Quiz SM202

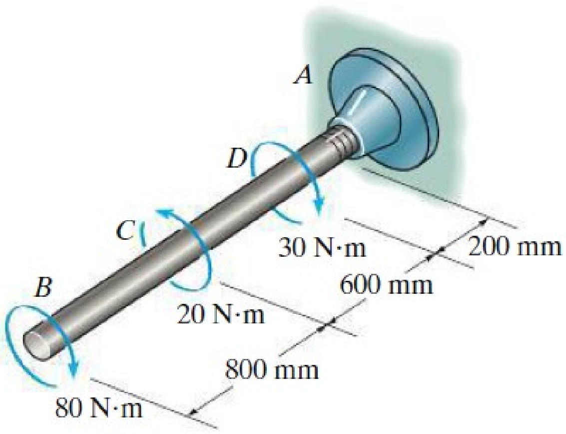
TORSION

Lucas Montogue

PROBLEMS

Problem 1 (Hibbeler, 2014, w/ permission)

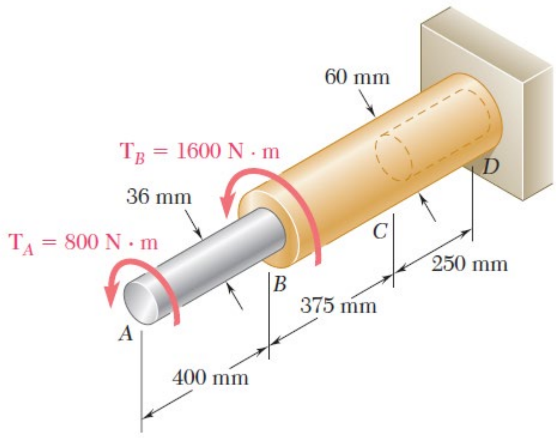
The 20-mm diameter steel shaft (G = 75 GPa) is subjected to the torques shown. Determine the absolute value of the angle of twist at end B.



A) $\phi = 1.85°$
B) $\phi = 3.66°$
C) $\phi = 5.73°$
D) $\phi = 7.54°$

Problem 2 (Beer et al., 2012, w/ permission)

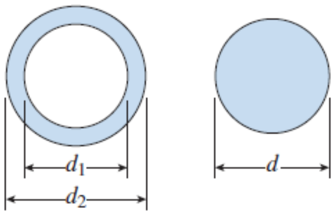
The aluminum rod AB (G = 27 GPa) is bonded to the brass rod (G = 39 GPa). Knowing that portion CD of the brass rod is hollow and has an inner diameter of 40 mm, determine the angle of twist at end A.



A) $\phi = 2.09°$
B) $\phi = 4.13°$
C) $\phi = 6.02°$
D) $\phi = 8.10°$

Problem **3A** (Gere & Goodno, 2009, w/ permission)

A hollow aluminum tube used in a roof structure has an outside diameter d_2 = 104 mm and an inside diameter d_1 = 82 mm (see figure). The tube is 2.75 m long, and the aluminum has shear modulus G = 28 GPa. If the tube is twisted in pure torsion by torques acting at the ends, what is the angle of twist (in degrees) when the maximum shear stress is 48 MPa?



A) $\phi = 3.75°$
B) $\phi = 5.20°$
C) $\phi = 6.75°$
D) $\phi = 8.20°$

Problem **3B**

What diameter d is required for a solid shaft (see preceding figure) to resist the same torque with the same maximum stress?

A) $d = 66.5$ mm
B) $d = 77.3$ mm
C) $d = 88.4$ mm
D) $d = 99.2$ mm

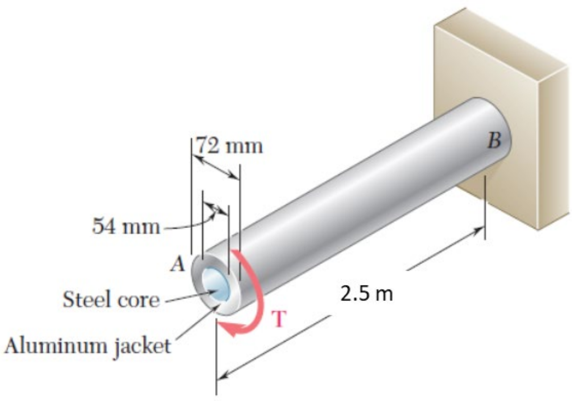
Problem **3C**

What is the ratio ρ of the weight of the hollow tube to the weight of the solid shaft?

A) $\rho = 0.524$
B) $\rho = 0.636$
C) $\rho = 0.745$
D) $\rho = 0.811$

Problem **4A** (Beer et al., 2012, w/ permission)

A torque of magnitude 4 kN·m is applied at end A of the 2.5-m long composite shaft shown. Knowing that the shear modulus is 77 GPa for steel and 27 GPa for aluminum, determine the maximum shear stress at the steel core.



A) $\tau_{st} = 12.4$ MPa
B) $\tau_{st} = 32.4$ MPa
C) $\tau_{st} = 53.6$ MPa
D) $\tau_{st} = 73.6$ MPa

Problem 4B

Determine the angle of twist at *A*.

A) $\phi = 3.09°$

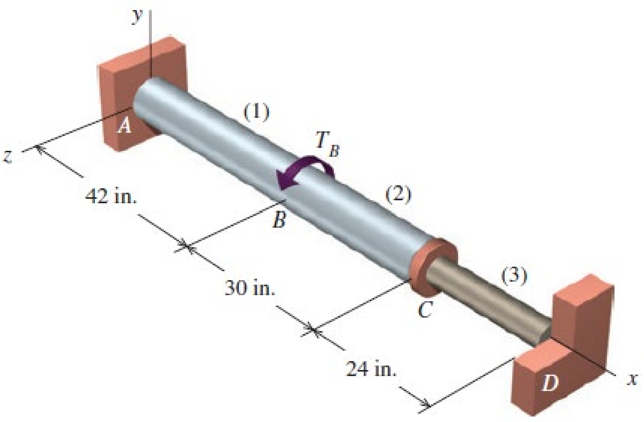
B) $\phi = 5.07°$

C) $\phi = 7.05°$

D) $\phi = 9.03°$

Problem 5A (Philpot, 2013, w/ permission)

The torsional assembly of the next figure consists of a cold-rolled stainless steel tube connected to a solid cold-rolled brass segment at flange *C*. The assembly is securely fastened to rigid supports at *A* and *D*. Stainless steel tube (1) and (2) has an outside diameter of 3.50 in., a wall thickness of 0.120 in., and a shear modulus of $G = 12{,}500$ ksi. The solid brass segment (3) has a diameter of 2.00 in. and a shear modulus of $G = 5600$ ksi. A concentrated torque of $T_B = 6$ kip-ft is applied to the stainless steel pipe at *B*. Determine the maximum shear stress magnitude in the stainless steel tube.



