Quiz VB104
Two-Degree-of-Freedom Systems
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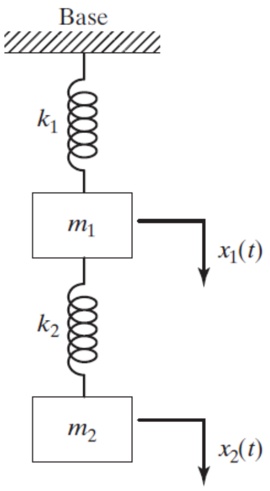
PROBLEMS

Problem 1 (Rao, 2011, w/ permission)

Find the natural frequencies of the system shown in the next figure, with $m_1 = m$, $m_2 = 2m$, $k_1 = k$, and $k_2 = 2k$. Which of the following quantities is one of the natural frequencies of this system if m = 20 kg and k = 1000 N/m?



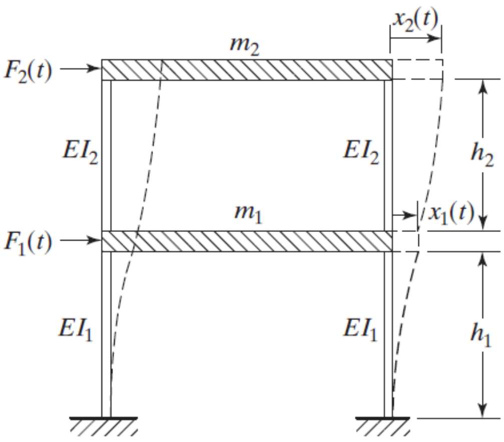
A) ω_n = 1.65 rad/s
B) ω_n = 3.66 rad/s
C) ω_n = 5.59 rad/s
D) ω_n = 7.60 rad/s

Problem 2 (Rao, 2011, w/ permission)

A two-story building frame is modeled as shown below. The girders are assumed to be rigid, and the columns have flexural rigidities EI_1 and EI_2, with negligible masses. The stiffness of each column can be computed as

$$\frac{24EI_i}{h_i^3} \quad i = 1, 2$$

for $m_1 = 2m$, $m_2 = m$, $h_1 = h_2 = h$, and $EI_1 = EI_2 = EI$, which of the following is one of the natural frequencies of this system?



A) $\omega_n = 0.54\sqrt{\frac{EI}{mh^3}}$

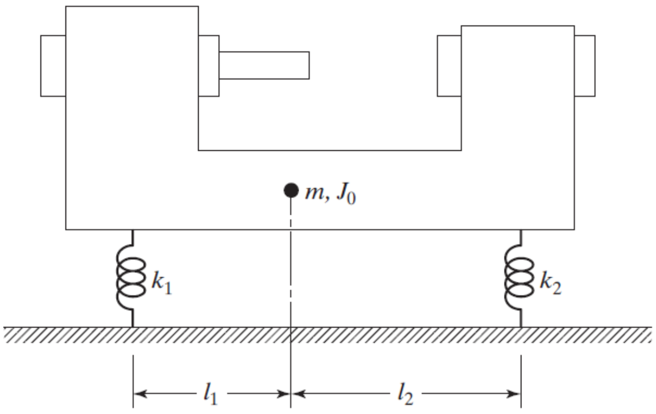
B) $\omega_n = 3.75\sqrt{\frac{EI}{mh^3}}$

C) $\omega_n = 6.98\sqrt{\dfrac{EI}{mh^3}}$

D) $\omega_n = 11.4\sqrt{\dfrac{EI}{mh^3}}$

Problem 3 (Rao, 2011, w/ permission)

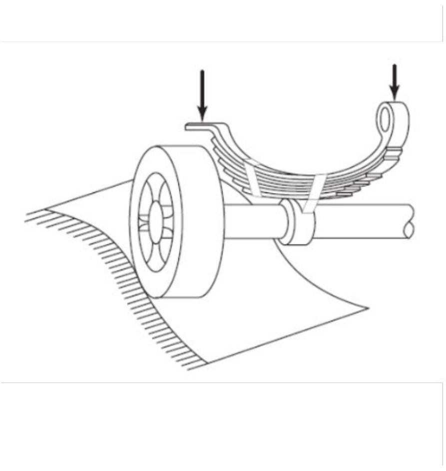
A machine tool, having a mass of m = 1000 kg and a mass moment of inertia of J_o = 300 kg·m², is supported on elastic supports, as shown. If the stiffnesses of the supports are k_1 = 3000 N/mm and k_2 = 2000 N/mm, and the supports are located at l_1 = 0.5 m and l_2 = 0.8 m, which of the following is one of the natural frequencies of this system?

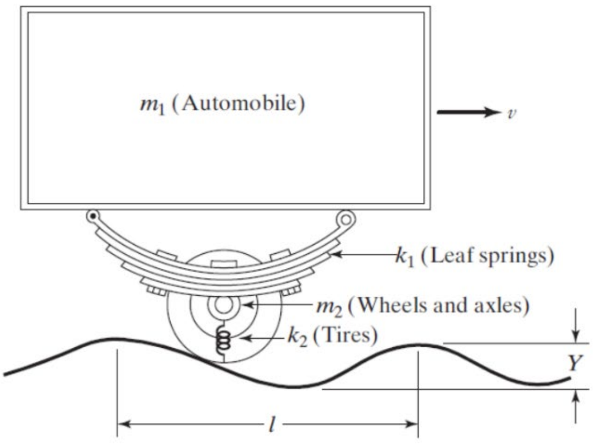


A) ω_n = 4.85 rad/s
B) ω_n = 30.4 rad/s
C) ω_n = 56.4 rad/s
D) ω_n = 82.8 rad/s

Problem 4 (Rao, 2011, w/ permission)

One of the wheels and leaf springs of an automobile, traveling over a rough road, is illustrated below. For simplicity, all the wheels can be assumed to be identical and the system can be idealized as shown in the next figure. The automobile has a mass of m_1 = 1000 kg and the leaf springs have a total stiffness of k_2 = 500 kN/m. The wheels and axles have a mass of m_2 = 300 kg and the tires have a stiffness of k_2 = 500 kN/m. If the road surface varies sinusoidally with an amplitude of Y = 0.1 m and a period of l = 6 m, find the first (lower) critical velocity of the automobile.

