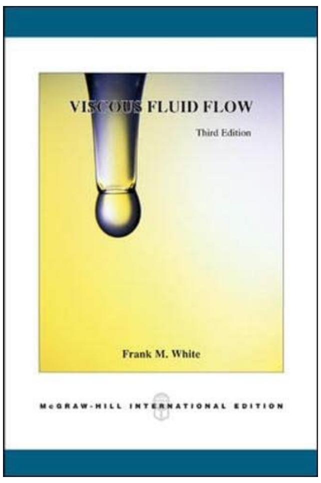
Montogue

Quiz FM113



Reviewed Solutions to White's *Viscous Fluid Flow*, 3rd Edition

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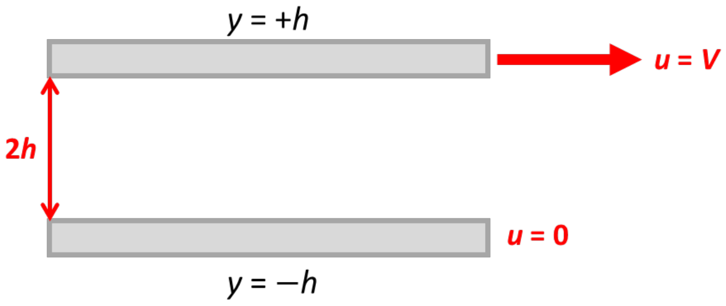
Chapter	Problems Covered
3	3.1, 3.2, 3.3, 3.4, 3.5, 3.8, 3.10, 3.12, 3.15, 3.16, 3.17, 3.23, 3.25, 3.27, 3.28, 3.31, 3.32, 3.33, 3.34
4	4.1, 4.3, 4.4, 4.10, 4.12, 4.13, 4.17, 4.18, 4.23, 4.24, 4.25
5	5.1, 5.2, 5.7, 5.9, 5.11, 5.15
6	6.4, 6.6, 6.7, 6.9, 6.13, 6.14, 6.15, 6.17, 6.18, 6.20, 6.21, 6.22, 6.30, 6.31, 6.32

PROBLEMS – CHAPTER 3

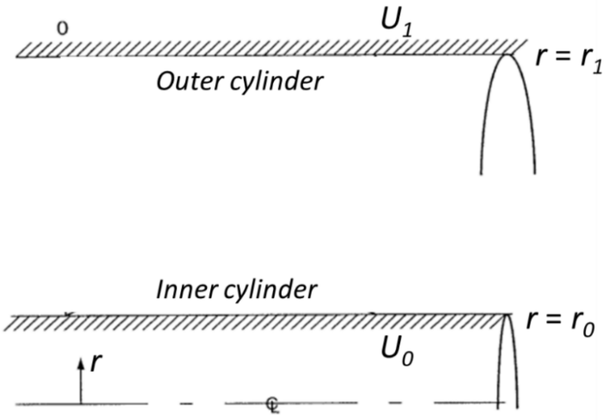
Problem 3.1

Solve for constant-pressure Couette flow between parallel plates, as illustrated below, for a non-Newtonian fluid such that $\tau = K(du/dy)^n$, where $n \neq 1$. Compare with the Newtonian solution. Assuming constant pressure and temperature, solve for the velocity distribution $u(y)$ between the plates if (*a*) $n < 1$ and (*b*) $n > 1$, and compare with the Newtonian solution, Eq. (3-6). Comment on the results.



Problem 3.2

Consider the axial Couette flow of Fig. 3-3 with both cylinders moving. Find the velocity distribution $u(r)$ and plot it for (*a*) $U_1 = U_0$, (*b*) $U_1 = -U_0$, and (*c*) $U_1 = 2U_0$. Comment on the results.



Problem 3.3

Consider the axial Couette flow of Fig. 3-3 with the inner cylinder moving at speed U_0 and the outer cylinder fixed. Solve for the temperature distribution $T(r)$ in the fluid if the inner and outer cylinder walls are at temperatures T_0 and T_1, respectively.

▶ Problem 3.4

A long thin rod of radius R is pulled axially at speed U through an infinite expanse of still fluid. Solve the Navier-Stokes equation for the velocity distribution $u(r)$ in the fluid and comment on a possible paradox.

▶ Problem 3.5

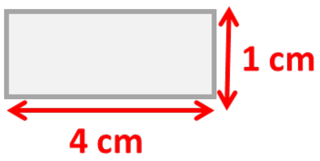
A circular cylinder of radius R is rotating at steady angular rate ω in an infinite fluid of constant ρ and μ. Assuming purely circular streamlines, find the velocity and pressure distribution in the fluid and compare with the flow field of an inviscid "potential" vortex.

▶ Problem 3.8

Air at 20°C and 1 atm is driven between two parallel plates 1 cm apart by an imposed pressure gradient (dp/dx) and by the upper plate moving at 20 cm/s. Find (*a*) the volume flow rate (in cm^3/s per meter of width) if $dp/dx = -0.3$ Pa/m and (*b*) the value of (dp/dx) (in Pa/m) which causes the shear stress at the lower plate to be zero.

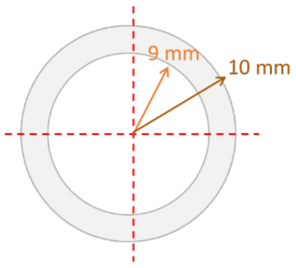
▶ Problem 3.10

Air at 20°C and approximately 1 atm flows at an average velocity of 1.7 m/s through a rectangular 1 × 4 cm duct. Estimate the pressure drop (in Pa/m) by (*a*) an exact calculation and (*b*) the hydraulic diameter approximation.



▶ Problem 3.12

If light oil viscosity (0.02 to 0.1 Pa·s) is measured by passing 1 m^3/h of fluid through an annulus 30 cm long, with inner and outer radii of 9 mm and 10 mm, find Δp.

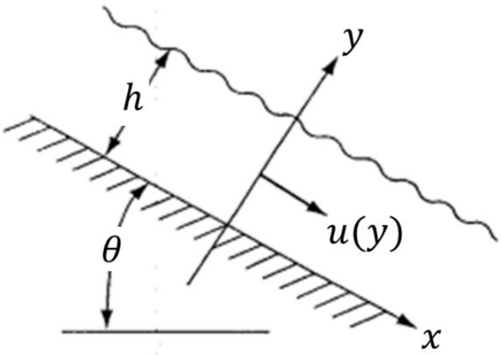


▶ Problem 3.15

Consider a wide liquid film of constant thickness h flowing steadily due to gravity down an inclined plane at angle θ, as shown in Fig. P3-15. The atmosphere exerts constant pressure and negligible shear on the free surface. Show that the velocity distribution is given by

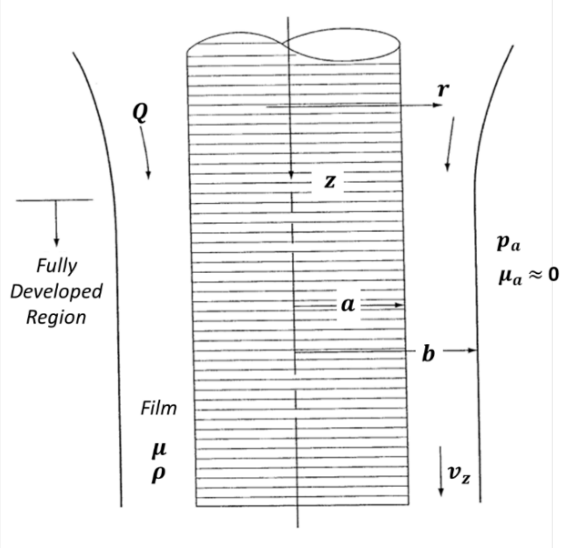
$$u = \frac{\rho g \sin\theta}{2\mu} y(2h - y)$$

and that the volume flow rate per unit width is $Q = \rho g h^3 \sin(\theta)/3\mu$. Compare this result with flow between parallel plates, Eqs. (3-44) and (3.45).



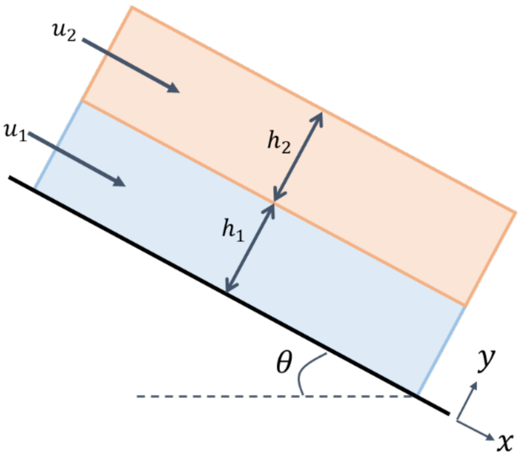
▶ Problem 3.16

Consider a film of liquid draining at volume flow rate Q down the outside of a vertical rod of radius a, as shown in Fig. P3-16. Some distance down the rod, a fully developed open region is reached where fluid shear balances gravity and the film thickness remains constant. Find formulas for the velocity distribution and the flow rate.



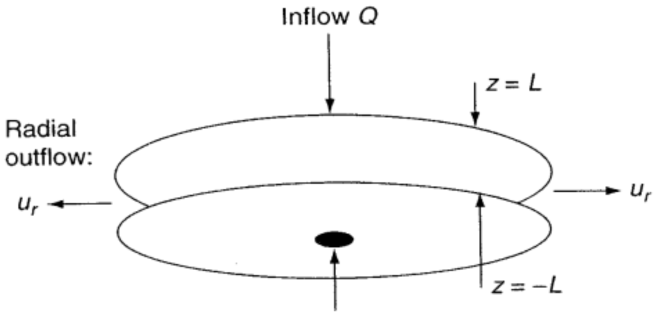
Problem 3.17

By extension of Prob. 3-15, consider a *double* layer of immiscible fluids 1 and 2, flowing steadily down an inclined plane, as in Fig. P3-17. The atmosphere exerts no shear stress on the surface and is at constant pressure. Find the laminar velocity distribution in the two layers.



