal is illustrated below.



Clearly,

$$X(f) = \begin{cases} 1/W \;\; ; \;\; \text{for} \left| f \right| \le 1000\,\text{Hz} \\ 0 \;\; ; \;\; \text{elsewhere} \end{cases}$$

The minimum sampling rate is $f_s = 2f_m$ = 2000 Hz.

P.4 ➡ Solution

Problem 4.1: The cosine component with greater frequency (= 2000 rad) dictates the maximum time interval for perfect signal reproduction. A circular frequency of 2000 rad corresponds to a linear frequency f_m = 2000/2π = 318 Hz. The sampling frequency must be such that $f_s \geq 2f_m$ = 636 Hz, leading to a maximum time interval

$$T_s = \frac{1}{f_s} = \frac{1}{636} = 0.00157\,\text{s} = \boxed{1.57\,\text{ms}}$$

Problem 4.2: Multiplying the sampling frequency by 1 hour, we find that 636 × 3600 = 2.29×10^6 samples are required.

P.5 ➡ Solution

Problem 5.1: If the bandwidth is 15 kHz, the Nyquist rate equals f_s = 2 × 15 = 30 kHz.

Problem 5.2: If L denotes the number of quantization levels and n is the number of bits required to represent each quantization level, we may write

$$2^n = L \rightarrow 2^n = 65{,}536$$
$$\therefore n = \log_2 65{,}536 = \boxed{16}$$

That is, 16 bits are needed to encode a sample.

Problem 5.3: The SQNR is expressed as

$$SQNR = 3L^2 \frac{P_m}{m_p^2}$$

Here, L = 65,536 is the number of quantization levels, P_m = 0.1 W is signal power, and m_p = 1 V is signal peak voltage, giving

$$SQNR = 3 \times 65{,}536^2 \times \frac{0.1}{1.0^2} = 1.29 \times 10^9$$

Expressing the result in dB,

$$(SQNR)_{\text{dB}} = 10\log_{10}\left(1.29 \times 10^9\right) = \boxed{91.1\,\text{dB}}$$

Problem 5.4: The number of binary digits per second required to encode the audio signal equals the product of Nyquist rate, f_s = 30 kHz, and the number of bits required to represent each quantization level, n = 16,

$$N = 30{,}000 \times 16 = \boxed{480{,}000\,\text{bits/second}}$$

Problem 5.5: The number of bits per second required to encode the signal is n = 44,100 × 16 = 705,600 bits/sec. The minimum bandwidth required to transmit the encoded signal is half of this rate, that is, $n/2$ = 352,800 = 352.8 kHz.

P.6 ➡ *Solution*

Problem 6.1: The Nyquist rate of the television signal equals twice the bandwidth, that is, $f_s = 2 \times 4.5 = 9.0$ MHz. The signal is to be sampled at a rate 20% greater than the Nyquist rate, so

$$\text{Sampling rate} = 1.2 \times 9.0 = \boxed{10.8\,\text{MHz}}$$

Problem 6.2: If L denotes the number of quantization levels and n is the number of bits required to represent each quantization level, it is easily seen that

$$L = 2^n \rightarrow n = \log_2 L$$

$$\therefore n = \log_2 1024 = \boxed{10\,\text{bits}}$$

Ten binary pulses are needed to encode each sample.

Problem 6.3: For a sampling rate of 10.8 MHz and a number of bits per sample equal to 10, the pulse rate is found to be 10.8 × 10 = 108 Mbit/sec. Dividing this pulse rate by 2 gives the minimum bandwidth required to transmit the signal,

$$B = \frac{108}{2} = \boxed{54.0\,\text{MHz}}$$

P.7 ➡ *Solution*

The number of levels (steps) in an n-bit PCM word is

$$M = 2^n$$

The step size is given by the ratio of voltage to M,

$$\Delta = \frac{V_{pp}}{M}$$

or

$$\Delta = \frac{V_{pp}}{2^n} \quad \text{(I)}$$

The quantization noise is, if the device is not to exceed $\pm P$ percent of the peak-to-peak analog level,

$$|n_q| \le \frac{\Delta}{2} \le \frac{P}{100} V_{pp}$$

$$\Delta \le \frac{P}{50} V_{pp}$$

or, using (I),

$$\frac{V_{pp}}{2^n} \le \frac{P}{50} V_{pp} \rightarrow \frac{1}{2^n} \le \frac{P}{50}$$