A) $\tau_{\max} = 7.40$ ksi

B) $\tau_{\max} = 9.81$ ksi

C) $\tau_{\max} = 14.4$ ksi

D) $\tau_{\max} = 27.2$ ksi

Problem 5B

Determine the rotation angle of flange *C*.

A) $\phi = 2.41°$

B) $\phi = 4.22°$

C) $\phi = 6.33°$

D) $\phi = 8.14°$

Problem 6A (Beer et al., 2012)

A solid shaft and a hollow shaft are made of the same material and are of the same weight and length. Assume that the angle of twist is also the same for each shaft. Denoting by *n* the ratio c_1/c_2, where c_1 and c_2 are the inner and outer radii of the hollow shaft, respectively, show that the ratio of torques $T_{\text{solid}}/T_{hollow}$ is such that

$$\frac{T_s}{T_h} = \frac{\sqrt{1-n^2}}{1+n^2}$$

Problem 6B

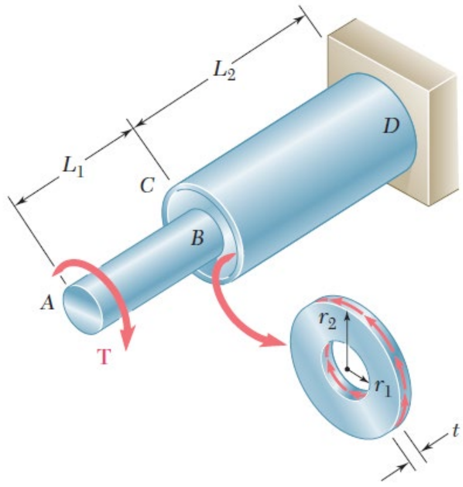
Suppose now that the angle of twist is the same in each shaft. Show that the ratio of torques now becomes

$$\frac{T_s}{T_h} = \frac{1-n^2}{1+n^2}$$

Problem 7 (Beer et al., 2012, w/ permission)

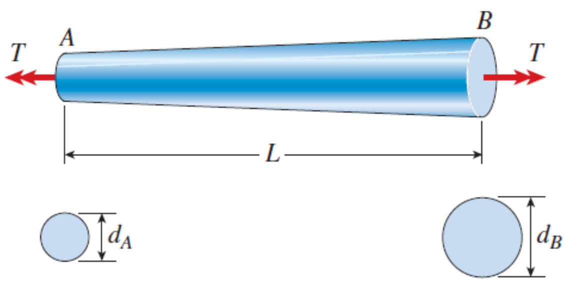
An annular plate of thickness t and modulus G is used to connect shaft AB of radius r_1 to the tube CD of radius r_2. Knowing that a torque T is applied to end A of shaft AB and that end D of tube CD is fixed, show that the angle through which end B of the shaft rotates with respect to end C of the tube is

$$\phi_{BC} = \frac{T}{4\pi G t}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$



Problem 8A (Gere & Goodno, 2009, w/ permission)

A tapered bar of solid circular section is twisted by torques T applied at the ends. The diameter of the bar varies linearly from d_A at the left-hand end to d_B at the right-hand end, with d_B assumed to be greater than d_A. Derive a formula for the angle of twist of the bar.



Problem 8B

For what ratio d_B/d_A will the angle of twist of the tapered bar be one-half the angle of twist of a prismatic bar of diameter d_A? The prismatic bar is made of the same material, has the same length, and is subjected to the same torque as the tapered bar.

A) $d_B/d_A = 1.13$

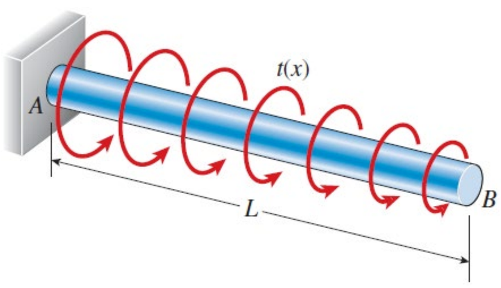
B) $d_B/d_A = 1.45$

C) $d_B/d_A = 1.72$

D) $d_B/d_A = 2.04$

Problem 9A (Gere & Goodno, 2012, w/ permission)

A prismatic bar AB of solid cross-section (diameter d) is loaded by a distributed torque (see figure). The torque per unit distance, is denoted by $t(x)$ and varies linearly from a maximum value at A to zero at end B. The length of the bar is L, and the shear modulus of elasticity of the material is G. Determine the maximum shear stress in the bar.



A) $\tau_{\max} = 2t_A L/(\pi d^3)$
B) $\tau_{\max} = 4t_A L/(\pi d^3)$
C) $\tau_{\max} = 8t_A L/(\pi d^3)$
D) $\tau_{\max} = 16t_A L/(\pi d^3)$

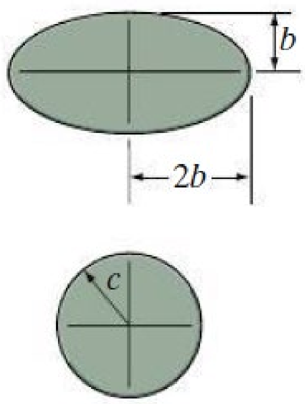
Problem 9B

Determine the angle of twist ϕ between the ends of the bar.