Problem 3.23

Consider radial outflow between two parallel disks fed by symmetric entrance holes, as shown below. Assume that velocity components $v_z = v_\theta = 0$ and $v_r = f(r,z)$ with constant density ρ and viscosity μ and pressure as a function of radial distance only, i.e., $p = p(r)$. Neglect gravity and entrance effects at $r = 0$. Set up the appropriate differential equation and boundary conditions and solve as far as possible – numerical (e.g., Runge-Kutta) integration may be needed for a complete solution. Sketch the expected velocity profile shape.



Problem 3.25

Consider the problem of steady flow induced by a circular cylinder of radius r_0 rotating at surface vorticity ω_0 and having a wall-suction velocity $v_r(r = r_0) = -v_w = const$. Set up the problem in polar coordinates assuming no circumferential variations $\partial/\partial\theta = 0$, and show that the vorticity in the fluid is given by

$$\omega = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = \omega_0 \left(\frac{r_0}{r}\right)^{\mathrm{Re}}$$

where $Re = r_0 v_w / \nu$ is the wall-suction Reynolds number of the cylinder. Integrate this relation to find the velocity distribution $v_\theta(r)$ in the fluid and show that the character of the solution is quite different for the three cases of the wall Reynolds number Re less than, equal to, or greater than 2.0.

Problem 3.27

The practical difficulty with the Ekman spiral solution, Eq. (3-144), is that it assumes laminar flow whereas the real ocean is turbulent. One approximate remedy is to replace kinematic viscosity ν everywhere by a (constant) turbulent or "eddy" viscosity correlated with wind shear and penetration depth using a suggestion by Clauser (1956):

$$\nu_{\text{turb}} \approx 0.04 D \left(\frac{\tau_0}{\rho}\right)^{1/2}$$

Repeat our text example, $V_{\text{wind}} = 6$ m/s over a 20°C air-water interface of 41°N latitude. Compute penetration depth D and surface velocity V_0.

Problem 3.28

Repeat the analysis of the Ekman flow, Sec. 3-7.2, for shallow water, that is, apply the bottom boundary condition, Eq. (3-143), at $z = -h$. Find the velocity components and show that the surface velocity is no longer at a 45° angle to the wind but rather satisfies the equation as follows for the surface angle θ:

$$\tan\theta = \frac{\sinh(2\pi h/D) - \sin(2\pi h/D)}{\sinh(2\pi h/D) + \sin(2\pi h/D)} \quad [\text{Ekman}(1905)]$$

Find the value of h/D for which θ = 20°.

Problem 3.31

The rotating disk is sometimes called von Kármán's centrifugal pump, since it brings in fluid axially and throws it out radially. Consider one side of a 50 cm disk rotating at 1200 rpm in air 20°C and 1 atm. Assuming laminar flow, compute (*a*) the flow rate, (*b*) the torque and power required, and (*c*) the maximum radial velocity at the disk edge.

Problem 3.32

Solve the Jeffery-Hamel wedge-flow relation, Eq. (3-195), for creeping flow, $Re = 0$ but $\alpha \neq 0$. Show that the proper solution is

$$f(\eta) = 1 + \frac{1}{2}\csc^2\alpha\left[\sin\left(\frac{\pi}{2} - 2\alpha\eta\right) - 1\right]$$

Show also that the constant $C = 4\alpha^2 \cot^2\alpha$ and sketch a few profiles. Show that backflow always occurs for $\alpha > 90°$.

Problem 3.33

In spherical polar coordinates, when the variations $\partial/\partial\phi$ vanish, an incompressible stream function $\psi(r,\theta)$ can be defined such that

$$u_r \equiv \frac{\partial\psi/\partial\theta}{r^2\sin\theta} \;;\; u_\theta \equiv -\frac{\partial\psi/\partial r}{r\sin\theta}$$

The particular stream function

$$\psi(r,\theta) = \frac{2\nu r\sin^2\theta}{1 + a - \cos\theta} \;;\; a = \text{const.}$$

is an exact solution of the Navier-Stokes equations and represents a round jet issuing from the origin. Sketch the streamlines in the upper two quadrants for a particular value of a between 0.001 and 0.1. (Various values could be distributed among a group.) Sketch the jet profile shape $u_r(1,\theta)$ and determine how the jet width δ (where $u_r = 0.01u_{max}$) and jet mass flow vary with r.

Problem 3.34

A sphere of specific gravity 7.8 is dropped into oil of specific gravity 0.88 and viscosity μ = 0.15 Pa·s. Estimate the terminal velocity of the sphere if its diameter is (*a*) 0.1 mm, (*b*) 1 mm, and (*c*) 10 mm. Which of these is a creeping motion?

PROBLEMS – CHAPTER 4

Problem 4.1

Repeat the flat-plate integral analysis of Sect. 4-1 for the cubic velocity profile

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta_u}\right) - \frac{1}{2}\left(\frac{y}{\delta_u}\right)^3$$

where δ_u is the velocity boundary-layer thickness. Is this profile any more (or less) realistic than the approximation of Eq. (4-11)? For the above profile, compute (*a*) $(\theta/x)\sqrt{Re_x}$; (*b*) $(\delta^*/x)\sqrt{Re_x}$; (*c*) $(\delta/x)\sqrt{Re_x}$ (*d*) $C_f\sqrt{Re_x}$ (*e*) $C_D\sqrt{Re_x}$.

▶ Problem 4.3

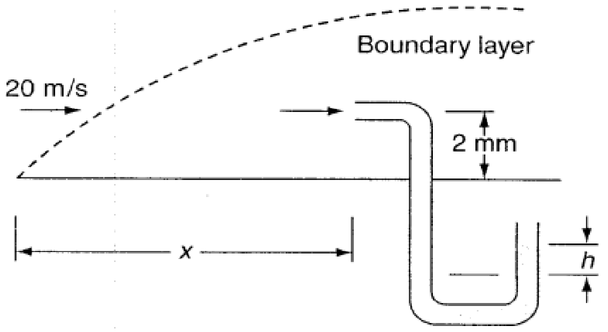
Schlichting (1979, p. 206) points out that the simple flat-plate velocity profile approximation

$$u \approx U \sin\left(\frac{\pi y}{2\delta}\right)$$

gives much better accurate values of c_f, θ, and δ^* (± 2 percent) than the parabolic profile of Eq. (4-11). Verify this by computing c_f. Does this sine-wave shape satisfy any additional boundary conditions compared to Eq. (4-11)?

▶ Problem 4.4

Air at 20°C and 1 atm flows past a smooth flat plate as in Fig. P4-4. A pitot stagnation tube, placed 2 mm from the wall, develops a water manometer head h = 21 mm. Use this information with the Blausius solution, Table 4-1, to estimate the position x of the pitot tube. Check to see if the flow is laminar.

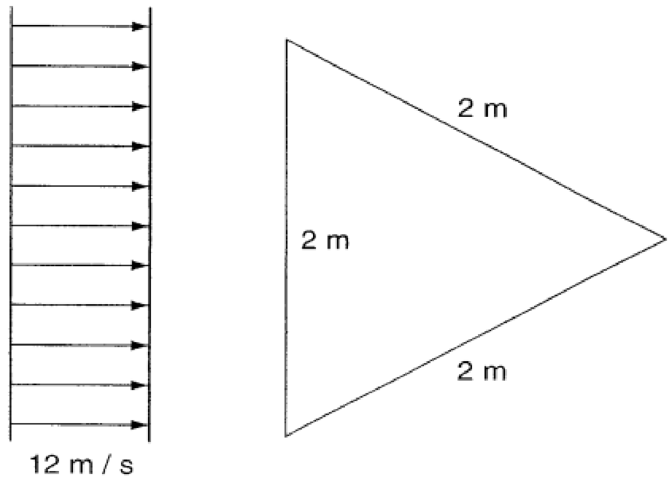


▶ Problem 4.10

The quantity $(\delta^*/\tau_w)(dp/dx)$ is called Clauser's parameter. It compares an external pressure gradient to wall friction and is very useful for turbulent boundary layers. Show that this parameter is a constant for a given laminar Falkner-Skan wedge-flow boundary layer. What value does this parameter have at the separation condition?

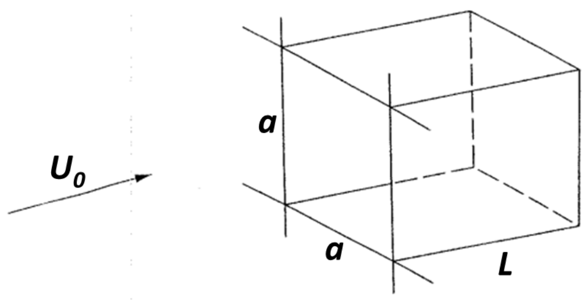
▶ Problem 4.12

A thin equilateral triangle plate is immersed parallel to a 12 m/s stream of air at 20°C and 1 atm, as in Fig. P4-12. Assuming laminar flow, estimate the drag of this plate (in N).



▶ Problem 4.13

Flow straighteners consist of arrays of narrow ducts place in a flow to remove swirl and other transverse (secondary) velocities. One element can be idealized as a square box with thin sides as in Fig. P4-13. Using laminar flat-plate theory, derive a formula for the pressure drop Δp across an $N \times N$ bundle of such boxes.



▶ Problem 4.17

Air at 20°C and 1 atm issues from a narrow slot and forms a two-dimensional laminar jet. At 50 cm downstream of the slot the maximum velocity is 20 cm/s. Estimate, at this position, (*a*) the jet width, (*b*) the jet mass flow per unit depth, and (*c*) an appropriate Reynolds number for the jet.

▶ Problem 4.18

Air at 20°C and 1 atm flows at 1 m/s past a slender two-dimensional body of length L = 30 cm and C_D = 0.05 based on 'plan' area (bL). At 3 m downstream of the trailing edge, estimate (*a*) the maximum wake velocity defect (in cm/s), (*b*) the "one percent" wake thickness (in cm), and (*c*) the wake-thickness Reynolds number.