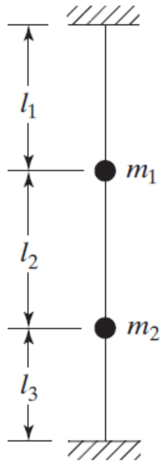




A) $V_{c,1}$ = 49.9 km/h
B) $V_{c,1}$ = 78.8 km/h
C) $V_{c,1}$ = 117 km/h
D) $V_{c,1}$ = 151 km/h

Problem 5 (Rao, 2011, w/ permission)

The following figure shows a system of two masses attached to a tightly stretched spring, fixed at both ends. Suppose that $m_1 = m_2 = m$ and $l_1 = l_2 = l_3 = l$. Let T be the tension in the spring. Which of the following is *false*?



A) One of the natural frequencies is $\omega_n = \sqrt{\frac{T}{ml}}$.

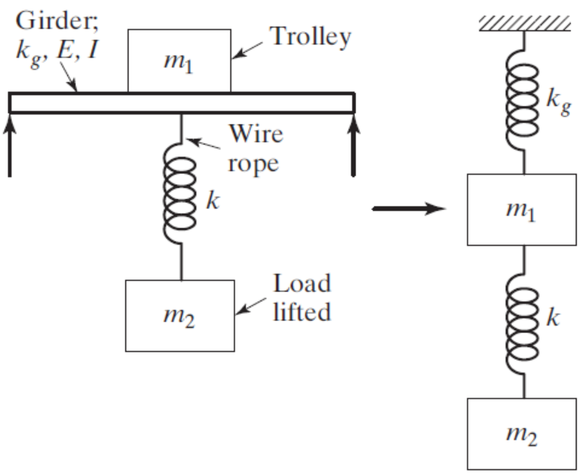
B) One of the natural frequencies is $\omega_n = \sqrt{\frac{3T}{ml}}$.

C) One of the amplitude ratios is $r = +1.0$.

D) One of the amplitude ratios is $r = -0.5$.

Problem 6 (Rao, 2011, w/ permission)

An electric overhead traveling crane, consisting of a girder, trolley, and wire rope, is shown in the figure below. The girder has a flexural rigidity (*EI*) of 6×10^{12} lb-in.², and a span (*L*) of 30 ft. The rope is made of steel and has a length (*l*) of 20 ft. The weighs of the trolley and the load lifted are 8000 lb and 2000 lb, respectively. Suppose we wish to find an area of cross-section of the rope such that the natural frequency is greater than 20 Hz. Which of the following is the lowest area that satisfies this specification?



A) A = 0.59 in.²

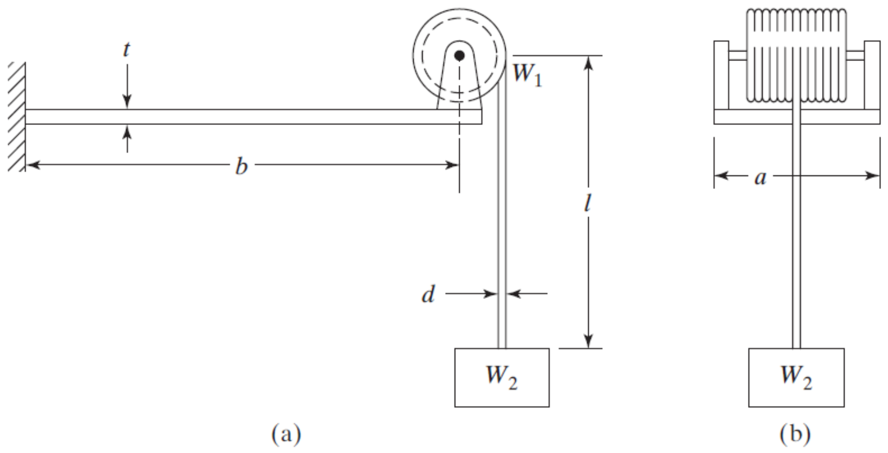
B) A = 1.05 in.²

C) A = 1.58 in.²

D) A = 2.09 in.²

Problem 7A (Rao, 2011, w/ permission)

A hoisting drum, having a weight W_1, is mounted at the end of a steel cantilever beam of thickness t, width a, and length b, as shown in the next figure. The wire rope is made of steel and has a diameter d and a suspended length of l. If the load hanging at the end of the rope is W_2, derive expressions for the natural frequencies of the system.



Problem 7B

Determine the minimum thickness of the cantilever beam supporting the hoisting drum and the wire rope carrying the load in the previous problem in order for the system to have natural frequencies greater than 10 Hz. Let W_1 = 1000 lb, W_2 = 500 lb, b = 30 in., l = 60 in, a = 10t, and d = t. Use $E = 30{\times}10^6$ psi for steel and g = 386.4 in./s².

A) t_{min} = 0.71 in.

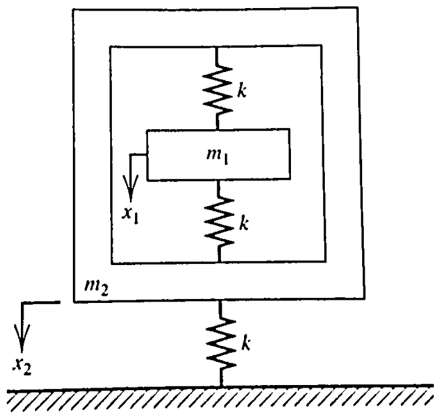
B) t_{min} = 1.10 in.

C) t_{min} = 1.53 in.

D) t_{min} = 1.96 in.

Problem 8 (Palm, 2007, w/ permission)

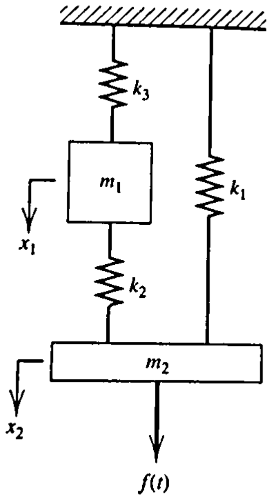
Refer to the following figure. Obtain expressions for the natural frequencies and mode shapes for the case where $m_1 = m_2 = m$. Regarding this system, which of the following is *false*?



A) One of the natural frequencies is $\omega = 0.662\omega_n$.
B) One of the natural frequencies is $\omega = 2.14\omega_n$.
C) One of the mode shapes is $X_2 = 0.187X_1$.
D) One of the mode shapes is $X_2 = -1.29X_1$.