A) $\phi = 16t_A L^2/(3\pi G d^4)$
B) $\phi = 16t_A L^2/(\pi G d^4)$
C) $\phi = 32t_A L^2/(3\pi G d^4)$
D) $\phi = 32t_A L^2/(\pi G d^4)$

Problem 10 (Hibbeler, 2014, w/ permission)

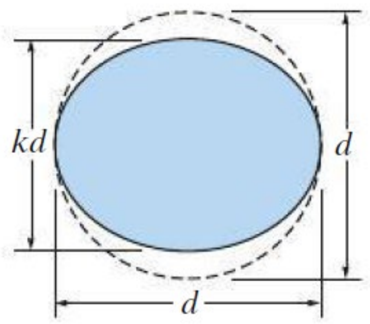
The shafts have elliptical and circular cross-sections and are to be made from the same amount of a similar material. Determine the percent increase in the maximum shear stress compared to the circular shaft when both shafts are subjected to the same torque and have the same length.



A) % increase in maximum shear stress = 23.3%
B) % increase in maximum shear stress = 41.4%
C) % increase in maximum shear stress = 62.1%
D) % increase in maximum shear stress = 80.2%

Problem 11 (Hibbeler, 2014, w/ permission)

It is intended to manufacture a circular bar to resist torque. However, the bar is made elliptical in the process of manufacturing, with one dimension smaller than the other by a factor of k, as shown. Determine the factor by which the maximum shear stress is increased.

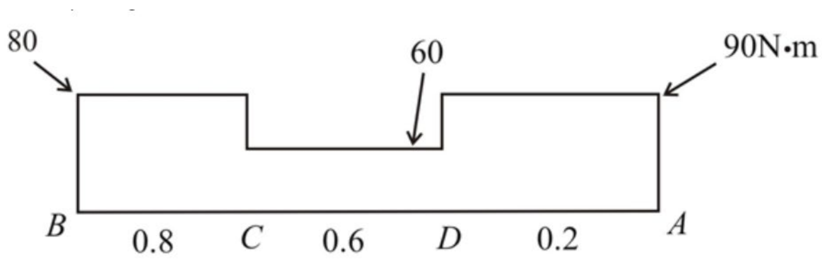


A) $f = 1/2k$
B) $f = 1/k$
C) $f = 1/2k^2$
D) $f = 1/k^2$

SOLUTIONS

P.1 ➡ *Solution*

The torque diagram for the shaft is provided below.



The total angle of twist is the sum of the angle of twist associated with each shaft segment; that is,

$$\phi = \sum_i \frac{T_i L_i}{G_i J_i}$$

Performing the summation and using the proper signs, we obtain

$$\phi = \frac{T_{BC} L_{BC}}{GJ} + \frac{T_{CD} L_{CD}}{GJ} + \frac{T_{DA} L_{DA}}{GJ}$$

$$\therefore \phi = \frac{1}{\left(75\times10^9\right)\times\left(\frac{\pi}{2}\times0.01^4\right)}\times\left(-80\times0.8-60\times0.6-90\times0.2\right)$$

$$\therefore \phi = -0.1 \text{ rad}$$

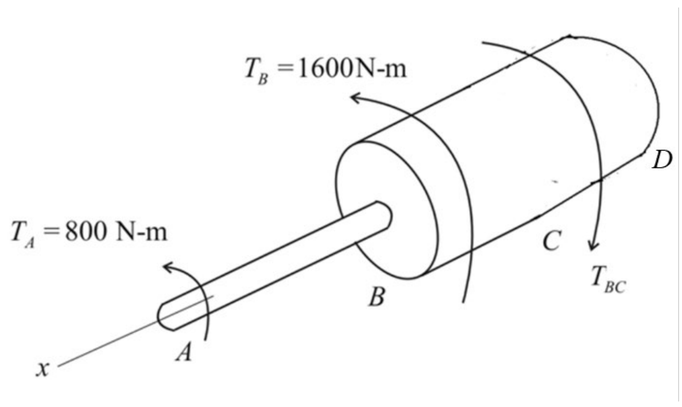
$$\therefore \boxed{|\phi| = 5.73^\circ}$$

The angle of twist at free end *B* is about 5.7 degrees.

The correct answer is **C**.

P.2 ➡ *Solution*

The structure in question is illustrated below.



The polar moments of inertia of segments *AB*, *BC*, and *CD* are, respectively,

$$J_{AB} = \frac{\pi}{2}\times0.018^4 = 1.65\times10^{-7} \text{ m}^4$$

$$J_{BC} = \frac{\pi}{2}\times0.03^4 = 1.27\times10^{-6} \text{ m}^4$$

$$J_{CD} = \frac{\pi}{2}\times\left(0.03^4-0.02^4\right) = 1.02\times10^{-6} \text{ m}^4$$

Let T_{BC} be the torque developed in shaft *BC*. Taking moments about the *x*-axis, we have

$$\Sigma M = 0 \rightarrow 800+1600-T_{BC} = 0$$

$$\therefore T_{BC} = 2400 \text{ N}\cdot\text{m}$$

Since no torque is applied at *C*, torque T_{CD} at the hollow segment of the brass rod is such that

$$T_{CD} = T_{BC} = 2400 \text{ N} \cdot \text{m}$$

The angle of twist at *A* is given by the sum of the angles of rotation in segments AB, BC, and CD, as follows,

$$\phi = \frac{T_{AB}L_{AB}}{J_{AB}G_{AB}} + \frac{T_{BC}L_{BC}}{J_{BC}G_{BC}} + \frac{T_{CD}L_{CD}}{J_{CD}G_{CD}}$$

$$\therefore \phi = \frac{800 \times 0.4}{\left(1.65 \times 10^{-7}\right) \times \left(27 \times 10^{9}\right)} + \frac{2400 \times 0.375}{\left(1.27 \times 10^{-6}\right) \times \left(39 \times 10^{9}\right)}$$

$$+ \frac{2400 \times 0.25}{\left(1.02 \times 10^{-6}\right) \times \left(39 \times 10^{9}\right)} = 0.105 \text{ rad}$$

$$\boxed{\phi = 6.02^{\circ}}$$

The end *A* of the structure will twist by about 6 degrees.

The correct answer is **C**.

P.3 ➡ *Solution*

Part A: The angle of twist for a cylindrical member under pure torsion is

$$\phi = \frac{TL}{GJ}$$

Manipulating the expression above gives

$$\phi = \underbrace{\left(\frac{Td_2}{2J}\right)}_{\tau_{\max}} \times \frac{2L}{Gd_2} = \tau_{\max} \times \frac{2L}{Gd_2}$$

where $\tau_{\max}$ is the shear stress at the outermost region of the cross-section, located a distance $d_2/2$ from the center. In the present case, $\tau_{\max} = 48$ MPa; substituting this and other pertaining variables yields

$$\phi = \tau_{\max} \times \frac{2L}{Gd_2} = 48 \times \frac{2 \times 2750}{28,000 \times 104} = 0.0907 \text{ rad} = \boxed{5.20^{\circ}}$$

Under a shear stress of 48 MPa, the bar will rotate by 5.2 degrees.