▶ Problem 4.23

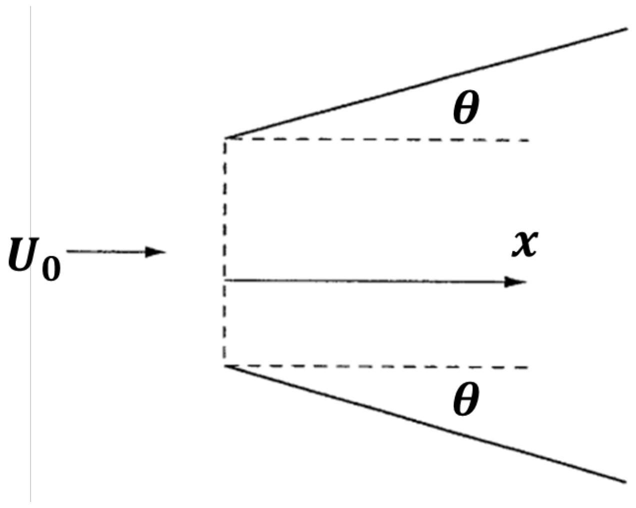
Apply the method of Thwaites, Sect. 4-6.6, to boundary-layer flow on a cylinder, using either the inviscid Eq. (4-143) or measured Eq. (4-144) freestream velocity distributions. Compare the computed local wall friction with Fig. 4-24*b*.

▶ Problem 4.24

Apply the Thwaites' integral method to one of the laminar-flow test cases in Table 4-5 (for best results have each member of the class take a different case). Compute and plot the local friction distribution $c_f\sqrt{\mathrm{Re}_x}$ and compare the predicted separation point with Table 4-5.

▶ Problem 4.25

Consider a two-dimensional thin-walled diffuser, as in Fig. P4-25. Assume incompressible flow with a one-dimensional freestream velocity $U(x)$ and entrance velocity $U_0(x)$. The entrance height is W and the constant depth into the paper is b. Using Thwaites' method, find an expression for the angle θ at which separation will occur at $x = L$. What is the value of θ if $L = 1.5W$?



PROBLEMS – CHAPTER 5

▶ Problem 5.1

While holding (g, ρ_1, ρ_2, $\mathfrak{T}$) constant, show that the right-hand side of Eq. (5-9) has a minimum at the wave number $\alpha = [g(\rho_1 - \rho_2)/\mathfrak{T}]^{1/2}$. Find experimental data somewhere and estimate this "critical" wavelength and velocity difference for air blowing over gasoline.

▶ Problem 5.2

Show that, if the upper and lower velocities in Fig. 5-2 are negligible and if surface tension is neglected, a disturbance of the interface will propagate at the phase speed

$$c = \sqrt{\frac{g\lambda(\rho_1 - \rho_2)}{2\pi(\rho_1 + \rho_2)}}$$

where λ is the wavelength of the disturbance. Discuss what might happen if $\rho_1 < \rho_2$. Estimate this propagation speed for an air-water interface when the wavelength is 3 m.

▶ Problem 5.7

For stagnation boundary-layer flow, $U = Kx$, estimate the position Re_x where instability first occurs.

▶ Problem 5.9

For the Howarth freestream velocity $U = U_0(1 - x/L)$, if $U_0L/\nu = 10^6$, estimate the point (x/L) where boundary-layer instability first occurs. Assume a low subsonic Mach number.

▶ Problem 5.11

For potential freestream flow across a cylinder, $U = 2U_0 sin(x/a)$, if $Re_D = 10^6$, estimate the position $(x/a)_{crit}$ where boundary-layer instability first occurs.

▶ Problem 5.15

For the separating Falkner-Skan wedge-flow boundary layer, $\beta = -0.19884$, estimate the position Re_x where instability first occurs.

PROBLEMS – CHAPTER 6

▶ Problem 6.4

The experiment of Clauser (1954), flow 2200 of Coles and Hirst (1968), used air at 24°C and 1 atm. At the first station, x = 6.92 ft, the turbulent-boundary-layer velocity data are as follows:

y (in.)	u (ft/s)	y (in.)	u (ft/s)
0.1	16.14	0.8	22.88
0.15	17.02	0.9	23.70
0.2	17.54	1.0	24.38
0.25	18.16	1.25	26.51
0.3	18.69	1.5	28.21
0.4	19.60	2.0	31.22
0.5	20.49	2.5	32.27
0.6	21.24	3.0	32.44
0.7	22.03	3.5	32.50

The boundary layer thickness was 3.5 in., and the local freestream velocity gradient was $dU/dx \approx -1.06\ s^{-1}$. Analyze these data, with suitable plots and formulas, to establish (*a*) the inner law and wall shear stress, (*b*) the outer law with Clauser's parameter β and the Coles parameter Π, and (*c*) the logarithmic overlap.

▶ Problem 6.6

For developed turbulent smooth wall pipe flow, assuming that the log-law analysis of Sec. 6-5.1 is valid with $\kappa = 0.41$, show that the maximum pipe velocity may be computed from

$$\frac{u_{max}}{u_{avg}} \approx 1 + 1.29\Lambda^{1/2}$$

where Λ is the friction factor.

▶ Problem 6.7

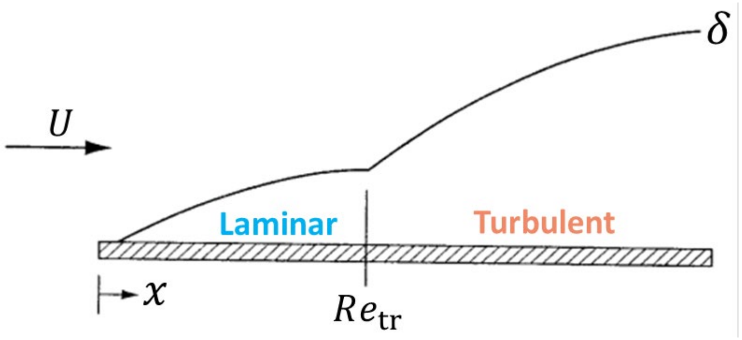
Water at 20°C flows through a smooth pipe of diameter 3 cm at 30 m^3/h. Assuming developed flow, estimate (*a*) the wall shear stress (in Pa), (*b*) the pressure drop (in Pa/m), and (*c*) the centerline velocity in the pipe. What is the maximum flow rate for which this flow would be laminar? What flow rate would give $\tau_w = 100$ Pa?

▶ Problem 6.9

Consider fully developed turbulent flow through a duct of square cross-section. Taking advantage of the double symmetry, analyze this problem using the log-law, Eq. (6-38*a*), plus a suitable assumption about variation of shear stress around the cross-section. Compare your result for Λ with the hydraulic radius concept.

▶ Problem 6.13

The flat-plate formulas of Sec. 6-6 assume turbulent flow beginning at the leading edge ($x = 0$). More likely, there is an initial region of laminar flow, as in Fig. P6-13.



Devise a scheme to compare $\delta(x)$ and $c_f(x)$ in the turbulent region, $Re > Re_{tr}$, by accounting for the laminar part of the flow.

▶ Problem 6.14

Water at 20°C and 1 atm flows at 6 m/s past a smooth flat plate 1 m long and 60 cm wide. Estimate (*a*) the trailing-edge displacement thickness, (*b*) the trailing-edge wall shear stress, and (*c*) the drag of one side of the plate, if $Re_{x,\mathrm{tr}} = 10^6$.

▶ Problem 6.15

Repeat Prob. 6-14 if the plate average roughness is 0.1 mm. Estimate ΔB at $x = L$.

▶ Problem 6.17

Rewrite Stevenson's relation Eq. (6-86) in the form of a wall-friction law with suction or blowing. Show that the ratio of C_f to the impermeable-wall value C_{f0} is approximately a function only of a "blowing parameter" $\beta = (2v_w)/(U_e C_f)$. Plot C_f/C_{f0} vs. β in the range $-0.5 < \beta < 2.0$ and compare with the correlation $[ln(1+\beta)]/\beta$ recommended by Kays and Crawford (1980, p. 181).

▶ Problem 6.18

Water at 20°C flows through a smooth permeable pipe of diameter 8 cm. The volume flow rate is 0.06 m^3/s. Estimate the wall shear stress, in pascals, if the wall velocity is (*a*) 0.01 m/s blowing; (*b*) 0 m/s; and (*c*) 0.01 m/s suction. To avoid excessive iteration, assume that the ratio of average to centerline velocity is 0.85.

▶ Problem 6.20

As an alternative to Eq. (6-62), Bergstrom *et al.* (2002) suggest the following formula for the downshift of the log-law Eq. (6-60) due to uniform surface roughness of height k:

$$\Delta B \approx \frac{1}{\kappa}\ln\left(k^+\right) - 3.5 \;\; ; \;\; \text{for } k^+ \geq 4.2$$

First compare this correlation with a sketch or graph, to Eq. (6-62). Then apply this correlation to derive a formula for pipe-friction factor Λ, similar to Eq. (6-64).

▶ Problem 6.21

Use numerical quadrature to evaluate and sketch Eq. (6-96) for zero pressure gradient. Compare your results with Eq. (6-41) and Fig. 6-11.

▶ Problem 6.22

Use the log-law, Eq. (6-38*a*), to analyze Couette flow between parallel plates a distance $2h$ apart, with the upper plate moving at velocity U. Show that the turbulent-flow velocity profile is S-shaped, as in Fig. 3-5. Sketch the profile for $Uh/\nu = 10^5$ and compute the ratio $\tau_w h/\mu U$ for this condition.

▶ Problem 6.30

At a certain section of a developed turbulent plane water jet, the maximum velocity is 3 m/s and the mass flow is 800 kg/s per meter of width. Estimate (*a*) the jet width, (*b*) maximum velocity, and (*c*) total mass flow, at a position 2 m further downstream.

▶ Problem 6.31

Air at 20°C and 1 atm issues at 0.001 kg/s from a 4 mm diameter orifice into still air. At a section in the jet 1 m downstream of the orifice, estimate (*a*) the maximum velocity, (*b*) the jet width, and (*c*) the ratio μ_t/μ.