Problem 9 (Palm, 2007, w/ permission)

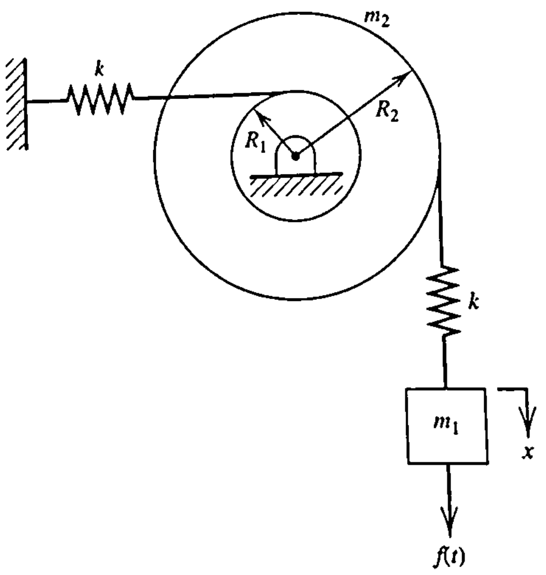
Refer to the following figure. Obtain expressions for the natural frequencies and mode shapes for the case where $k_1 = k$, $k_2 = k_3 = 2k$, and $m_1 = m_2 = m$. Regarding this system, which of the following is *false*?



A) One of the natural frequencies is $\omega = 0.60\omega_n$.
B) One of the natural frequencies is $\omega = 2.36\omega_n$.
C) One of the mode shapes is $X_2 = 1.28X_1$.
D) One of the mode shapes is $X_2 = -0.785X_1$.

Problem 10 (Palm, 2007, w/ permission)

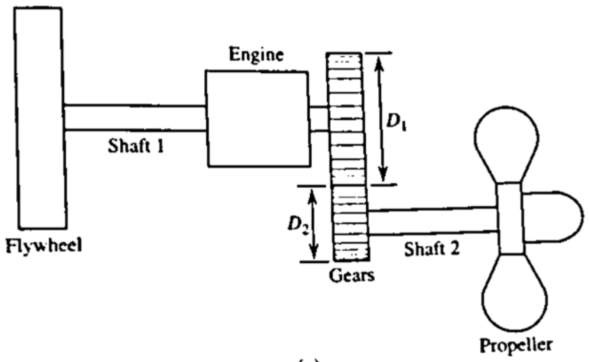
Refer to the following figure. Obtain expressions for the natural frequencies for the case where $R_2 = 2R_1$, $m_1 = m$, and $m_2 = 2m$. Which of the following is a valid expression for the one of the natural frequencies of this system?



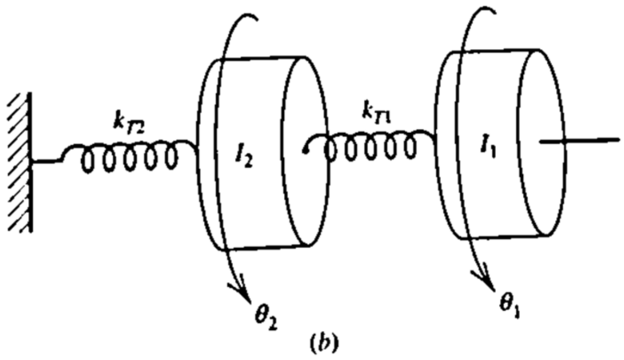
A) $\omega = 0.057\omega_n$
B) $\omega = 0.342\omega_n$
C) $\omega = 0.594\omega_n$
D) $\omega = 0.805\omega_n$

Problem 11 (Palm, 2007, w/ permission)

Refer to the following figure, which shows a ship's propeller, drive train, engine, and flywheel. The diameter ratio of the gears is $D_2/D_1 = 1.5$. The inertias in kg·m² of gear 1 and gear 2 are 500 and 100, respectively. The flywheel, engine, and propeller inertias are 10,000, 1000, and 2500 respectively. The torsional stiffness of shaft 1 is 5×10^6 N·m/rad, and that of shaft 2 is 10^6 N·m/rad. Since the flywheel inertia is so much larger than the other inertias, a simpler model of the shaft vibrations can be obtained by assuming the flywheel does not rotate. In addition, since the shaft between the engine and gears is short, we will assume that it is very stiff compared to the other shafts. If we also neglect the shaft inertias, the resulting model consists of two inertias, one obtained by lumping the engine and gear inertias, and one for the propeller, as shown. Using these assumptions, obtain the natural frequencies and mode shapes of the system.



(a)



(b)

SOLUTIONS

P.1 Solution

The equations of motion for this system are

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$
$$m_2\ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$

Let the response of the system be given by $x_i(t) = X_i \cos(\omega t + \phi)$, with i = 1,2. Substituting in the relations above and manipulating, we get the matrix equation

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The frequency equation follows from the determinant

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

which becomes

$$\omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)\omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

One of the solutions to this equation is

$$\omega_1^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} - \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}} \quad \text{(I)}$$

This relation has been labeled equation (I) because it will be useful in future problems. With $m_1 = m$ = 20 kg, m_2 = 2×20 = 40 kg, $k_1 = k$ = 1000 N/m, and $k_2 = 2k$ = 2×1000 = 2000 N/m, frequency ω_1 is determined to be

$$\omega_1^2 = \frac{1000 + 2000}{2 \times 20} + \frac{2000}{2 \times 40} - \sqrt{\frac{1}{4}\left(\frac{1000 + 2000}{20} + \frac{2000}{40}\right)^2 - \frac{1000 \times 2000}{20 \times 40}}$$

$$\therefore \omega_1^2 = 13.4$$

$$\therefore \omega_1 = 3.66 \text{ rad/s}$$

In a similar manner, the second natural frequency could have been determined as ω_2^2 = 187 and hence ω_2 = 13.7 rad/s. We could've also established the amplitude ratios r_1 and r_2 as

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + k_1 + k_2}{k_2} = \frac{-20 \times 3.66^2 + 1000 + 2000}{2000} = 1.37$$

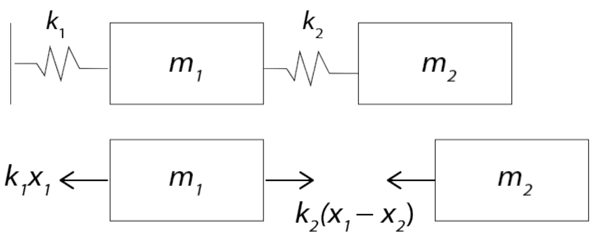
and

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + k_1 + k_2}{k_2} = \frac{-20 \times 13.7^2 + 1000 + 2000}{2000} = -0.377$$

▸ The correct answer is **B**.