The correct answer is **B**.

Part B: If we were to replace the hollow shaft with a solid shaft to resist the same torque and with the same maximum stress, such a stress would be determined as

$$\tau_{\max} = \frac{Tr}{J} = \frac{T \times \frac{d}{2}}{\frac{\pi}{32 d^4}} = \frac{16T}{\pi d^3}$$

In the hollow section, the same stress is calculated as

$$\tau_{\max} = \frac{16Td_2^2}{\pi\left(d_2^4 - d_1^4\right)}$$

Equating the two previous equations and solving for *d*, it follows that

$$\frac{16\cancel{T}}{\cancel{\pi}d^3} = \frac{16\cancel{T}d_2}{\cancel{\pi}\left(d_2^4 - d_1^4\right)} \rightarrow d^3 = \frac{\left(d_2^4 - d_1^4\right)}{d_2}$$

$$\therefore d = \left(\frac{d_2^4 - d_1^4}{d_2}\right)^{\frac{1}{3}}$$

Substituting $d_1 = 82$ mm and $d_2 = 104$ mm, we find that

$$d = \left(\frac{104^4 - 82^4}{104}\right)^{\frac{1}{3}} = \boxed{88.4\ \text{mm}}$$

The diameter of the solid shaft is just above 8.8 centimeters.

The correct answer is **C**.

Part C: Recall that the weight of each shaft is proportional to its cross-sectional area; consequently, a comparison of weights is obtained by dividing the cross-sectional area of one section by that of the other. For the hollow section,

$$A_H = \frac{\pi}{4}\times\left(104^2 - 82^2\right) = 3213.9\ \text{mm}^2$$

while for the solid section,

$$A_S = \frac{\pi}{4}\times 88.4^2 = 6137.5\ \text{mm}^2$$

so that the ratio of weights, ρ, is determined as

$$\rho = \frac{A_H}{A_S} = \frac{3213.9}{6137.5} = \boxed{0.524}$$

That is to say, the hollow structure necessitates only about half as much material as the solid structure to withstand the same torque with the same maximum stress.

The correct answer is **A**.

P.4 ➡ *Solution*

Part A: The polar moments of inertia of the steel core, J_1, and the aluminum jacket, J_2, are respectively

$$J_1 = \frac{\pi}{2}\times 0.027^4 = 8.35\times 10^{-7}\ \text{m}^4$$

$$J_2 = \frac{\pi}{2}\times\left(0.036^4 - 0.027^4\right) = 1.80\times 10^{-6}\ \text{m}^4$$

Let T_1 be the torque exerted on the steel core and T_2 be the torque imparted on the aluminum jacket. We can propose the relation

$$T_1 + T_2 = T \quad \text{(I)}$$

where $T = 4$ kN·m is the torque applied at end A. Since this is the only equation we can obtain from statics, a compatibility equation is needed. Because the core and the jacket are concentric and perfectly fitted, the angle of twist is the same for both members, i.e.,

$$\phi_1 = \phi_2$$

Using the torque-twist relationship, we have

$$\frac{T_1 \cancel{L}}{J_1 G_{\text{st}}} = \frac{T_2 \cancel{L}}{J_2 G_{\text{Al}}} \rightarrow \frac{T_1}{J_1 G_{\text{st}}} = \frac{T_2}{J_2 G_{\text{Al}}} \quad \text{(II)}$$

where L cancels out because the core and sleeve have the same length. Substituting the pertaining variables, we obtain

$$\frac{T_1}{J_1 G_{\text{st}}} = \frac{T_2}{J_2 G_{\text{Al}}} \rightarrow \frac{T_1}{\left(8.35\times 10^{-7}\right)\times 77} = \frac{T_2}{\left(1.80\times 10^{-6}\right)\times 27}$$

$$\therefore T_1 = 1.32 T_2$$

Backsubstituting into equation (I) gives

$$T_1 + T_2 = T \to (1.32T_2) + T_2 = 4000$$

$$\therefore T_2 = \frac{4000}{2.32} = 1724.1 \text{ N}\cdot\text{m}$$

and, consequently,

$$T_1 = 4000 - T_2 = 4000 - 1724.1 = 2275.9 \text{ N}\cdot\text{m}$$

Finally, the maximum shear stress in the steel core, τ_{st}, is determined as

$$\tau_{\text{st}} = \frac{T_1 r_1}{J_1} = \frac{2275.9 \times 0.027}{\left(8.35\times10^{-7}\right)} = \boxed{73.6 \text{ MPa}}$$

If need be, we can also determine the stress in the aluminum sleeve using T_2 above.

The correct answer is **D**.

Part B: The angle of twist in question can be determined via either of the following equations,

$$\phi_1 = \frac{T_1 L_1}{G_{\text{st}} J_1} \;;\; \phi_2 = \frac{T_2 L_2}{G_{\text{Al}} J_2}$$

since both expressions should yield the same result, in accordance with the compatibility equation. Let us make use of the equation to the left. Substituting the pertaining variables brings to

$$\phi_1 = \frac{T_1 L_1}{G_{\text{st}} J_1} = \frac{2275.9\times2.5}{\left(77\times10^9\right)\times\left(8.35\times10^{-7}\right)} = 0.0885 \text{ rad}$$

$$\therefore \boxed{\phi = 5.07^\circ}$$

The correct answer is **B**.