$$\therefore 2^n \ge \frac{50}{P}$$

Applying base-2 logarithms,

$$2^n \ge \frac{50}{P} \rightarrow \log_2 2^n \ge \log_2 \left(\frac{50}{P}\right)$$

$$\therefore n \ge \log_2 \left(\frac{50}{P}\right)$$

Changing base,

$$n \ge \log_2 \left(\frac{50}{P}\right) \rightarrow n \ge \underbrace{\log_2 10}_{=3.32} \log_{10} \left(\frac{50}{P}\right)$$

$$\therefore \boxed{n \ge 3.32 \log_{10} \left(\frac{50}{P}\right)}$$

P.8 ➡ *Solution*

Problem 8.1: The sampling rate f_s is related to the maximum analog frequency B by the simple equation

$$f_s = 2B = 2 \times 100 = \boxed{200\,\text{Hz}}$$

A sampling rate of 200 Hz is required for this scheme.

Problem 8.2: For an accuracy $P = \pm 0.2\%$, the number of bits needed in each PCM word can be obtained with the relation derived in Problem 7,

$$n \geq 3.32\log_{10}\left(\frac{50}{P}\right) = 3.32\log_{10}\left(\frac{50}{0.1}\right) = 8.96 \approx \boxed{9\,\text{bits}}$$

That is, each PCM word must have 9 bits.

Problem 8.3: To find the bit rate, multiply the PCM word length by the sampling rate,

$$R = nf_s = 9 \times 200 = 1800\,\text{bits/s} = \boxed{1.8\,\text{kbit/s}}$$

P.9 ➡ Solution

The signal to noise ratio of a PCM system can be estimated as

$$(\text{SNR})_{\text{dB}} = 1.76 + 6n$$

Substituting *SNR* = 30 dB gives the number of bits in each PCM word,

$$(\text{SNR})_{\text{dB}} = 1.76 + 6n = 30 \rightarrow n = \frac{30 - 1.76}{6} = 4.71$$

$$\therefore n_{\min} = \lceil n \rceil = 5\,\text{bits}$$

The byte rate is

$$R = nf_s = 5 \times 8000 = 40{,}000\ \text{bits/s}$$

$$\therefore R = 40\,\frac{\text{kbits}}{\text{s}} \times \frac{1}{8}\,\frac{\text{byte}}{\text{bit}} = 5\,\text{kbyte/s}$$

The duration of conversation data that can be stored in the disk is then

$$t = \frac{\text{Memory}}{60 \times R} = \frac{700 \times 10^6\ \cancel{\text{byte}}}{60\,\frac{\cancel{\text{s}}}{\text{min}} \times \left(5000\,\frac{\cancel{\text{byte}}}{\cancel{\text{s}}}\right)} = 2330\ \text{min}$$

$$\therefore t = 2330\ \cancel{\text{min}} \times \frac{1}{60}\,\frac{\text{h}}{\cancel{\text{min}}} = \boxed{38.8\,\text{h}}$$

The compact disk can store about 39 hours of VF conversation data.

P.10 ➡ Solution

The number of levels for a 13-bit PCM word is 2^{13}. Evoking the expression for signal to noise ratio and adding a subscript 0 to denote initial conditions, we write

$$SNR_0 = 10\log_{10}\left[P_m\left(\frac{3L^2}{m_p^2}\right)\right] = 30\,\text{dB}$$

$$\therefore P_m\left(\frac{3L^2}{m_p^2}\right) = 10^3$$

$$\therefore P_m\left[\frac{3 \times \left(2^{13}\right)^2}{m_p^2}\right] = 10^3$$

$$\therefore \frac{P_m}{m_p^2} = \frac{10^3}{3 \times 2^{26}} \quad \text{(I)}$$

Now, adding a subscript 1 to denote final conditions, that is, conditions at the desired SNR of 42 dB, we proceed to write

$$SNR_1 = 10\log_{10}\left[P_m\left(\frac{3L_1^2}{m_p^2}\right)\right] = 42\,\text{dB}$$

$$\therefore \log_{10}\left[P_m\left(\frac{3L_1^2}{m_p^2}\right)\right] = 4.2$$

$$\therefore P_m\left(\frac{3L_1^2}{m_p^2}\right) = 10^{4.2}$$

$$\therefore \frac{P_m}{m_p^2} = \frac{10^{4.2}}{3L_1^2} \quad \text{(II)}$$

Because the signal power P_m and the peak voltage m_p are assumed constant, we can equate (I) and (II) and solve for the new number of levels L_1,

$$\frac{10^3}{3\times 2^{26}} = \frac{10^{4.2}}{3L_1^2} \to L_1 = \sqrt{\frac{10^{4.2}\times 2^{26}}{10^3}}$$
$$\therefore L_1 = 32,600$$