P.2 Solution

The equivalent system is illustrated in the next figure.



Stiffnesses k_1 and k_2 are given by

$$k_{1,2} = 2 \times \frac{24EI_{1,2}}{h_{1,2}^3} = \frac{48EI_{1,2}}{h_{1,2}^3}$$

or simply

$$k_1 = k_2 = k = \frac{48EI}{h^3}$$

Referring to the previous figure, the equations of motion for the girders are

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

For harmonic motion, $x_i(t) = X_i \cos(\omega t + \phi)$, with i = 1,2. Substituting in the expressions above and manipulating, we get the matrix equation

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The frequency equation follows from the determinant

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

$$\therefore m_1m_2\omega^4 - (m_2k_1 + m_2k_2 + m_1k_2)\omega^2 + k_1k_2 = 0$$

The values of ω^2 are then

$$\omega^2 = \frac{(m_2k_1 + m_2k_2 + m_1k_2) \mp \sqrt{(m_2k_1 + m_2k_2 + m_1k_2)^2 - 4m_1m_2k_1k_2}}{2m_1m_2}$$

At this point, we can substitute m_1 = 2m, m_2 = m, and k_1 = k_2 = k, so that

$$\omega_{1,2}^2 = \frac{(mk + mk + 2mk) \mp \sqrt{(mk + mk + 2mk)^2 - 8m^2k^2}}{4m^2}$$

$$\therefore \omega_{1,2}^2 = \frac{4mk \mp \sqrt{16m^2k^2 - 8m^2k^2}}{4m^2}$$

$$\therefore \omega_{1,2}^2 = \frac{4mk \mp 2\sqrt{2}mk}{4m^2} = \left(1 \mp \frac{\sqrt{2}}{2}\right)\frac{k}{m}$$

Finally,

$$\omega_1 = \sqrt{\left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{k}{m}\right)} = 0.541\sqrt{\frac{k}{m}} = \boxed{3.75\sqrt{\frac{EI}{mh^3}}}$$

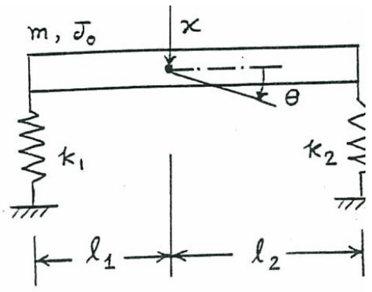
We could just as well compute the other natural frequency,

$$\omega_2 = \sqrt{\left(1 + \frac{1}{\sqrt{2}}\right)\left(\frac{k}{m}\right)} = 1.31\sqrt{\frac{k}{m}} = 9.05\sqrt{\frac{EI}{mh^3}}$$

▸ The correct answer is **B**.

P.3 › Solution

The system can be modeled by a vertical displacement coordinate x and an angular displacement coordinate θ, as shown.



The equations of motion in terms of x and θ are

$$m\ddot{x}+k_1\left(x-l_1\theta\right)+k_2\left(x-l_2\theta\right)=0$$

$$J_0\ddot{\theta}-k_1l_1\left(x-l_1\theta\right)+k_2l_2\left(x+l_2\theta\right)=0$$

For free vibration, the motions in the x- and θ-directions are described by $x(t)=X\cos(\omega t+\phi)$ and $\theta(t)=\Theta\cos(\omega t+\phi)$, respectively. Substituting these expressions in the equations of motion and manipulating, we get the matrix equation

$$\begin{bmatrix}-m\omega^2+k_1+k_2 & -\left(k_1l_1-k_2l_2\right)\\ -\left(k_1l_1-k_2l_2\right) & -J_0\omega^2+k_1l_1^2+k_2l_2^2\end{bmatrix}\begin{bmatrix}X\\ \Theta\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

The frequency equation is obtained by dint of the determinant

$$\begin{vmatrix}-m\omega^2+k_1+k_2 & -\left(k_1l_1-k_2l_2\right)\\ -\left(k_1l_1-k_2l_2\right) & -J_0\omega^2+k_1l_1^2+k_2l_2^2\end{vmatrix}=0$$

$$\therefore\begin{vmatrix}-1000\times\omega^2+3\times10^6+2\times10^6 & -\left(3\times10^6\times0.5-2\times10^6\times0.8\right)\\ -\left(3\times10^6\times0.5-2\times10^6\times0.8\right) & -300\times\omega^2+3\times10^6\times0.5+2\times10^6\times0.8^2\end{vmatrix}=0$$

$$\therefore\begin{vmatrix}-\omega^2+5000 & 100\\ 100 & -0.3\omega^2+2030\end{vmatrix}=0$$

$$\therefore 0.3\omega^4-3530\omega^2+10.1\times10^6=0$$

Solving this equation yields

$$\omega_1^2=4910\rightarrow\omega_1=70.1\text{ rad/s}$$

and

$$\omega_2^2=6856\rightarrow\boxed{\omega_2=82.8\text{ rad/s}}$$

We could just as well have obtained the mode shapes, which follow from the relation

$$\left(-1000\times\omega_1^2+5\times10^6\right)X+0.1\times10^6\Theta=0$$

with the result that

$$\left.\frac{X}{\Theta}\right|_{\omega_1}=\frac{-0.1\times10^6}{-1000\omega_1^2+5\times10^6}=-1.16$$

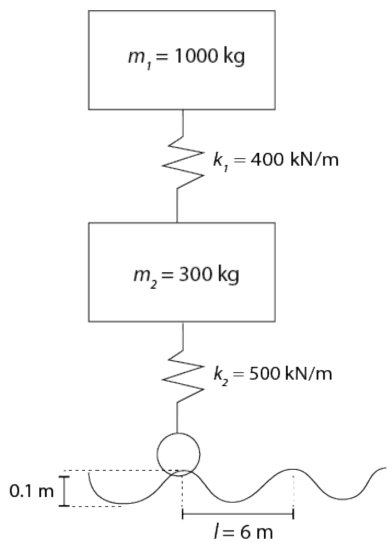
and

$$\left.\frac{X}{\Theta}\right|_{\omega_2}=\frac{-0.1\times10^6}{-1000\omega_2^2+5\times10^6}=0.0539$$

The correct answer is **D**.

P.4 Solution

The situation in question is illustrated below.