▶ Problem 6.32

A long 5-m-diameter vertical cylinder is placed in the ocean where the current is 60 cm/s across the cylinder. At 1 km downstream of the cylinder, estimate (*a*) the wake width; and (*b*) the wake velocity defect.

SOLUTIONS

P.3.1 ➡ *Solution*

A fluid in which the shear stress-strain relationship is described by the power law $\tau = K(du/dy)^n$ is an example of non-Newtonian fluid. The non-Newtonian version of Eq. (3-4) in the text is

$$0 = -\frac{dp}{dx} + \frac{\partial \tau}{\partial y} = 0 + \frac{\partial \tau}{\partial y}$$

since the pressure is constant. Integrating once with respect to *y* gives

$$\frac{du}{dy} = \text{constant}$$

Integrating a second time brings to

$$u(y) = ay + b$$

where *a* and *b* are integration constants. One boundary condition is no-slip at the lower wall, i.e., $u(-h) = 0$:

$$-ah + b = 0 \quad \text{(I)}$$

The other BC pertains to velocity in the upper wall, i.e., $u(+h) = V$:

$$ah + b = V \quad \text{(II)}$$

Equations (I) and (II) form a system of linear equations on *a* and *b*:

$$\begin{cases} -ah + b = 0 \\ ah + b = V \end{cases}$$

Adding one equation to the other gives

$$-\cancel{ah} + b + \cancel{ah} + b = 0 + V$$

$$\therefore 2b = V$$

$$\therefore b = \frac{V}{2}$$

Substituting *b* in (I) and solving for *a*:

$$-ah + \frac{V}{2} = 0$$

$$\therefore a = \frac{V}{2h}$$

so that

$$u(y)=\frac{V}{2h}y+\frac{V}{2}=\boxed{\frac{V}{2}\left(1+\frac{y}{h}\right)}$$

Notice that this is identical to the linear velocity profile expected for a Newtonian fluid. Accordingly, the velocity profile that describes parallel-plate (Couette) flow of a power-law non-Newtonian fluid is identical to that of a Newtonian parallel-plate flow.

P.3.2 ➡ *Solution*

With velocity *u* as a function of *r* only, the analysis follows Section 3-2.2:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=0\rightarrow u(r)=C_1\ln(r)+C_2$$

The two BCs are $u(r_0)=U_0$ and $u(r_1)=U_1$; substituting these into the equation for $u(r)$ yields:

$$\begin{cases}C_1\ln(r_0)+C_2=U_0 & \text{(I)}\\ C_1\ln(r_1)+C_2=U_1 & \text{(II)}\end{cases}$$

Subtracting (II) from (I), we get

$$C_1\ln(r_0)+C_2-C_1\ln(r_1)-C_2=U_0-U_1$$

$$\therefore C_1\ln\left(\frac{r_0}{r_1}\right)=U_0-U_1$$

$$\therefore C_1\ln\left(\frac{r_1}{r_0}\right)=U_1-U_0$$

$$\therefore C_1=\frac{U_1-U_0}{\ln(r_1/r_0)}$$

Solving (I) for C_2 yields

$$C_1\ln(r_0)+C_2=U_0\rightarrow C_2=U_0-C_1\ln(r_0)$$

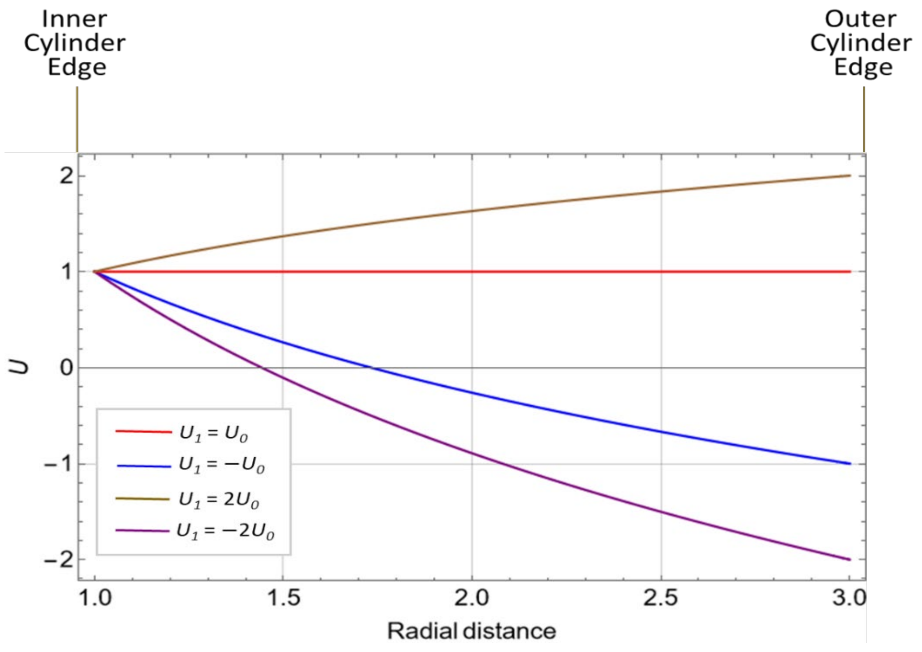
$$\therefore C_2=U_0-\left[\frac{U_1-U_0}{\ln(r_1/r_0)}\right]\ln(r_0)$$

so that

$$u(r)=U_0\frac{\ln(r_1/r)}{\ln(r_1/r_0)}+U_1\frac{\ln(r/r_0)}{\ln(r_1/r_0)}$$

As noted by White, this is simply the sum of the separate solutions for moving inner and outer cylinders, as described by Eqs. (3-18) and (3-19). This superposition is possible because the Navier-Stokes equations are linear for this particular flow.

Now, we were told to plot velocity profiles for (*a*) $U_1=U_0$, (*b*) $U_1=-U_0$, and (*c*) $U_1=2U_0$. As an example, let $U_0=1$ (arbitrary units) and $r_0=1$, $r_1=3r_0=3$ (also in arbitrary units). For illustrative purposes, we add a case (*d*) corresponding to $U_1=-2U_0$.



P.3.3 ➡ *Solution*

The temperature is assumed to be a function of radial distance only, i.e., $T = T(r)$; there are no radial or circumferential velocities, so the energy equation, Eq. (B-9) in Appendix B of the textbook, reduces to

$$0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu\left(\frac{du}{dr}\right)^2$$

where, referring to Eq. (3-18), velocity $u(r)$ is given by

$$u(r) = U_0 \frac{\ln(r_1/r)}{\ln(r_1/r_0)}$$

Introducing $u(r)$, separating variables, and integrating once, we obtain

$$\frac{dT}{dr} = -\frac{\mu U_0^2}{k\ln^2(r_1/r_0)}\frac{\ln(r)}{r} + \frac{C_1}{r}$$

where C_1 is an integration constant. Integrating a second time and noting that $\int \ln(r)/r dr = \ln^2(r)/2$,

$$T(r) = -\frac{\mu U_0^2}{2k}\frac{\ln^2(r/r_0)}{\ln^2(r_1/r_0)} + C_1 \ln r + C_2$$

The boundary conditions are $T(r_0) = T_0$ and $T(r_1) = T_1$. Substituting the former BC gives

$$T(r_0) = \underbrace{-\frac{\mu U_0^2}{2k}\frac{\ln^2(r_0/r_0)}{\ln^2(r_1/r_0)}}_{=0} + C_1 \ln r_0 + C_2 = T_0$$

$$\therefore C_1 \ln r_0 + C_2 = T_0 \quad \text{(I)}$$

Substituting the second BC, in turn, we have

$$T(r_1) = -\frac{\mu U_0^2}{2k}\frac{\ln^2(r_1/r_0)}{\ln^2(r_1/r_0)} + C_1 \ln r_1 + C_2 = T_1$$

$$\therefore -\frac{\mu U_0^2}{2k} + C_1 \ln r_1 + C_2 = T_1 \quad \text{(II)}$$

Subtracting (II) from (I) and manipulating,

$$\therefore C_1 \ln r_0 + C_2 + \frac{\mu U_0^2}{2k} - C_1 \ln r_1 - C_2 = T_0 - T_1 \quad \text{(I)}$$

$$\therefore C_1 \ln\left(\frac{r_0}{r_1}\right) + \frac{\mu U_0^2}{2k} = T_0 - T_1$$

$$\therefore C_1 = \frac{T_0 - T_1 - \frac{\mu U_0^2}{2k}}{\ln(r_0/r_1)}$$

Substituting C_1 in (I) and solving for C_2,

$$\frac{\left[T_0 - T_1 - \frac{\mu U_0^2}{2k}\right]}{\ln(r_0/r_1)}\ln r_0 + C_2 = T_0$$

$$\therefore C_2 = T_0 - \frac{\left[T_0 - T_1 - \frac{\mu U_0^2}{2k}\right]}{\ln(r_0/r_1)}\ln r_0$$