P.5 ➡ *Solution*

Part A: Consider first the polar moments of inertia of the stainless steel tube, J_1, and the solid brass tube, J_3,

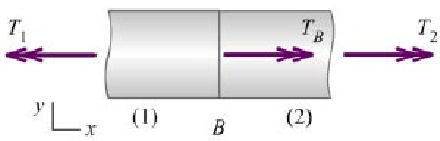
$$J_1 = \frac{\pi}{32}\times\left(3.5^4 - 3.26^4\right) = 3.64 \text{ in.}^4$$

$$J_3 = \frac{\pi}{32}\times2.0^4 = 1.57 \text{ in.}^4$$

Consider a free-body diagram cut around section *B*, as shown. From equilibrium of moments, we can write

$$\Sigma M = 0 \to -T_1 + T_2 + T_B = 0$$

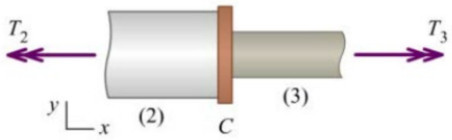
$$\therefore T_1 = T_2 + T_B \;\; \text{(I)}$$



Similarly, an equilibrium of moments around section *C* yields

$$\Sigma M = 0 \to -T_2 + T_3 = 0$$

$$\therefore T_3 = T_2 \;\; \text{(II)}$$



Since these are all the equations we can glean from statics, one equation of compatibility is in order. Noting that the two ends of the torsion structure are securely attached to fixed supports at *A* and *D*, the sum of the angles of twist in the three shafts must equal zero; that is,

$$\phi_1 + \phi_2 + \phi_3 = 0$$

Using the torsion formula, we have

$$\frac{T_1 L_1}{J_1 G_1} + \frac{T_2 L_2}{J_2 G_2} + \frac{T_3 L_3}{J_3 G_3} = 0$$

Using equations (I) and (II), we can replace T_1 and T_3,

$$\frac{(T_2 + T_B) L_1}{J_1 G_1} + \frac{T_2 L_2}{J_2 G_2} + \frac{T_2 L_3}{J_3 G_3} = 0$$

$$\therefore \frac{T_2 L_1}{J_1 G_1} + \frac{T_B L_1}{J_1 G_1} + \frac{T_2 L_2}{J_2 G_2} + \frac{T_2 L_3}{J_3 G_3} = 0$$

$$\therefore T_2 \left(\frac{L_1}{J_1 G_1} + \frac{L_2}{J_2 G_2} + \frac{L_3}{J_3 G_3} \right) = -\frac{T_B L_1}{J_1 G_1}$$

$$\therefore T_2 = -\frac{\dfrac{T_B L_1}{J_1 G_1}}{\left(\dfrac{L_1}{J_1 G_1} + \dfrac{L_2}{J_2 G_2} + \dfrac{L_3}{J_3 G_3} \right)}$$

However, $J_1 = J_2$ and $G_1 = G_2$, which simplifies the equation a bit further, giving

$$T_2 = -\frac{\dfrac{T_B L_1}{J_1 G_1}}{\dfrac{1}{J_1 G_1}(L_1 + L_2) + \dfrac{L_3}{J_3 G_3}} = -\frac{\dfrac{(6 \times 12) \times 42}{3.64 \times 12{,}500}}{\dfrac{1}{3.64 \times 12{,}500} \times (42 + 30) + \dfrac{24}{1.57 \times 5600}} = -15.4 \text{ kip-in}$$

We can then return to equations (I) and (II) and determine the remaining torques,

$$T_1 = T_2 + T_B \rightarrow T_1 = -15.4 + (6 \times 12) = 56.6 \text{ kip-in.}$$

and

$$T_3 = T_2 = -15.4 \text{ kip-in.}$$

The maximum shear stress magnitudes in the three segments are

$$\tau_1 = \frac{T_1 r_1}{J_1} = \frac{56.6 \times 3.5/2}{3.64} = 27.2 \text{ ksi}$$

$$\tau_2 = \frac{T_2 r_2}{J_2} = \frac{15.4 \times 3.5/2}{3.64} = 7.40 \text{ ksi}$$

$$\tau_3 = \frac{T_3 r_3}{J_3} = \frac{15.4 \times 2.0/2}{1.57} = 9.81 \text{ ksi}$$

We were asked to determine the maximum shear stress magnitude in the stainless steel tube; accordingly, such a stress is $\tau_1 = \tau_{max} = 27.2$ ksi.

The correct answer is **D**.

Part B: The angle of rotation in member (1) can be defined by the difference in the rotation at ends *A* and *B*. Mathematically,

$$\phi_1 = \phi_B - \phi_A \rightarrow \phi_B = \phi_1 + \phi_A \text{ (III)}$$

Similarly, the angle of rotation of segment (2) is given by

$$\phi_2 = \phi_C - \phi_B \rightarrow \phi_C = \phi_2 + \phi_B$$

Substituting from equation (III), we obtain

$$\phi_C = \phi_2 + \phi_B \rightarrow \phi_C = \phi_2 + \left(\phi_1 + \phi_A\right)$$

Since joint *A* is restrained from rotating, the equation simplifies to

$$\phi_C = \phi_2 + \phi_1 + \cancel{\phi_A} \rightarrow \phi_C = \phi_1 + \phi_2 \text{ (IV)}$$

Now, the angle of twist associated with member (1) is

$$\phi_1 = \frac{T_1 L_1}{J_1 G_1} = \frac{56.6 \times 42}{3.64 \times 12{,}500} = 0.0522 \text{ rad}$$

while that of segment (2) is

$$\phi_2 = \frac{T_2 L_2}{J_2 G_2} = -\frac{15.4 \times 30}{3.64 \times 12{,}500} = -0.0102 \text{ rad}$$

Substituting these results into the equation for ϕ_C, the desired angle of twist is calculated as

$$\phi_C = 0.0522 - 0.0102 = 0.0420 \text{ rad}$$

$$\therefore \boxed{\phi = 2.41^{\circ}}$$

The correct answer is **A**.