The pertaining equations of motion are

$$m_1\ddot{x}_1 + k_1x_1 - k_1x_2 = 0$$
$$m_2\ddot{x}_2 + \left(k_1 + k_2\right)x_2 - k_1x_1 = 0$$

Let the response of the system be given by $x_i(t) = X_i\cos(\omega t + \phi)$, with $i = 1, 2$. Substituting in the equations above and manipulating, we obtain the matrix equation

$$\begin{bmatrix} -m_1\omega^2 + k_1 & -k_1 \\ -k_1 & -m_2\omega^2 + k_1 + k_2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The frequency equation follows from the determinant

$$\begin{vmatrix} -m_1\omega^2 + k_1 & -k_1 \\ -k_1 & -m_2\omega^2 + k_1 + k_2 \end{vmatrix} = 0$$

$$\therefore \left(-m_1\omega^2 + k_1\right)\times\left(-m_2\omega^2 + k_1 + k_2\right) - \left(-k_1\right)\times\left(-k_1\right) = 0$$

$$\therefore m_1m_2\omega^4 - \left(m_1k_1 + m_1k_2\right)\omega^2 - k_1m_2\omega^2 + k_1^2 + k_1k_2 - k_1^2 = 0$$

$$\therefore m_1m_2\omega^4 - \left(m_1k_1 + m_1k_2 + k_1m_2\right)\omega^2 + k_1k_2 = 0$$

The values of ω^2 are then

$$\omega^2 = \frac{\left[\left(m_1k_1 + m_1k_2 + m_2k_1\right) \pm \left(\begin{matrix} m_1^2k_1^2 + m_1^2k_2^2 + m_2^2k_1^2 + 2m_1^2k_1k_2 \\ -2m_1m_2k_1k_2 + 2m_1m_2k_1^2 \end{matrix}\right)^{\frac{1}{2}}\right]}{2m_1m_2}$$

At this point, we can substitute the data we received and obtain ω_1 = 14.5 rad/s and ω_2 = 56.5 rad/s. The corresponding linear frequencies are $f_1 = 14.5/2\pi$ = 2.31 Hz and f_2 = 8.99 Hz. The first critical velocity $V_{c,1}$ in km/h follows as

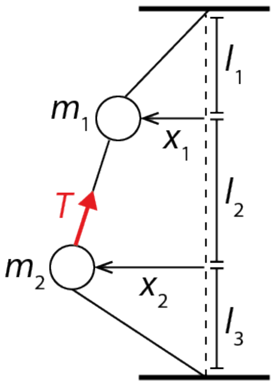
$$\frac{V_{c,1}\times 1000}{3600}\times\left(\frac{1}{l}\right) = 2.31 \rightarrow V_{c,1} = \frac{2.31\times 3600\times 6}{1000} = \boxed{49.9\text{ km/h}}$$

Similarly, the second critical velocity is calculated to be $V_{c,2}$ = 194 km/h.

▶ The correct answer is **A**.

P.5 › Solution

Refer to the figure below.



The horizontal (along the x_1-direction) components of tension in the string lying above and below m_1 are $-x_1T/l_1$ and $-(x_1 + x_2)T/l_2$, respectively. Thus, we can write Newton's second law for mass m_1 as

$$m_1\ddot{x}_1 = -\frac{x_1T}{l_1} - \frac{\left(x_1 - x_2\right)T}{l_2}$$

or

$$m_1\ddot{x}_1 + \frac{x_1T}{l_1} + \frac{\left(x_1 - x_2\right)T}{l_2} = 0$$

Likewise, we have, for mass m_2,

$$m\ddot{x}_2 + \frac{x_2 T}{l_3} - \left(\frac{x_1 - x_2}{l_2}\right)T = 0$$

Positing a harmonic solution of the form $x_i(t) = X_i \cos(\omega t + \phi)$, with i = 1,2, and substitute in the foregoing equations, we obtain the matrix equation

$$\begin{bmatrix} -m\omega^2 + \frac{2T}{l} & -\frac{T}{l} \\ -\frac{T}{l} & -m\omega^2 + \frac{2T}{l} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The frequency equation is found from the determinant

$$\begin{vmatrix} -m\omega^2 + \frac{2T}{l} & -\frac{T}{l} \\ -\frac{T}{l} & -m\omega^2 + \frac{2T}{l} \end{vmatrix} = 0$$

$$\therefore \left(-m\omega^2 + \frac{2T}{l}\right)^2 - \left(\frac{T}{l}\right)^2 = 0$$

$$\therefore \left(-m\omega^2 + \frac{T}{l}\right)\left(-m\omega^2 + \frac{3T}{l}\right) = 0$$

The solutions to the latter equation are

$$m\omega_1^2 = \frac{T}{l} \rightarrow \omega_1 = \sqrt{\frac{T}{ml}}$$

and

$$m\omega_2^2 = \frac{3T}{l} \rightarrow \omega_2 = \sqrt{\frac{3T}{ml}}$$

It remains to calculate the mode shapes. One of them is

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m\omega_1^2 + 2T/l}{T/l} = 1$$

while the other is

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m\omega_2^2 + 2T/l}{T/l} = -1$$

► The false statement is **D**.

P.6 › Solution

The stiffness of the girder, k_g, is given by

$$k_g = \frac{48EI}{l_g^3} = \frac{48 \times \left(6 \times 10^{12}\right)}{\left(30 \times 12\right)^3} = 6.17 \times 10^6 \text{ lb/in.}$$

The stiffness of the rope, in turn, is

$$k_r = \frac{AE}{l_r} = \frac{A \times \left(30 \times 10^6\right)}{\left(20 \times 12\right)} = 1.25 \times 10^5 A \text{ lb/in.}$$

In addition, the mass of the trolley is m_1 = 8000/386.4 = 20.7 lb-sec²/in. and the mass of the lifted load is m_2 = 2000/386.4 = 5.18 lb-sec²/in. The desired minimum frequency value is 20 Hz. Accordingly, let ω_1 = 25 Hz = 157 rad/s, or ω_1^2 = 24,650 (rad/s)². Given the mechanical similarity between the systems, the natural frequency can be determined from equation (I) of Problem 1, repeated below for convenience.

$$\omega_1^2 = \frac{k_g + k_r}{2m_1} + \frac{k_r}{2m_2} - \sqrt{\frac{1}{4}\left(\frac{k_g + k_r}{m_1} + \frac{k_r}{m_2}\right)^2 - \frac{k_g k_r}{m_1 m_2}}$$

Using the known values of k_g, m_1, and m_2, some trial values of A are suggested and used to evaluate the natural frequency ω_1, as shown.