The corresponding number of bits required for quantization is $\log_2$ 32,600 = 14.99 ≈ 15. Transitioning from 13 to 15 bits will require increasing the transmission bandwidth by (15 – 13)/13 × 100% ≈ 15.4%.

P.11 ➡ *Solution*

The signal to noise ratio corresponding to 45 dB is $10^{4.5}$ = 31,600. The number of quantization levels L associated with a μ-law compandor can be established as

$$\frac{S_0}{N_0} = \frac{3L^2}{\left[\ln(\mu+1)\right]^2} = 31,600 \to L = \sqrt{\frac{31,600\times\left[\ln(\mu+1)\right]^2}{3}}$$

$$\therefore L = \sqrt{\frac{31,600\times\left[\ln(100+1)\right]^2}{3}} = 473$$

The lowest number of bits that can accommodate this number of quantization levels is n = 9, giving an output L such that $L = 2^9 = 512$. Lastly, the output SNR is

$$\frac{S_0}{N_0} = \frac{3L^2}{\left[\ln(\mu+1)\right]^2} = \frac{3\times 512^2}{\left[\ln(100+1)\right]^2} = 36,900$$
$$\therefore SNR = 10\log_{10}(S_0/N_0) = 10\log_{10}36,900 = \boxed{45.7\,\text{dB}}$$

P.12 ➡ *Solution* (Answer from Chegg)

Problem 12.1: To determine the SNR, simply substitute the pertaining variables into the SQNR relation for a μ-law quantizer,

$$(SQNR)_{\text{dB}} = 10\log_{10}\frac{3L^2}{\left[\ln(\mu+1)\right]^2} = 10\log_{10}\left\{\frac{3\times 256^2}{\left[\ln(255+1)\right]^2}\right\} = \boxed{38.057\,\text{dB}}$$

Problem 12.2: The Nyquist sampling rate is 2 × 1 MHz = 2.0 MHz. Sampling at a rate 25% greater than the NSR would yield a frequency of 2.5 MHz. We know that the transmission bandwidth also takes into account the number of bits used for quantizing each sample, so decreasing the sample rate allows us to increase the number of bits per sample and still keep the same bandwidth. The minimum number of bits required is $n = \log_2 256 = 8$.

If we decide that 20% higher than the Nyquist rate is enough, that means we can sample at 2.4 MHz, which is a 4% decrease from 2.5 MHz. This allows us to increase the number of bits by 4%. Considering each sample only uses 8 bits, this doesn't even let us raise our sample length to 9 bits. It follows that we are not able to increase the SNR by changing the sampling rate to 20% above the Nyquist rate.

Since we cannot continuously increase the quantization bits, the next increase in SNR occurs only when we move from 8 bits to 9 bits – which means twice as many quantization levels. This requires an increase in bandwidth of 12.5%, unless we manage to decrease the bandwidth by decreasing the sampling rate. This requires a change from sampling 25% over the Nyquist rate, to sampling only 11% over the Nyquist rate. Accordingly, we set $L = 2^9 = 512$ and compute

$$(SQNR)_{\text{dB}} = 10\log_{10}\frac{3L^2}{\left[\ln(1+\mu)\right]^2} = 10\log_{10}\left\{\frac{3\times 512^2}{\left[\ln(1+255)\right]^2}\right\} = \boxed{44.078\,\text{dB}}$$

The updated scheme leads to a raise of about 6 dB. The Chegg solution notes that if we're allowed to sample as low as exactly the Nyquist rate, we'd have 10 bits for quantization, adding another 6 dB to the SNR. In summary, sampling at the Nyquist rate enables us to attain 12 dB more SNR than if we sample at 25% over the Nyquist rate.

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