P.6 ➡ *Solution*

Part A: Since the solid shaft and the hollow shaft are of same weight and length, their cross-sectional areas must also be the same. This means that

$$A_s = A_h \rightarrow \pi \times c^2 = \pi \times \left(c_2^2 - c_1^2\right)$$

where c is the radius of the solid shaft, c_1 is the inner radius of the hollow shaft, and c_2 is the outer radius of the hollow shaft. We can adjust this expression as

$$\pi \times c^2 = \pi \times \left(c_2^2 - c_1^2\right) \rightarrow \cancel{\pi} \times c^2 = \cancel{\pi} \times c_2^2 \times \left[1 - \left(\frac{c_1}{c_2}\right)^2\right]$$

$$\therefore c^2 = c_2^2 \times \left[1 - \left(\frac{c_1}{c_2}\right)^2\right]$$

However, $c_1/c_2 = n$, so that

$$c^2 = c_2^2 \times \left(1 - n^2\right) \text{ (I)}$$

Now, the maximum shear stress in the solid shaft is

$$\tau_s = \frac{T_s c}{J_s} \text{ (II)}$$

where J_s is the polar moment of inertia, which is given by

$$J_s = \frac{\pi}{2} c^4$$

From equation (I), we get

$$J_s = \frac{\pi}{2} c^4 \rightarrow J_s = \frac{\pi}{2} \times c_2^4 \left(1 - n^2\right)^2$$

Proceeding similarly with the hollow shaft, we have

$$\tau_h = \frac{T_h c_2}{J_h} \text{ (III)}$$

where

$$J_h = \frac{\pi}{2} \times \left(c_2^4 - c_1^4\right)$$

$$\therefore J_h = \frac{\pi}{2} \times c_2^4 \left[1 - \left(\frac{c_1}{c_2}\right)^4\right]$$

$$\therefore J_h = \frac{\pi}{2} \times c_2^4 \left(1 - n^4\right)$$

The shear stress for both shafts should be the same, so that, equating (II) and (III), we obtain

$$\frac{T_s c}{J_s} = \frac{T_h c_2}{J_h} \rightarrow \frac{T_s}{T_h} = \frac{J_s c_2}{J_h c}$$

Substituting the expressions obtained for the polar moments, we get

$$\frac{T_s}{T_h} = \frac{J_s c_2}{J_h c} \rightarrow \frac{T_s}{T_h} = \frac{\frac{\pi}{2} c_2^4 \left(1 - n^2\right)^2 c_2}{\frac{\pi}{2} c_2^4 \left(1 - n^4\right) c}$$

From equation (I), $c = c_2\sqrt{1 - n^2}$; that is,

$$\frac{T_s}{T_h} = \frac{\frac{\pi}{2} c_2^4 \left(1 - n^2\right)^2 c_2}{\frac{\pi}{2} c_2^4 \left(1 - n^4\right) c} \rightarrow \frac{T_s}{T_h} = \frac{\frac{\pi}{2} c_2^4 \left(1 - n^2\right)^2 c_2}{\frac{\pi}{2} c_2^4 \left(1 - n^4\right)\left(c_2\sqrt{1 - n^2}\right)}$$

Noting that $(1 - n^4) = (1 - n^2)(1 + n^2)$ and canceling common terms in the numerator and the denominator, it follows that

$$\frac{T_s}{T_h} = \frac{\cancel{\frac{\pi}{2}} \cancel{c_2^4} \left(1 - n^2\right)^2 \cancel{c_2}}{\cancel{\frac{\pi}{2}} \cancel{c_2^4} \left(1 - n^4\right)\left(\cancel{c_2}\sqrt{1 - n^2}\right)} \rightarrow \frac{T_s}{T_h} = \frac{\left(1 - n^2\right)^2}{\left(1 - n^2\right)\left(1 + n^2\right)\sqrt{1 - n^2}}$$

$$\therefore \boxed{\frac{T_s}{T_h} = \frac{\sqrt{1 - n^2}}{1 + n^2}}$$

This concludes the proof.

Part B: The angle of rotation can be obtained with the torque-twist relationship,

$$\phi = \frac{Tr}{J}$$

Since the angles of twist of the solid shaft, ϕ_s, and of the hollow shaft, ϕ_h, must be the same, we can write

$$\phi_s = \phi_h \rightarrow \frac{T_s \cancel{L}}{J_s \cancel{G}} = \frac{T_h \cancel{L}}{J_h \cancel{G}}$$

$$\therefore \frac{T_s}{T_h} = \frac{J_s}{J_h}$$

Substituting the polar moments from previous results gives

$$\frac{T_s}{T_h} = \frac{J_s}{J_h} \rightarrow \frac{T_s}{T_h} = \frac{\cancel{\frac{\pi}{2}c_2^4}\left(1-n^2\right)^2}{\cancel{\frac{\pi}{2}c_2^4}\left(1-n^4\right)} = \frac{\left(1-n^2\right)^2}{\left(1-n^4\right)}$$

Noting that $(1-n^4) = (1-n^2)(1+n^2)$, we conclude that

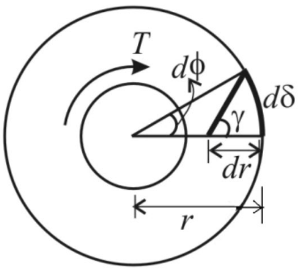
$$\frac{T_s}{T_h} == \frac{\left(1-n^2\right)^2}{\left(1-n^4\right)} \rightarrow \frac{T_s}{T_h} = \frac{\left(1-n^2\right)^{\cancel{2}}}{\cancel{\left(1-n^2\right)}\left(1+n^2\right)}$$

$$\therefore \boxed{\frac{T_s}{T_h} = \frac{1-n^2}{1+n^2}}$$

as we intended to show.

P.7 ➡ *Solution*

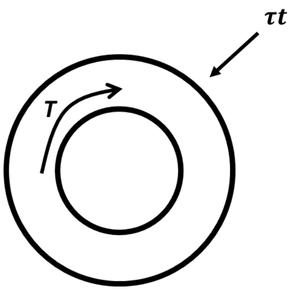
Consider the following elemental strip along the circumference of the rod. The inner circle is the cross-section of segment *AB*, while the outer circle is the section of *BC*.



The force per unit length on the bar is given by the product of shear stress and thickness,

$$F = \tau \times t$$

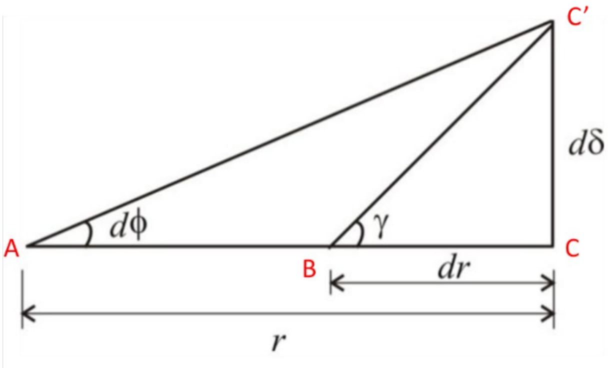
Consider now a free-body diagram for the bar.