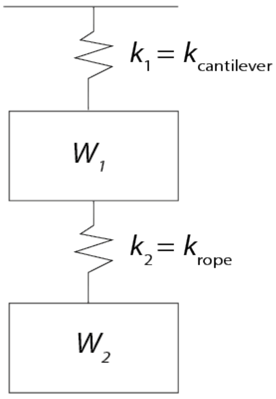
A (in.^{2})	k_r (lb/in.)	ω_1^2 (rad/s)^2	ω_1 (rad/s)
0.6	75000	14296	119.6
0.7	87500	16642	129.0
0.8	100000	18977	137.8
0.9	112500	21300	145.9
1	125000	23612	153.7
1.05	131250	24763	157.4
1.15	143750	27058	164.5
1.25	156250	29340	171.3
1.3	162500	30477	174.6

As highlighted above, A = 1.05 in.^{2} is the first frequency that satisfies the specifications.

▶ The correct answer is **B**.

P.7 › Solution

Part A: The system in question can be represented as follows.



The stiffness of the cantilever beam, $k_1 = k_{\text{cantilever}}$, is given by

$$k_1 = k_{\text{cantilever}} = \frac{3EI}{b^3} = \frac{3E \times \left(\frac{1}{12} \times a \times t^3\right)}{b^3} = \frac{Eat^3}{4b^3}$$

The stiffness of the rope, $k_2 = k_{\text{rope}}$ is in turn

$$k_2 = k_{\text{rope}} = \frac{AE}{l} = \frac{\pi d^2 E}{4l}$$

Note that this system is mechanically similar to the spring-mass system introduced in Problem 1. Consequently, the natural frequencies can be determined with equation (I),

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

Noting that $m_1 = W_1/g$ and $m_2 = W_2/g$ and substituting the spring constants, we obtain

$$\boxed{\omega_{1,2}^2 = \left(\frac{Eat^3}{4b^3} + \frac{\pi d^2 E}{4l}\right)\frac{g}{2W_1} + \frac{\pi d^2 Eg}{8lW_2} \mp \sqrt{\frac{1}{4}\left[\left(\frac{Eat^3}{4b^3} + \frac{\pi d^2 E}{4l}\right)\frac{g}{W_1} + \frac{\pi d^2 Eg}{4lW_2}\right]^2 - \frac{E^2 at^3 \pi d^2 g^2}{16lb^3 W_1 W_2}}}$$

Part B: The squared circular frequency that corresponds to 10 Hz is

$$\omega_1^2 = (2\pi \times 10)^2 = 3948\ (\text{rad/s})^2$$

Substituting each variable into the relation derived in the previous part, we have

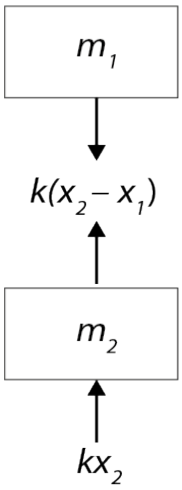
$$\left[\frac{(30\times10^6)\times10t\times t^3}{4\times30^3} + \frac{\pi\times t^2\times(30\times10^6)}{4\times60}\right]\times\frac{386.4}{2\times1000} + \frac{\pi\times t^2\times(30\times10^6)\times386.4}{8\times60\times500}$$
$$-\sqrt{\frac{1}{4}\times\left\{\left[\frac{(30\times10^6)\times10t\times t^3}{4\times30^3} + \frac{\pi\times t^2\times(30\times10^6)}{4\times60}\right]\times\frac{386.4}{1000} + \frac{\pi\times t^2\times(30\times10^6)\times386.4}{4\times60\times500}\right\}^2 - \frac{(30\times10^6)^2\times10t\times t^3\times\pi\times t^2\times386.4^2}{16\times60\times30^3\times1000\times500}} \geq 3948$$

Solving the inequality above with a CAS such as Mathematica gives t_{min} = 1.53 in. That is, in order for the system to have natural frequencies greater than 10 Hz, the thickness of the cantilever beam should be greater than about one and a half inches.

▸ The correct answer is **C**.

P.8 › Solution

The masses can be represented by the following free-body diagram.



Let the free responses be described by $x_1 = X_1 e^{st}$ and $x_2 = X_2 e^{st}$. Then, applying Newton's second law to mass m_1, we can write

$$m_1\ddot{x}_1 = 2k\left(x_2 - x_1\right)$$

$$\therefore m_1\ddot{x}_1 - 2k\left(x_2 - x_1\right) = 0$$

or, substituting the spatial variables,

$$\left(s^2 + 2\omega_n^2\right)X_1 - 2\omega_n^2 X_2 = 0 \text{ (I)}$$

Likewise, for mass m_2,

$$m_2\ddot{x}_2 = -2k\left(x_2 - x_1\right) - kx_2$$

$$\therefore m_2\ddot{x}_2 + 2k\left(x_2 - x_1\right) + kx_2 = 0$$

which becomes

$$\left(s^2 + 3\omega_n^2\right)X_2 - 2\omega_n^2 X_1 = 0 \text{ (II)}$$

Equations (I) and (II) can be cast in matrix form as

$$\begin{bmatrix} s^2 + 2\omega_n^2 & -2\omega_n^2 \\ -2\omega_n^2 & s^2 + 3\omega_n^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The frequency equation is obtained by means of the determinant

$$\begin{vmatrix} s^2 + 2\omega_n^2 & -2\omega_n^2 \\ -2\omega_n^2 & s^2 + 3\omega_n^2 \end{vmatrix} = 0$$

$$\therefore \left(s^2 + 2\omega_n^2\right)\left(s^2 + 3\omega_n^2\right) - \left(-2\omega_n^2\right)\left(-2\omega_n^2\right) = 0$$

$$\therefore s^4 + 5\omega_n^2 s^2 + 2\omega_n^4 = 0$$

The roots of the equation are

$$\omega_1 = 0.662\omega_n$$

and

$$\omega_2 = 2.14\omega_n$$

We proceed to determine the mode shapes. The ratio of the X coefficients is

$$\left(s^2+2\omega_n^2\right)X_1-2\omega_n^2X_2=0$$

$$\therefore 2\omega_n^2X_2=\left(s^2+2\omega_n^2\right)X_1$$

$$\therefore X_2=\left(\frac{s^2+2\omega_n^2}{2\omega_n^2}\right)X_1$$

With $s^2 = -0.438\omega_n^2$, we obtain, for the first mode shape,

$$X_2=0.781X_1$$

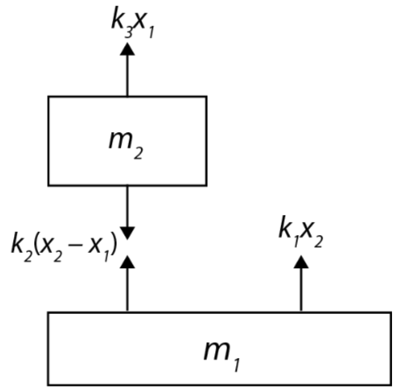
while, with $s^2 = -4.58\omega_n^2$, for the second mode shape,

$$X_2=-1.29X_1$$

▸ The false statement is **C**.