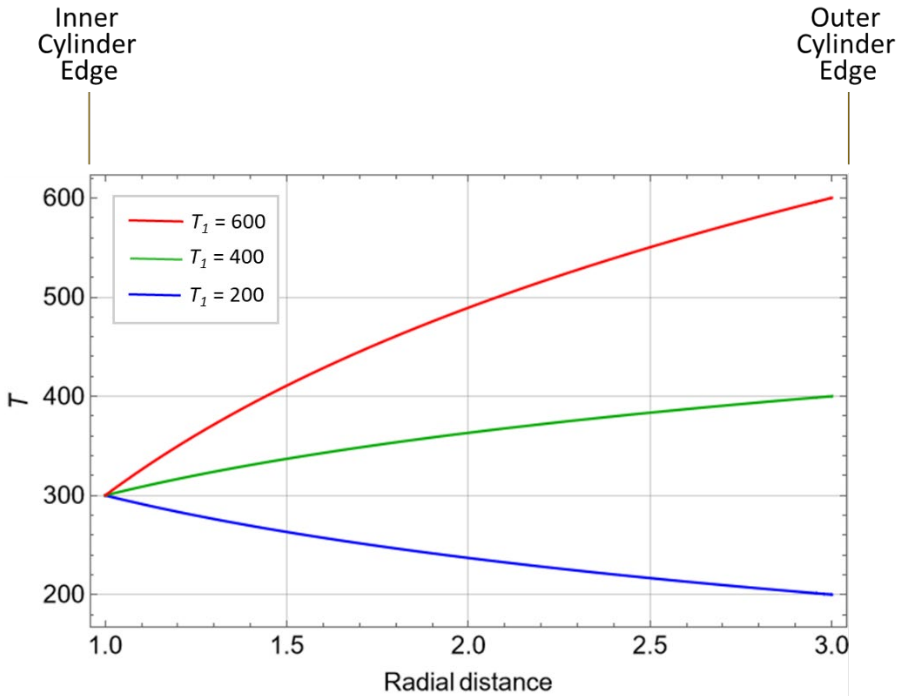
Therefore, the temperature profile is described by the equation

$$T(r) = -\frac{\mu U_0^2}{2k}\frac{\ln^2(r/r_0)}{\ln^2(r_1/r_0)} + \frac{\left[T_0 - T_1 - \frac{\mu U_0^2}{2k}\right]}{\ln(r_0/r_1)}\ln r + T_0 - \frac{\left[T_0 - T_1 - \frac{\mu U_0^2}{2k}\right]}{\ln(r_0/r_1)}\ln r_0$$

$$\therefore T(r) = -\frac{\mu U_0^2}{2k} \frac{\ln^2(r/r_0)}{\ln^2(r_1/r_0)} + \left(T_0 - T_1 - \frac{\mu U_0^2}{2k}\right) \frac{\ln(r/r_0)}{\ln(r_0/r_1)} + T_0$$

$$\therefore T(r) = T_0 - \frac{\mu U_0^2}{2k} \frac{\ln^2(r/r_0)}{\ln^2(r_1/r_0)} + \left(T_1 - T_0 + \frac{\mu U_0^2}{2k}\right) \frac{\ln(r/r_0)}{\ln(r_1/r_0)}$$

As an example, let's plot the temperature profile for a case in which the inner cylinder is moving at a speed U_0 = 1 arbitrary unit; also, let r_0 = 1, r_1 = $3r_0$ = 3, T_0 = 300, μ = 0.01, and k = 0.05. The outer cylinder temperature T_1 is taken as 200, 400, or 600.



P.3.4 ➡ *Solution*

The equation to integrate is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$$

which becomes

$$u = C_1 \ln(r) + C_2$$

The first boundary condition is that the velocity at radial distance R must equal U:

$$u(R) = C_1 \ln(R) + C_2 = U$$

The second boundary condition is that the velocity at a very large distance from the centerline of the rod should equal zero:

$$u(r \to \infty) = C_1 \ln(\infty) + C_2 = 0 \quad (?)$$

However, no constants C_1 and C_2 can be found that make the second BC valid. It is impossible to find steady-flow constants if the rod moves through an *infinite* expanse of fluid. Physically, it would require the finite-diameter rod to deliver an infinite amount of kinetic energy to the fluid with only finite wall shear stress.

P.3.5 ➡ *Solution*

The velocity follows from the θ-momentum equation in Section 3-2.3:

$$\frac{d^2 u_\theta}{dr^2} + \frac{d}{dr}\left(\frac{u_\theta}{r}\right) = 0$$

which can be integrated to yield

$$u_\theta = C_1 r + \frac{C_2}{r}$$

One of the boundary conditions is that u_θ should be zero at a very large radial distance,

$$u_\theta(r \to \infty) = C_1 \infty + \frac{C_2}{\infty} = 0$$

The only way to satisfy this equality is to have $C_1 = 0$. The other boundary condition is that the circumferential velocity must equal ωR at a distance R from the centerline of the cylinder:

$$u_\theta(r = R) = \omega R = 0 \times R + \frac{C_2}{R}$$

$$\therefore C_2 = \omega R^2$$

Thus, u_θ is described by the equation

$$\boxed{u_\theta = \frac{\omega R^2}{r}}$$

This is the solution given by Eq. (3-25). It does indeed correspond to a potential vortex. The pressure is found from the *r*-momentum equation:

$$\frac{dp}{dr} = \frac{\rho u_\theta^2}{r} = \frac{\rho}{r} \times \frac{\omega^2 R^4}{r^2} = \frac{\rho \omega^2 R^4}{r^3}$$

Integrating:

$$p(r) = -\frac{\rho \omega^2 R^4}{2r^2} + C \quad \text{(I)}$$

The boundary condition is that $p = p_0$ at $r = R$, which brings to

$$p(R) = -\frac{\rho \omega^2 R^4}{2R^2} + C = p_0$$

$$\therefore -\frac{\rho \omega^2 R^2}{2} + C = p_0$$

$$\therefore C = p_0 + \frac{\rho \omega^2 R^2}{2}$$

Lastly, we substitute *C* into (I) to obtain

$$p(r) = -\frac{\rho \omega^2 R^4}{2r^2} + p_0 + \frac{\rho \omega^2 R^2}{2}$$

$$\therefore \boxed{p(r) = p_0 + \frac{1}{2}\rho \omega^2 R^2 \left(1 - \frac{R^2}{r^2}\right)}$$

This is exactly what one would find by using Bernoulli's equation for a potential vortex.

P.3.8 ➡ *Solution*

The velocity distribution is given by Eq. (3-42) in the textbook:

$$u(y) = \frac{U}{2}\left(1 + \frac{y}{h}\right) + \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\left(1 - \frac{y^2}{h^2}\right)$$

The volume flow per unit depth is found by integrating this across the fluid between plates:

```
In[986]:= Integrate[U/2 * (1 + y/h) + h^2/(2 * μ) * Δp * (1 - y^2/h^2), {y, -h, h}, Assumptions → h > 0]

Out[986]= h U + (2 h^3 Δp)/(3 μ)
```

We are given h = 0.5 cm = 0.005 m, U = 20 cm/s = 0.2 m/s, and dp/dx = −0.3 Pa/m. The viscosity of air at 20°C may be taken as 1.80×10^{-5} Pa·s. Substituting in the expression above, we obtain

$$Q = hU + \frac{2h^3}{3\mu}\left(-\frac{dp}{dx}\right)$$

$$\therefore Q = 0.005 \times 0.2 + \frac{2 \times 0.005^3}{3 \times \left(1.80 \times 10^{-5}\right)} \times (-0.3) = 0.00238\,\text{m}^2/\text{s}$$

$$\therefore \boxed{Q = 2380\,\text{cm}^3/\text{s/m}}$$

To find the pressure gradient for which the shear stress at the lower plate is zero, we appeal to the 'separation criterion' mentioned in Sect. 3-3.2:

$$\frac{dp}{dx} = \frac{2\mu U}{(2h)^2} = \frac{2 \times \left(1.80 \times 10^{-5}\right) \times 0.20}{(2 \times 0.005)^2} = \boxed{0.0724\,\text{Pa/m}}$$

A small, positive pressure gradient is needed to cause flow separation.

P.3.10 ➡ *Solution*

We first compute the hydraulic diameter D_h,

$$D_h = \frac{4A}{P} = \frac{4 \times (0.01 \times 0.04)}{\left[2 \times (0.01 + 0.04)\right]} = 0.016\,\text{m}$$

and then the Reynolds number (the density and viscosity of air at 20°C may be taken as 1.20 kg/m^3 and 1.80×10^{-5} N·s/m^2, respectively):

$$\text{Re}_{D_h} = \frac{\rho V D_h}{\mu} = \frac{1.20 \times 1.7 \times 0.016}{1.80 \times 10^{-5}} = 1810$$

which is less than 2000, hence the flow is laminar. For air flowing at 1.7 m/s in a 1 × 4 cm duct, the flow rate is

$$Q = VA = 1.7 \times (0.01 \times 0.04) = 6.80 \times 10^{-4}\,\text{m}^3/\text{s}$$

In the exact analysis, the flow rate can be obtained with equation (3.48) in the textbook,

$$Q = \frac{4ba^3}{3\mu}\left(-\frac{dp}{dx}\right)\left[1 - \frac{192a}{\pi^5 b}\sum_{i=1,3,5,\ldots}^{\infty}\frac{\tanh(i\pi b/2a)}{i^5}\right] \quad \text{(I)}$$

In the case at hand, a = 4 cm/2 = 2 cm and b = 1 cm/2 = 0.5 cm; using only the first three terms in the series, we obtain

$$\sum_{i=1,3,5,\ldots}^{\infty}\frac{\tanh(i\pi b/2a)}{i^5} = \frac{\tanh\left(1 \times \pi \times 0.005/(2 \times 0.02)\right)}{1^5} + \frac{\tanh\left(3 \times \pi \times 0.005/(2 \times 0.02)\right)}{3^5}$$
$$+ \frac{\tanh\left(5 \times \pi \times 0.005/(2 \times 0.02)\right)}{5^5} = 0.377$$

so that, substituting in (I) and solving for pressure drop, we obtain

$$Q = \frac{4 \times 0.005 \times 0.02^3}{3 \times \left(1.80 \times 10^{-5}\right)}\left(-\frac{dp}{dx}\right)\left[1 - \frac{192 \times 0.02}{\pi^5 \times 0.005} \times 0.377\right] = 6.8 \times 10^{-4}$$

$$\therefore 1.60 \times 10^{-4}\left(-\frac{dp}{dx}\right) = 6.8 \times 10^{-4}$$

$$\therefore -\frac{dp}{dx} = -\frac{6.8 \times 10^{-4}}{1.60 \times 10^{-4}} = \boxed{-4.25\,\text{Pa}}$$

Now, in the approximate approach, the rectangular duct is modelled as though it were a circular duct with hydraulic diameter $D_h = 4A/P = 0.016$ m. The friction factor is

$$C_f = \frac{16}{\text{Re}} = \frac{16}{1810} = 0.00884$$

The average shear stress is obtained by multiplying the friction factor by the dynamic pressure:

$$\tau_{avg} = C_f \frac{1}{2}\rho V^2 = 0.00884 \times \frac{1}{2} \times 1.20 \times 1.7^2 = 0.0153\,\text{Pa}$$

However, the average shear can also be expressed as

$$\tau_{avg} = \frac{D_h}{4}\left(-\frac{dp}{dx}\right)$$

so that, solving for pressure drop,

$$\frac{dp}{dx} = -\frac{4\tau_{avg}}{D_h} = -\frac{4 \times 0.0153}{0.016} = \boxed{-3.83\ \text{Pa}}$$

The circular-duct approximation underestimates the (absolute value of) pressure drop obtained via the exact solution by approximately 10%.