Summing moments about the center of the bar, we obtain

$$\tau t \times 2\pi r \times r - T = 0 \rightarrow \tau = \frac{T}{2\pi t r^2} \text{ (I)}$$

The equation above provides the maximum shear stress in the structure. The next figure shows the geometry of an elemental strip taken from the structure.



Here, $d\phi$ is the angle of twist of the strip, $d\delta$ is the circumferential displacement, and γ is the shear strain. From the stress-strain relationship for shear, we can write

$$\tau = G\gamma \to \gamma = \frac{\tau}{G}$$

Substituting τ from equation (I) gives

$$\gamma = \frac{\tau}{G} = \frac{1}{G} \times \frac{T}{2\pi t r^2} \to \gamma = \frac{T}{2\pi t r^2 G} \text{ (II)}$$

Since *ACC'* is a right triangle, we see that

$$\tan d\phi = \frac{d\delta}{r}$$

For small angles, however, we can use the approximation $\tan d\phi \approx d\phi$. Accordingly,

$$d\phi \approx \frac{d\delta}{r} \to d\delta = r d\phi$$

Because *BCC'* is also a right triangle, we have

$$\tan\gamma = \frac{d\delta}{dr}$$

Using the approximation $\tan\gamma \approx \gamma$ yields

$$\gamma = \frac{d\delta}{dr}$$

Substituting $d\delta = r d\phi$, the equation becomes

$$\gamma = \frac{d\delta}{dr} = r\frac{d\phi}{dr} \to d\phi = \gamma\frac{dr}{r}$$

At this point, we can replace the shear strain with equation (II),

$$d\phi = \gamma\frac{dr}{r} = \frac{T}{2\pi t G}\frac{dr}{r^3}$$

To obtain the angle of twist ϕ_{BC} of end *B* with respect to end *C*, all we have to do is integrate the expression above from r_1 to r_2; that is,

$$\phi_{BC} = \frac{T}{2\pi t G}\int_{r_1}^{r_2}\frac{dr}{r^3} \to \phi_{BC} = \frac{T}{2\pi t G}\left[-\frac{1}{2r^2}\right]_{r=r_1}^{r=r_2}$$

$$\therefore \boxed{\phi_{BC} = \frac{T}{4\pi t G}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)}$$

as expected.

P.8 ➡ *Solution*

Part A: We begin by setting up an expression that models the variation of the diameter *d* with distance *x* from the end *A*,

$$d(x) = d_A + \frac{d_B - d_A}{L}x$$

Using this expression, we can establish the polar moment of inertia as

$$J(x) = \frac{\pi d^4}{32} = \frac{\pi}{32} \times \left(d_A + \frac{d_B - d_A}{L}x\right)^4$$

Substituting this relation in the torsion formula, the angle of twist is determined as

$$\phi = \int_0^L \frac{Tdx}{GJ(x)} = \frac{32T}{\pi G}\int_0^L \frac{dx}{\left(d_A + \frac{d_B - d_A}{L}x\right)^4}$$

The integration above can be carried out with the standard result

$$\int \frac{dx}{(a+bx)^4} = -\frac{1}{3b(a+bx)^3}$$

or, in the present case,

$$\int_0^L \frac{dx}{\left(d_A + \frac{d_B - d_A}{L}x\right)^4} = \frac{L}{3d_A^3(d_B - d_A)} - \frac{L}{3d_B^3(d_B - d_A)} = \frac{L}{3(d_B - d_A)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right)$$

Returning to the expression for ϕ with this result, we conclude that

$$\boxed{\phi = \frac{32TL}{3\pi G(d_B - d_A)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right)}$$

This is the desired equation for the angle of twist of the tapered bar. As pointed out by Gere & Goodno, a convenient form in which to write this result is

$$\phi = \frac{TL}{GJ_A}\left(\frac{\alpha^2 + \alpha + 1}{3\alpha^3}\right)$$

where

$$\alpha = \frac{d_B}{d_A}\ ;\ J_A = \frac{\pi d_A^4}{32}$$

That is, α is the ratio of end diameters d_B/d_A and J_A is the polar moment of inertia of the cross-section at *A*. In the special case $\alpha = 1$, the diameters are equal and the equation reduces to

$$\phi = \frac{TL}{GJ_A}\underbrace{\left(\frac{1^2 + 1 + 1}{3 \times 1^3}\right)}_{=1} = \frac{TL}{GJ_A}$$

That is to say, the torque-twist relationship becomes the torsion formula as applied to a solid shaft of constant cross-section, in accordance with our expectations. For values of α greater than 1, the angle of rotation decreases because the larger diameter at end *B* produces an increase in the torsional stiffness (relatively to a prismatic bar).

Part B: The angle of twist of the hypothetical tapered bar, ϕ_1, is given by the relation derived in the previous part,

$$\phi_1 = \frac{TL}{GJ_A}\left(\frac{\alpha^2+\alpha+1}{3\alpha^3}\right)$$

where $\alpha = d_B/d_A$. For a prismatic bar, the angle of twist, ϕ_2, is determined as

$$\phi_2 = \frac{TL}{GJ_A}$$

According to the problem statement, we shall have $\phi_1 = (1/2)\phi_2$. Substituting from the two previous equations, we have

$$\phi_1 = \frac{1}{2}\phi_2 \rightarrow \cancel{\frac{TL}{GJ_A}}\left(\frac{\alpha^2+\alpha+1}{3\alpha^3}\right) = \frac{1}{2}\cancel{\frac{TL}{GJ_A}}$$

$$\therefore 2\alpha^2 + 2\alpha + 2 = 3\alpha^3$$

$$\therefore 3\alpha^3 - 2\alpha^2 - 2\alpha - 2 = 0$$

This is a third-degree polynomial, and as such can be solved by numerical methods. Using a CAS such as Mathematica, we verify that the only real solution is $\alpha = d_B/d_A \approx 1.45$. That is to say, the larger cross-section of the tapered bar should have 1.45 times the diameter of the smaller cross-section if the bar is to have half the angle of twist of its prismatic counterpart.

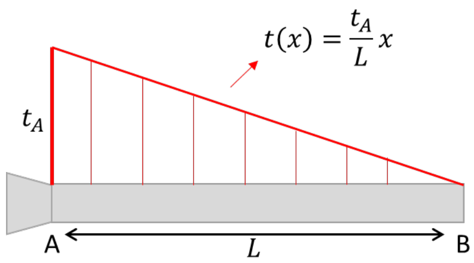
The correct answer is **B**.