P.9 › Solution

The masses can be represented by the following free-body diagram.



Let the free responses be described by $x_1 = X_1e^{st}$ and $x_2 = X_2e^{st}$. Applying Newton's second law to mass m_1, we can write

$$m_1\ddot{x}_1=-k_3x_1+k_2\left(x_2-x_1\right)$$

$$\therefore m_1\ddot{x}_1+x_1\left(k_2+k_3\right)-k_2x_2=0$$

Since $k_1 = k$, $k_2 = k_3 = 2k$, and $m_1 = m_2 = m$, the equation above becomes

$$m_1\ddot{x}_1+x_1\left(2k+2k\right)-2kx_2=0$$

$$\therefore m\ddot{x}_1+4kx_1-2kx_2=0$$

Substituting the spatial variables yields

$$\left(ms^2+4k\right)X_1-2kX_2=0 \quad \text{(I)}$$

The equation of motion for mass m_2, in turn, is

$$m_2\ddot{x}_2=f\left(t\right)-k_1x_2-k_2\left(x_2-x_1\right)$$

$$\therefore m_2\ddot{x}_2+x_2\left(k_1+k_2\right)-k_2x_1=f\left(t\right)$$

Substituting as before, we have

$$m\ddot{x}_2+3kx_2-2kx_1=f\left(t\right)$$

Replacing the spatial variables gives

$$-2kX_1+\left(ms^2+3k\right)X_2=F \quad \text{(II)}$$

Equations (I) and (II) can be cast in matrix form as

$$\begin{bmatrix} ms^2+4k & -2k \\ -2k & ms^2+3k \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The frequency equation is obtained by means of the determinant

$$\begin{vmatrix} ms^2+4k & -2k \\ -2k & ms^2+3k \end{vmatrix} = 0$$

$$\therefore \left(ms^2+4k\right)\left(ms^2+3k\right)-\left(-2k\right)\left(-2k\right)=0$$

$$\therefore s^4+7\omega_n^2 s^2+8\omega_n^4=0$$

The roots of the equation are

$$\omega_1 = 1.2\omega_n$$

and

$$\omega_2 = 2.36\omega_n$$

We proceed to determine the mode shapes. The ratio of the X coefficients is

$$\left(ms^2+4k\right)X_1-2kX_2=0$$

$$\therefore X_2=\left(\frac{ms^2+4k}{2k}\right)X_1$$

With $s^2 = -1.44\omega_n^2$, we obtain, for the first mode shape,

$$X_2 = 1.28X_1$$

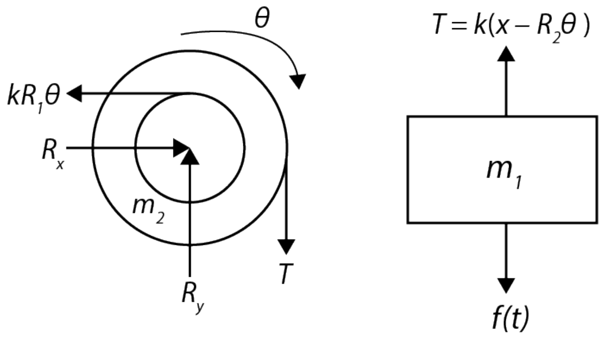
while, with $s^2 = -5.57\omega_n^2$, for the second mode shape,

$$X_2 = -0.785X_1$$

The false statement is **A**.

P.10 › Solution

The free body diagram for the two masses is shown in continuation.



The equation of motion for the pulley is

$$\frac{m_2R_2^2\ddot{\theta}}{2}=-kR_1^2\theta+R_2T$$

In this equation, $T = k(x - R_2\theta)$ is the tension in the vertical spring. Substituting and manipulating, we get

$$\frac{m_2R_2^2\ddot{\theta}}{2}=-kR_1^2\theta+R_2\left[k\left(x-R_2\theta\right)\right]$$

$$\therefore \frac{m_2R_2^2\ddot{\theta}}{2}=-kR_1^2\theta+kR_2x-kR_2^2\theta$$

$$\therefore \frac{m_2R_2^2\ddot{\theta}}{2}=-k\left(R_1^2+R_2^2\right)\theta+kR_2x$$

$$\therefore \frac{m_2R_2^2\ddot{\theta}}{2}+k\left(R_1^2+R_2^2\right)\theta-kR_2x=0$$

At this point, we can substitute $R_2 = 2R_1$, $m_1 = m$, and $m_2 = 2m$, so that

$$\frac{2m\times\left(2R_1\right)^2\times\ddot{\theta}}{2}+k\left[R_1^2+\left(2R_1\right)^2\right]\theta-k\times\left(2R_1\right)\times x=0$$

$$\therefore 4mR_1^2\ddot{\theta}+5kR_1^2\theta-2kR_1x=0$$

Substituting $\theta = \Theta e^{st}$ and $x = Xe^{st}$, the relation above becomes

$$\left(4mR_1^2s^2 + 5kR_1^2\right)\Theta - 2kR_1X = 0 \text{ (I)}$$

Next, we write the equation of motion for the block of mass m_1,

$$m_1\ddot{x} = f(t) - T$$

$$\therefore m_1\ddot{x} = f(t) - k\left(x - R_2\theta\right)$$

$$\therefore m_1\ddot{x} = f(t) - kx + kR_2\theta$$

$$\therefore m_1\ddot{x} + kx - kR_2\theta = f(t)$$

Substituting as before gives

$$m\ddot{x} + kx - k \times 2R_1 \times \theta = f(t)$$

$$\therefore m\ddot{x} + kx - 2kR_1\theta = f(t)$$

and, replacing x and θ,

$$\left(ms^2 + k\right)X - 2kR_1\Theta = F \text{ (II)}$$

Gathering equations (I) and (II) brings to the matrix equation

$$\begin{bmatrix} 4mR_1^2s^2 + 5kR_1^2 & -2kR_1 \\ -2kR_1 & ms^2 + k \end{bmatrix}\begin{bmatrix} \Theta \\ X \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The frequency equation follows from the determinant