P.3.12 ➡ *Solution*

The hydraulic diameter of an annulus with outer radius a = 10 mm and inner radius b = 9 mm is calculated as $D_h = 2(a - b)$ = 2 mm = 0.002 m. The area of the annulus is $\pi(a^2 - b^2) = \pi \times (0.01^2 - 0.009^2) = 5.97\times10^{-5}$ m^2. The flow rate is

$$Q = 1\frac{\text{m}^3}{\cancel{\text{h}}} \times \frac{1}{3600}\frac{\cancel{\text{h}}}{\text{s}} = 2.78\times10^{-4}\ \text{m}^3/\text{s}$$

Dividing Q by the annular area gives the flow velocity:

$$V = \frac{Q}{A} = \frac{2.78\times10^{-4}}{5.97\times10^{-5}} = 4.66\,\text{m/s}$$

Assuming that the density of a typical light oil is 900 kg/m^3 and noting that the viscosity of tested oils varied from 0.02 Pa·s to 0.1 Pa·s, the corresponding range of Reynolds numbers is:

$$\text{Re}_{min} = \frac{\rho V D_h}{\mu_{max}} = \frac{900 \times 4.66 \times 0.002}{0.1} = 83.9$$
$$\text{Re}_{max} = \frac{\rho V D_h}{\mu_{min}} = \frac{900 \times 4.66 \times 0.002}{0.02} = 419$$

Even the greatest Reynolds number is well within the laminar range, so the pressure drop can be computed with equation (3-51) in the textbook:

$$Q = \frac{\pi}{8\mu}\left(-\frac{dp}{dx}\right)\left[a^4 - b^4 - \frac{\left(a^2 - b^2\right)^2}{\ln\left(a/b\right)}\right]$$

On one extreme, for an oil with viscosity μ_{min} = 0.02 Pa·s,

$$Q = \frac{\pi}{8 \times 0.02} \times \frac{\Delta p}{0.3} \times \left[0.01^4 - 0.009^4 - \frac{\left(0.01^2 - 0.009^2\right)^2}{\ln\left(0.01/0.009\right)}\right] = 2.78\times10^{-4}$$

$$\therefore \Delta p = \frac{2.78\times10^{-4}}{8.29\times10^{-10}} = 335{,}000\,\text{Pa} = \underline{335\,\text{kPa}}$$

On the other extreme, for an oil with viscosity μ_{max} = 0.1 Pa·s,

$$Q = \frac{\pi}{8 \times 0.1} \times \frac{\Delta p}{0.3} \times \left[0.01^4 - 0.009^4 - \frac{\left(0.01^2 - 0.009^2\right)^2}{\ln\left(0.01/0.009\right)}\right] = 2.78\times10^{-4}$$

$$\therefore \Delta p = \frac{2.78\times10^{-4}}{1.66\times10^{-10}} = 1{,}680{,}000\,\text{Pa} = \underline{1680\,\text{kPa}}$$

The pressure drop ranges from about 300 kilopascals to over 1600 kilopascals. For such a large pressure drop, a mechanical gage might be recommended.

P.3.15 ➡ *Solution*

Since the boundary conditions are independent of *x*, we may assume that *u* = *u(y)*, i.e., *u* is a function of the normal coordinate *y*, and that the other two velocity components, *v* and *w*, are both zero. Also, ∂*p*/∂*x* = 0. The momentum equation then reduces to

$$u\frac{d^2u}{dy^2}=-\rho g\sin\theta=\text{constant}$$

Integrating twice brings to

$$u=-\frac{\rho g\sin\theta}{2\mu}y^2+C_1y+C_2\quad\text{(I)}$$

where C_1 and C_2 are integration constants. One of the boundary conditions relates to no-slip at the wall, that is, *u*(*y* = 0) = 0, with the result that

$$u(y=0)=-\frac{\rho g\sin\theta}{2\mu}\times 0^2+C_1\times 0+C_2=0$$

$$\therefore C_2=0$$

As a second boundary condition, we note that there is negligible shear at the free surface due to weak interaction with the constant-pressure, low-density atmosphere:

$$\tau_{xy}(y=h)=0\rightarrow\mu\left.\frac{\partial u}{\partial y}\right|_{y=h}=-\frac{\rho g\sin\theta}{2}\times(2h)+\mu C_1=0$$

$$\therefore C_1=\frac{\rho gh\sin\theta}{\mu}$$

Substituting C_1 and C_2 into (I) and rearranging, we get

$$u=-\frac{\rho g\sin\theta}{2\mu}y^2+\frac{\rho gh\sin\theta}{\mu}y+0$$

$$\therefore \boxed{u=\frac{\rho g\sin\theta}{2\mu}y(2h-y)}$$

This expression can be integrated to yield the volume flow per unit width of film:

```
In[465]:= Integrate[(ρ*g*Sin[θ])/(2*μ) *y (2*h-y), {y, 0, h}, Assumptions → h > 0]

Out[465]= (g h^3 ρ Sin[θ])/(3 μ)
```

That is,

$$Q=\frac{\rho gh^3\sin\theta}{3\mu}$$

The flow rate varies as h^3, hence the rate of draining is highly dependent upon the film thickness.

P.3.16 ➡ *Solution*

Assume, due to the constant-pressure atmosphere outside the film, that ∂*p*/∂*z* = 0, and, for a fully developed film, u_z = *u(r)* only. The *z*-momentum equation, in this case, reduces to

$$0=\rho g+\frac{u}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)$$

This second-order ordinary differential equation is integrated twice to yield

$$u=-\frac{\rho gr^2}{4\mu}+C_1\ln(r)+C_2\quad\text{(I)}$$

The first boundary condition is no-slip at the wall, that is, $u(r = a) = 0$, which can be substituted in (I) to yield

$$u(r=a) = -\frac{\rho g a^2}{4\mu} + C_1 \ln(a) + C_2 = 0$$

$$\therefore C_2 = \frac{\rho g a^2}{4\mu} - C_1 \ln(a) \quad \text{(II)}$$

The second boundary condition we can use is the absence of surface shear, that is:

$$\left.\frac{du}{dr}\right|_{y=b} = 0 \rightarrow \left.\left(-\frac{\rho g r}{2\mu} + \frac{C_1}{r}\right)\right|_{r=b} = 0$$

$$\therefore -\frac{\rho g b}{2\mu} + \frac{C_1}{b} = 0$$

$$\therefore C_1 = \frac{\rho g b^2}{2\mu}$$

Substituting C_1 in (II) yields

$$C_2 = \frac{\rho g a^2}{4\mu} - \frac{\rho g b^2}{2\mu}\ln(a)$$

Substituting C_1 and C_2 in (I) and rearranging gives the velocity profile we're looking for:

$$u(r) = -\frac{\rho g r^2}{4\mu} + \frac{\rho g b^2}{2\mu}\ln(r) + \frac{\rho g a^2}{4\mu} - \frac{\rho g b^2}{2\mu}\ln(a)$$

$$\therefore \boxed{u(r) = \frac{\rho g b^2}{4\mu}\left[2\ln\left(\frac{r}{a}\right) - \left(\frac{r}{b}\right)^2 + \left(\frac{a}{b}\right)^2\right]}$$

Lastly, the flow rate can be obtained by evaluating the integral

$$Q = \int_a^b u(r) 2\pi r dr$$

The integration is tedious and can be obtained with Mathematica:

```
In[466]:= Simplify[Integrate[((ρ * g * bb^2)/(4 * μ) * (2 * Log[r/aa] - (r/bb)^2 + (aa/bb)^2)) * 2 * Pi * r,
    {r, aa, bb}, Assumptions → bb > aa > 0]]
Out[466]=
 -(g π ρ (aa^4 - 4 aa^2 bb^2 + 3 bb^4 - 4 bb^4 Log[bb/aa]))/(8 μ)
```

As can be seen, the result is (note the tiny negative sign in the output)

$$Q = -\frac{\pi \rho g}{8\mu}\left[-4b^4 \ln\left(\frac{b}{a}\right) - 4a^2 b^2 + a^4 + 3b^4\right]$$

In closing, we define $b/a = \beta$ and factor out a^4 to obtain

$$Q = \frac{\pi \rho g a^4}{8\mu}\left[4\beta^4 \ln(\beta) + 4\beta^2 - 1 - 3\beta^4\right] \; ; \; \beta = \frac{b}{a} > 1$$

For thin films approximating a flat wall, $1.0 < b/a < 1.2$, Q increases (approximately) as the cube of the film thickness (see also Problem 3.15). For $b/a > 1.2$, Q increases even faster than the cube of the thickness, as film area increases with radius.