P.9 ➡ *Solution*

Part A: According to the torsion formula, shear stress is proportional to the applied torque. In the present case, the shear stress reaches a maximum of $t_A L/2$ when $x = L$, that is, at section A. The maximum shear stress is, accordingly,

$$\tau_{\max} = \frac{16T_{\max}}{\pi d^3} = \frac{16\times(t_A L/2)}{\pi d^3}$$

$$\therefore \boxed{\tau_{\max} = \frac{8t_A L}{\pi d^3}}$$



The correct answer is **C**.

Part B: Let *T(x)* be the torque at a distance *x* from end *B*. From the geometry of the previous figure, we see that, in this loading configuration, the torque is

$$T(x) = \frac{t(x)x}{2} = \frac{t_A x}{L}\times\frac{x}{2} = \frac{t_A x^2}{2L}$$

Now, the infinitesimal angle of twist $d\phi$ is given by

$$d\phi = \frac{T(x)dx}{GJ}$$

which, substituting *T(x)* and the polar moment $J = \pi d^4/32$, becomes

$$\int_0^{\phi} d\phi = \frac{t_A}{2LGJ}\int_0^L x^2 dx \rightarrow \phi = \frac{t_A}{2LG\times\left(\frac{\pi d^4}{32}\right)}\times \frac{x^3}{3}\Bigg|_{x=0}^{x=L}$$

$$\therefore \phi = \frac{16t_A L^{\cancel{3}}}{3\pi G\cancel{L}d^4}$$

$$\therefore \boxed{\phi = \frac{16t_A L^2}{3\pi G d^4}}$$

Clearly, the angle of twist is proportional to the intensity of the distributed torque and the length of the beam.

The correct answer is **A**.

P.10 ➡ *Solution*

Since the elliptical and circular shafts are made of the same amount of material, their cross-sectional areas must be the same. Mathematically,

$$\cancel{\pi}\times b\times 2b = \cancel{\pi}c^2 \rightarrow c = \sqrt{2}b$$

(Recall that the area of an elliptical section is $A = \pi ab$, where a is the semi-major axis and b is the semi-minor axis). The circular shaft has a maximum shear stress $(\tau_{\max})_C$ such that

$$(\tau_{\max})_C = \frac{Tr}{J} = \frac{T\times\sqrt{2}b}{\frac{\pi}{2}\times\left(\sqrt{2}b\right)^4} = \frac{T}{\sqrt{2}\pi b^3}$$

while the elliptical member has a maximum shear stress $(\tau_{\max})_E$ given by

$$(\tau_{\max})_E = \frac{2T}{\pi ab^2} = \frac{2T}{\pi\times 2b\times b^2} = \frac{T}{\pi b^3}$$

Let ρ be the percentage increase of stress in the elliptical section relative to the circular section, which, in mathematical terms, equals

$$\rho = \frac{(\tau_{\max})_E - (\tau_{\max})_C}{(\tau_{\max})_C}$$

Substituting the results obtained above, it follows that

$$\rho = \frac{(\tau_{\max})_E - (\tau_{\max})_C}{(\tau_{\max})_C} = \frac{\frac{T}{\pi b^3} - \frac{T}{\sqrt{2}\pi b^3}}{\frac{T}{\sqrt{2}\pi b^3}} = \frac{\cancel{\frac{T}{\pi b^3}}\left(1-\frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}}\cancel{\frac{T}{\pi b^3}}} = 0.414$$

$$\therefore \boxed{\rho = 41.4\%}$$

That is, the elliptical section, albeit being made of the same amount of material, will have a maximum shear stress over 40% greater than an equivalent circular section. This problem illustrates the greater efficiency of the circular shaft in carrying torsional loads relatively to other types of geometry.

The correct answer is **B**.

P.11 ➡ *Solution*

For the circular shaft, the maximum shear stress is given by the well-known result

$$(\tau_{\max})_C = \frac{Tr}{J} = \frac{T\times\frac{d}{2}}{\frac{\pi}{2}\times\left(\frac{d}{2}\right)^4} = \frac{16T}{\pi d^3}$$

Knowing that $a = d/2$ is the semi-major axis and $a = kd/2$ is the semi-minor axis of the elliptical cross-section, the corresponding shear stress is determined as

$$(\tau_{\max})_E = \frac{2T}{\pi ab^2} \rightarrow (\tau_{\max})_E = \frac{2T}{\pi\times\left(\frac{d}{2}\right)\times\left(\frac{kd}{2}\right)^2} = \frac{16T}{\pi k^2 d^3}$$

The factor f of increase in maximum shear stress follows as

$$f = \frac{(\tau_{\max})_E}{(\tau_{\max})_C} = \frac{\frac{\cancel{16T}}{\cancel{\pi}k^2\cancel{d^3}}}{\frac{\cancel{16T}}{\cancel{\pi d^3}}} \rightarrow \boxed{f = \frac{1}{k^2}}$$

Note that, for the particular case in which $k = 1$, the ellipse restores a circular shape and $f = 1$, as one would expect.

The correct answer is **D**.

ANSWER SUMMARY

Problem 1		C
Problem 2		C
Problem 3	3A	B
	3B	C
	3C	A
Problem 4	4A	D
	4B	B
Problem 5	5A	D
	5B	A
Problem 6	6A	Open-ended pb.
	6B	Open-ended pb.
Problem 7		Open-ended pb.
Problem 8	8A	Open-ended pb.
	8B	B
Problem 9	9A	C
	9B	A
Problem 10		B
Problem 11		D

REFERENCES

- BEER, F., JOHNSTON, E., DEWOLF, J., and MAZUREK, D. (2012). *Mechanics of Materials*. 6th edition. New York: McGraw-Hill.
- GERE, J. and GOODNO, B. (2009). *Mechanics of Materials*. 7th edition. Stamford: Cengage Learning.
- HIBBELER, R. (2014). *Mechanics of Materials*. 9th edition. Upper Saddle River: Pearson.
- PHILPOT, T. (2013). *Mechanics of Materials: An Integrated Learning System*. 3rd edition. Hoboken: John Wiley and Sons.

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