$$\begin{vmatrix} 4mR_1^2s^2 + 5kR_1^2 & -2kR_1 \\ -2kR_1 & ms^2 + k \end{vmatrix} = 0$$

$$\therefore \left(4mR_1^2s^2 + 5kR_1^2\right)\left(ms^2 + k\right) - \left(-2kR_1\right)\left(-2kR_1\right) = 0$$

$$\therefore 4s^4 + 9\omega_n^2s^2 + \omega_n^4 = 0$$

$$\therefore s^4 + 2.25\omega_n^2s^2 + 0.25\omega_n^4 = 0$$

The roots of this equation are

$$\boxed{\omega_1 = 0.342\omega_n}$$

and

$$\omega_2 = 1.46\omega_n$$

The correct answer is **B**.

P.11 Solution

The equation of motion for the propeller is

$$I_1\ddot{\theta}_1 - k_{T,1}\left(\theta_2 - \theta_1\right) = 0$$

The equation of motion for the engine and gear, in turn, is

$$I_2\ddot{\theta}_2 + k_{T,2}\left(\theta_2 - \theta_1\right) + k_{T,2}\theta_2 = 0$$

Let the response of inertia 1 (the propeller) be given by $\theta_1(t) = \Theta_1 e^{st}$, and that of the engine and gear be given by $\theta_2(t) = \Theta_2 e^{st}$. Substituting in the first equation above gives

$$\left(I_1s^2 + k_{T,1}\right)\Theta_1 - k_{T,1}\Theta_2 = 0 \text{ (I)}$$

Substituting in the second equation of motion, we obtain

$$-k_{T,1}\Theta_1 - \left(I_2s^2 + k_{T,1} + k_{T,2}\right)\Theta_2 = 0 \text{ (II)}$$

Gathering the two preceding relations, we get the matrix equation

$$\begin{bmatrix} I_1 s^2 + k_{T,1} & -k_{T,1} \\ -k_{T,1} & I_2 s^2 + k_{T,1} + k_{T,2} \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The frequency equation is obtained by means of the determinant

$$\begin{vmatrix} I_1 s^2 + k_{T,1} & -k_{T,1} \\ -k_{T,1} & I_2 s^2 + k_{T,1} + k_{T,2} \end{vmatrix} = 0$$

$$\therefore \left(I_1 s^2 + k_{T,1}\right)\left(I_2 s^2 + k_{T,1} + k_{T,2}\right) - \left(-k_{T,1}\right)\left(-k_{T,1}\right) = 0$$

$$\therefore s^4 + \left(\frac{k_{T,1}}{I_1} + \frac{k_{T,1}}{I_2} + \frac{k_{T,2}}{I_2}\right) s^2 + \frac{k_{T,1} k_{T,2}}{I_1 I_2} = 0$$

Before proceeding, we require the inertia of the engine/gear ensemble, I_2. In pursuance of this quantity, we consider the kinetic energy of this part of the system,

$$T = \frac{1}{2} \times \left(I_{\text{gear},1}\right)\dot{\theta}_2^2 + \frac{1}{2} \times \left(I_{\text{gear},2}\right)\dot{\theta}_1^2 + \frac{1}{2} \times \left(I_{\text{engine}}\right)\dot{\theta}_2^2$$

Now, since

$$\dot{\theta}_1 = \left(D_1/D_2\right)\dot{\theta}_2 = 1.5\dot{\theta}_2$$

it follows that

$$T = \frac{1}{2} \times 500 \times \dot{\theta}_2^2 + \frac{1}{2} \times 100 \times \left(1.5\dot{\theta}_2^2\right)^2 + \frac{1}{2} \times 1000 \times \dot{\theta}_2^2$$

$$\therefore T = \frac{1}{2}\left(500\theta_2^2 + 100 \times 2.25\dot{\theta}_2^2 + 1000\right)$$

$$\therefore T = \frac{1}{2} \times 1725 \times \dot{\theta}_2^2$$

$$\therefore I_2 = 1725 \text{ kg} \cdot \text{m}^2$$

Substituting this and other data into the frequency equation, we obtain

$$s^4 + \left(\frac{5 \times 10^6}{2500} + \frac{5 \times 10^6}{1725} + \frac{10^6}{1725}\right) s^2 + \frac{\left(5 \times 10^6\right) \times 10^6}{2500 \times 1725} = 0$$

$$\therefore s^4 + 5478 s^2 + 1.16 \times 10^6 = 0$$

Solving this equation, we find that

$$\omega_1 = 14.9 \text{ rad/s}$$

and

$$\omega_2 = 72.5 \text{ rad/s}$$

We proceed to establish the mode shapes,

$$\left(I_1 \times s^2 + k_{T,1}\right)\Theta_1 - k_{T,2}\Theta_2 = 0$$

$$\therefore \Theta_2 = \left(\frac{I_1 s^2 + k_{T,1}}{k_{T,2}}\right)\Theta_1$$

Noting that $s^2 = -\omega_1^2 = -222$, we have, substituting this and other quantities,

$$\Theta_2 = \left[\frac{2500 \times (-222) + 5 \times 10^6}{10^6}\right]\Theta_1 \rightarrow \Theta_2 = 4.45\Theta_1$$

In a similar manner, if $s^2 = -\omega_2^2 = -5256$, it follows that

$$\Theta_2 = \left[\frac{2500 \times (-5256) + 5 \times 10^6}{10^6}\right]\Theta_1 \rightarrow \Theta_2 = -8.14\Theta_1$$

ANSWER SUMMARY

Problem 1		**B**
Problem 2		**B**
Problem 3		**D**
Problem 4		**A**
Problem 5		**D**
Problem 6		**B**
Problem 7	**7A**	*Open-ended pb.*
	7B	**C**
Problem 8		**C**
Problem 9		**A**
Problem 10		**B**
Problem 11		*Open-ended pb.*

REFERENCES

- PALM, W. (2007). *Mechanical Vibration*. Hoboken: John Wiley and Sons.
- RAO, S. (2011). *Mechanical Vibrations*. 5th edition. Upper Saddle River: Pearson.

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