P.3.17 ➡ *Solution*

We solve for the velocity distribution in each of the two layers and then check if they coincide at the boundary $y = h_1$. Assuming that velocities u_1 and u_2 are functions of the y-coordinate only, the layers satisfy the momentum equations

$$\underline{x-\text{momentum:}}\ \rho_1 g\sin\theta+\mu_1\frac{d^2u_1}{dy^2}=0$$

$$\underline{y-\text{momentum:}}\ \rho_1 g\cos\theta-\frac{\partial p_1}{\partial y}=0$$

Also, the pressure gradient $\partial p/\partial x = 0$ because the surface pressure is constant. Accordingly, the pressure is hydrostatic in $\rho_1 g\cos(\theta)$ and is uncoupled from velocity. Accordingly, we can ignore pressure and solve for velocity; integrating the *x*-momentum equation twice yields the profiles

$$u_1=-\frac{\rho_1 g\sin\theta}{2\mu_1}y^2+C_1y+D_1$$

$$u_2=-\frac{\rho_2 g\sin\theta}{2\mu_2}y^2+C_2y+D_2$$

where $C_{1,2}$ and $D_{1,2}$ are integration constants. By inspection, $D_1 = 0$ because the lower layer is bounded by a rigid surface in its lower end and hence the no-slip condition applies. The other three boundary conditions are negligible shear at the surface (weakly interacting atmosphere), matching velocities in the interface between the two layers, and matching shear in the interface between the two layers. In mathematical terms:

$$\underline{\text{Surface shear:}}\ \tau_2=0=\mu_2\left.\frac{du_2}{dy}\right|_{y=h_1+h_2}$$

$$\therefore \mu_2\left.\left(-\frac{\rho_2 g\sin\theta}{\mu_2}y+C_2\right)\right|_{y=h_1+h_2}=0$$

$$\therefore -\rho_2 g\sin\theta\left(h_1+h_2\right)+\mu_2C_2=0\ \ \text{(I)}$$

$$\underline{\text{Interface velocity:}}\ u_1\left(y=h_1\right)=u_2\left(y=h_1\right)$$

$$\therefore\ \ -\frac{\rho_1 g\sin\theta}{2\mu_1}h_1^2+C_1h_1=-\frac{\rho_2 g\sin\theta}{2\mu_2}h_1^2+C_2h_1+D_2\ \ \text{(II)}$$

$$\underline{\text{Interface shear:}}\ \tau_1\left(y=h_1\right)=\tau_2\left(y=h_1\right)$$

$$\therefore \mu_1\left.\left(-\frac{\rho_1 g\sin\theta}{\mu_1}y+C_1\right)\right|_{y=h_1}=\mu_2\left.\left(-\frac{\rho_2 g\sin\theta}{\mu_2}y+C_2\right)\right|_{y=h_1}$$

$$\therefore \mu_1\left(-\frac{\rho_1 g\sin\theta}{\mu_1}h_1+C_1\right)=\mu_2\left(-\frac{\rho_2 g\sin\theta}{\mu_2}h_1+C_2\right)\ \ \text{(III)}$$

Expressions (I) to (III) constitute a system of linear equations with three unknowns, namely C_1, C_2, and D_2. The most obvious equation to tackle is (I), which can be readily solved for C_2:

$$-\rho_2 g\sin\theta\left(h_1+h_2\right)+\mu_2C_2=0\rightarrow C_2=\frac{\rho_2 g\sin\theta}{\mu_2}\left(h_1+h_2\right)$$

Substituting C_2 in (III) and solving for C_1, we obtain:

$$\therefore \mu_1\left(-\frac{\rho_1 g\sin\theta}{\mu_1}h_1+C_1\right)=\mu_2\left[-\frac{\rho_2 g\sin\theta}{\mu_2}h_1+\frac{\rho_2 g\sin\theta}{\mu_2}\left(h_1+h_2\right)\right]$$

$$\therefore -\rho_1 g\sin\left(\theta\right)h_1+\mu_1C_1=-\rho_2 g\sin\left(\theta\right)h_1+\rho_2 g\sin\left(\theta\right)\left(h_1+h_2\right)$$

$$\therefore -\rho_1 g\sin\left(\theta\right)h_1+\mu_1C_1=\cancel{-\rho_2 g\sin\left(\theta\right)h_1}+\cancel{\rho_2 g\sin\left(\theta\right)h_1}+\rho_2 g\sin\left(\theta\right)h_2$$

$$\therefore C_1=\frac{g\sin\theta}{\mu_1}\left(\rho_1h_1+\rho_2h_2\right)$$

Lastly, we substitute C_1 and C_2 into (II) and solve for the remaining constant, D_2:

$$-\frac{\rho_1 g\sin\theta}{2\mu_1}h_1^2+\frac{g\sin\theta}{\mu_1}(\rho_1 h_1+\rho_2 h_2)h_1=-\frac{\rho_2 g\sin\theta}{2\mu_2}h_1^2+\frac{\rho_2 g\sin\theta}{\mu_2}(h_1+h_2)h_1+D_2$$

$$\therefore -\frac{\rho_1 g\sin\theta}{2\mu_1}h_1^2+\frac{\rho_1 g\sin\theta}{\mu_1}h_1^2+\frac{\rho_2 g\sin\theta}{\mu_1}h_1h_2=-\frac{\rho_2 g\sin\theta}{2\mu_2}h_1^2+\frac{\rho_2 g\sin\theta}{\mu_2}h_1^2+\frac{\rho_2 g\sin\theta}{\mu_2}h_1h_2+D_2$$

$$\therefore D_2=-\frac{\rho_1 g\sin\theta}{2\mu_1}h_1^2+\frac{\rho_1 g\sin\theta}{\mu_1}h_1^2+\frac{\rho_2 g\sin\theta}{\mu_1}h_1h_2-\frac{\rho_2 g\sin\theta}{2\mu_2}h_1^2-\frac{\rho_2 g\sin\theta}{\mu_2}h_1h_2$$

$$\therefore D_2=h_1 g\sin\theta\left(\frac{\rho_1 h_1}{2\mu_1}+\frac{\rho_2 h_2}{\mu_1}-\frac{\rho_2 h_1}{2\mu_2}-\frac{\rho_2 h_2}{\mu_2}\right)$$

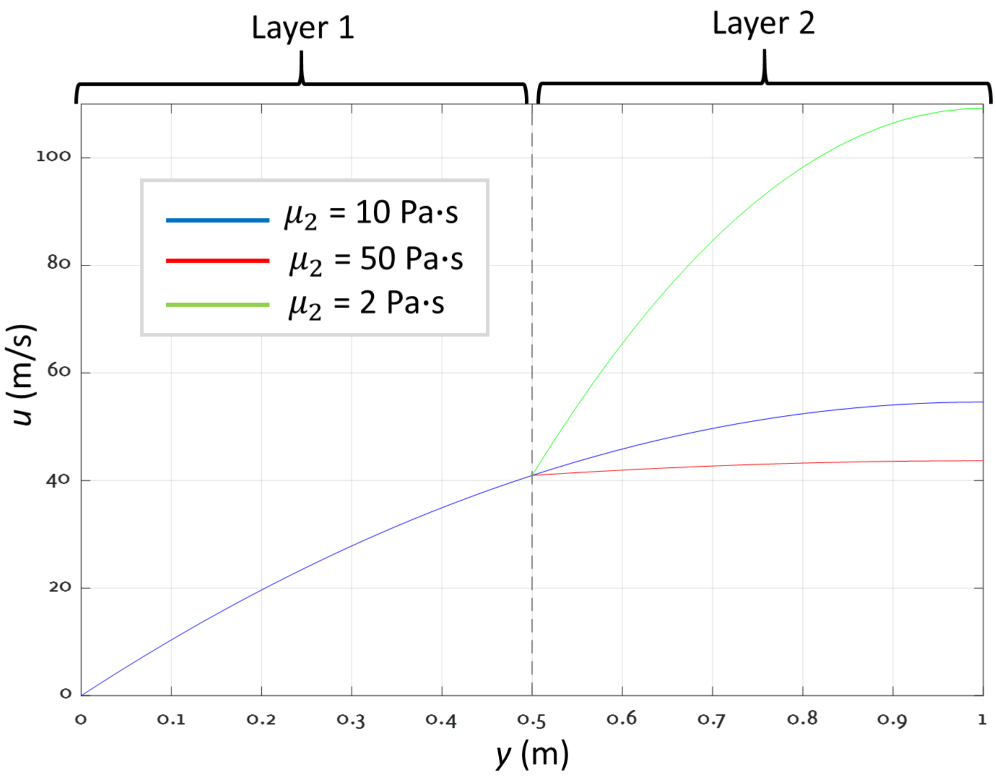
Thus, the velocity profiles $u_1(y)$ and $u_2(y)$ are described by the lengthy equations

$$u_1(y)=-\frac{\rho_1 g\sin\theta}{2\mu_1}y^2+\frac{g\sin\theta}{\mu_1}(\rho_1 h_1+\rho_2 h_2)y$$

$$u_2=-\frac{\rho_2 g\sin\theta}{2\mu_2}y^2+\frac{\rho_2 g\sin\theta}{\mu_2}(h_1+h_2)y+h_1 g\sin\theta\left(\frac{\rho_1 h_1}{2\mu_1}+\frac{\rho_2 h_2}{\mu_1}-\frac{\rho_2 h_1}{2\mu_2}-\frac{\rho_2 h_2}{\mu_2}\right)$$

Let us exemplify these profiles with a bilayer of thickness equal to 1 and such that $h_1 = h_2 = 0.5$. The density of the two fluids is the same, $\rho_1 = \rho_2 = 800$ kg/m^3, the inclination is $\theta = 8°$, and the viscosity of the lower fluid is $\mu_1 = 10$ Pa·s; the viscosity of the upper fluid is varied from $\mu_2 = \mu_1$ (case 1), $\mu_2 = 5\mu_1$ (case 2), and $\mu_2 = 0.2\mu_1$ (case 3). We apply the following MATLAB code to plot the three cases; the blue, red and green curves refer to cases 1, 2 and 3, respectively.

```
syms y
rho1 = 800;
rho2 = 800;
mu1 = 10;
mu2 = 10;
mu2Alt1 = 10*5;
mu2Alt2 = 10/5;
h1 = 0.5;
h2 = 0.5;
g = 9.81;
theta = 8*pi/180;
u = piecewise(0<y<0.5,-rho1*g*sin(theta)/(2*mu1)*y^2 +
g*sin(theta)/mu1*(rho1*h1+rho2*h2)*y,0.5 < y < 1,-
rho2*g*sin(theta)/(2*mu2)*y^2 +
rho2*g*sin(theta)/mu2*(h1+h2)*y+h1*g*sin(theta)*(rho1*h1/(2*m
u1)+rho2*h2/mu1-rho2*h1/(2*mu2)-rho2*h2/mu2));
fplot(u,'b');
xlim([0 1]);
grid on
hold on
uAlt1 = piecewise(0.5 < y < 1,-
rho2*g*sin(theta)/(2*mu2Alt1)*y^2 +
rho2*g*sin(theta)/mu2Alt1*(h1+h2)*y+h1*g*sin(theta)*(rho1*h1/
(2*mu1)+rho2*h2/mu1-rho2*h1/(2*mu2Alt1)-rho2*h2/mu2Alt1));
fplot(uAlt1,'r')
uAlt2 = piecewise(0.5 < y < 1,-
rho2*g*sin(theta)/(2*mu2Alt2)*y^2 +
rho2*g*sin(theta)/mu2Alt2*(h1+h2)*y+h1*g*sin(theta)*(rho1*h1/
(2*mu1)+rho2*h2/mu1-rho2*h1/(2*mu2Alt2)-rho2*h2/mu2Alt2));
fplot(uAlt2, 'g')
```



It is apparent that, if the upper fluid has viscosity equal to that of the lower layer, the velocity profile for the upper fluid is patched nicely with that of the lower one, and reaches a maximum speed of about 55 m/s. If, however, the upper fluid has viscosity substantially greater than the lower one, the maximum velocity reduces to about 42 m/s. Finally, having an upper fluid substantially *less* viscous than the lower one yields a much increased velocity profile in layer 2, with the fluid exceeding 100 m/s close to the free surface.

P.3.23 ➡ *Solution*

Firstly, the continuity equation reads

$$\frac{\partial}{\partial r}(r u_r) = 0$$

or

$$r u_r = f(z)$$

Substituting $u_r = f(z)/r$ into the r-momentum equation brings to

$$\rho u_r \frac{\partial u_r}{\partial r} = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) - \frac{u_r}{r^2} + \frac{\partial^2 u_z}{\partial z^2}\right]$$

or

$$-\frac{\rho\left[f(z)\right]^2}{r^2} = -\frac{dp}{dr} + \frac{u}{r}\frac{d^2 f}{dz^2} \quad \text{(I)}$$

which is to be solved for $f(z)$. For any given section r, the equation above is a nonlinear ordinary differential equation that can be solved for $f(z)$, subject to two boundary conditions:

$$\underline{\text{No-slip condition:}}\ f = 0 \text{ at } z = \pm L$$

$$\underline{\text{Symmetry condition:}}\ \frac{df}{dz} = 0 \text{ at } z = 0$$

The parameters involved in (I) become clearer if we integrate the equation in the radial direction from $r = r_1$ to $r = r_2$. The result is

$$\frac{\rho f^2}{2}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) = \Delta p + \mu \ln\left(\frac{r_2}{r_1}\right)\frac{d^2 f}{dz^2} \quad \text{(II)}$$

where $\Delta p = p_1 - p_2$. For "creeping" flow, the inertia term on the left-hand side is negligible, and the solution can be found immediately:

$$f(z) = ru_r = \frac{\Delta p L^2}{2\mu \ln(r_2/r_1)}\left(1 - \frac{z^2}{L^2}\right) \text{ (III)}$$

This parabolic profile is analogous to Poiseuille flow in a channel. At higher Reynolds numbers, the creeping flow simplification breaks down and (II) must be solved in its complete form. We may nondimensionalize (II) using solution (III) as a guide, so that

$$\frac{d^2\phi}{dz^{*2}} = -K\phi^2 - 2 \text{ (IV)}$$

where

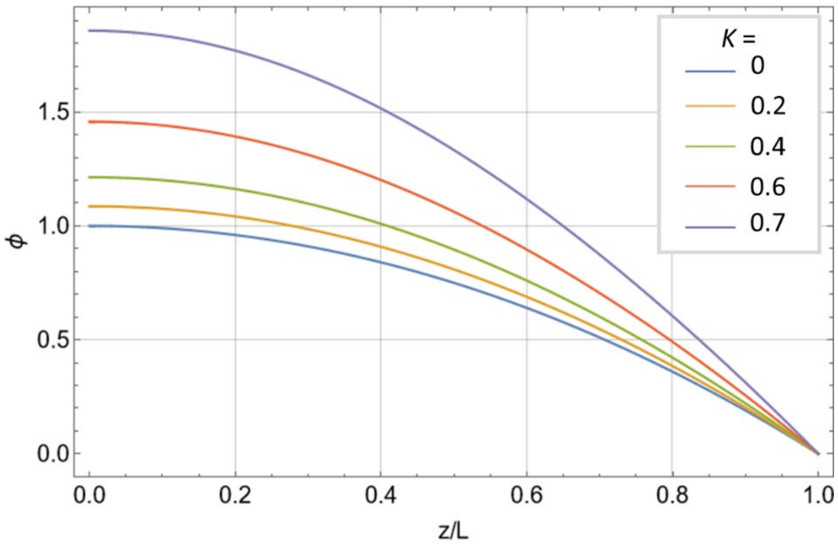
$$\phi = \frac{\Delta p L^2 f}{2\mu \ln(r_2/r_1)} \; ; \; z^* = \frac{z}{L}$$

and

$$K = \frac{\rho\left(r_1^2 - r_2^2\right)\Delta p L^4}{4\mu^2 \ln^2(r_2/r_1)}$$

The single parameter K is proportional to the Reynolds number of the flow. The boundary conditions become $\phi(\pm 1) = 0$ and $d\phi/dz^*(0) = 0$. Equation (IV) may be solved numerically via simple techniques such as the Runge-Kutta method. For a given K, the problem is to find the proper value of $f(0)$ that causes $f(1)$ to be zero. Solutions can be found for K ranging from 0.0 to 0.75, but no solutions can be found for $K \gtrsim 0.75$ approximately. Profiles for some values of K can be found with the following Mathematica code:

```
sol=ParametricNDSolve[{φ''[z]+k*φ[z]^2+2==0,φ[1]==0,φ'[0]==0},φ,{z,0,2},{k}]
Plot[Evaluate[Table[\[Phi]][k][z] /. sol, {k, {0, 0.2, 0.4, 0.6, 0.7}}]], {z, 0, 1}, PlotRange -> All, Frame -> True, GridLines -> Automatic]
```



As can be seen, the velocity profiles are approximately parabolic.

P.3.25 ➡ *Solution*

With pressure and velocities dependent only upon r, the continuity equation simplifies to

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 0$$

so that

$$v_r = -\frac{r_0 v_w}{r}$$

since $v_r|_{r=r_0} = -v_w$. With the radial velocity known, the θ (circumferential) momentum equation,

$$v_r \frac{dv_\theta}{dr} + \frac{v_r v_\theta}{r} = \frac{\nu}{r}\left[\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right) - \frac{v_\theta}{r}\right]$$

can be solved for tangential velocity. The general solution is

$$v_\theta = A \times r^{-1} + B \times r^{1-\mathrm{Re}}$$

where $Re = v_w r_0/\nu$ is the wall Reynolds number. The first term above is the 'potential' vortex of a solid cylinder. The second term is a combined effect of viscosity and wall suction and is unbounded if $Re < 1$, which is unrealistic. Further, the flow *circulation*, $\Gamma = 2\pi r v_\theta$, is proportional to r^{2-Re} and is unbounded if $Re < 2$, which is likewise unrealistic. Therefore, for physical realism,

$$B = 0 \;\text{ if }\; \mathrm{Re} \le 2$$

The vorticity in the flow is obtained by differentiation:

$$\omega = \frac{1}{r}\frac{d}{dr}\left(r v_\theta\right) = \frac{1}{r}\frac{d}{dr}\left[r \times \left(A r^{-1} + B r^{1-\mathrm{Re}}\right)\right] = B\left(2-\mathrm{Re}\right) r^{-\mathrm{Re}} \quad \text{(I)}$$

Utilizing the boundary condition for surface vorticity, $\omega = \omega_0$ at $r = r_0$, we obtain the constant B for $Re > 2$:

$$B\left(2-\mathrm{Re}\right) r_0^{-\mathrm{Re}} = \omega_0 \rightarrow B = \frac{\omega_0 r_0^{\mathrm{Re}}}{2-\mathrm{Re}}$$

Substituting in (I) gives the final solution for vorticity:

$$\omega = B\left(2-\mathrm{Re}\right) r^{-\mathrm{Re}} = \frac{\omega_0 r_0^{\mathrm{Re}}}{\cancel{2-\mathrm{Re}}} \cancel{\left(2-\mathrm{Re}\right)} r^{-\mathrm{Re}}$$

$$\therefore \omega = \omega_0 \left(\frac{r_0}{r}\right)^{\mathrm{Re}}$$

With B known, the constant A follows from the condition $v_\theta = r_0\omega_0$ at $r = r_0$. The final solution for velocity is a piecewise function conditioned by the range of Reynolds number:

$$v_\theta = \begin{cases} \dfrac{\omega_0 r_0^2}{r\left(2-\mathrm{Re}\right)}\left[1-\mathrm{Re}+\left(\dfrac{r_0}{r}\right)^{\mathrm{Re}-2}\right] & \text{if } \mathrm{Re} > 2 \\ \dfrac{\omega_0 r_0^2}{r} \;\; \text{if } \mathrm{Re} \le 2 \end{cases}$$

P.3.27 ➡ *Solution*

The Ekman solutions for surface velocity and penetration depth are

$$V_o = \frac{\tau_0/\rho}{\sqrt{2\omega\nu\sin\phi}} \;\; ; \;\; D = \pi\sqrt{\frac{\nu}{\omega\sin\phi}}$$

In the case at hand, the wind velocity is V_{wind} = 6 m/s and the air density at 20°C may be taken as $\rho_{\text{air}} \approx 1.205$ kg/m^3. The angular velocity of the Earth may be taken as $\omega = 2\pi/86{,}400 = 7.27\times10^{-5}$ s^{-1}. The surface wind shear may be estimated with equation (3-141):

$$\tau_0 = 0.002\rho_{\text{air}} V_{\text{wind}}^2 = 0.002 \times 1.205 \times 6.0^2 = 0.0868\,\text{Pa}$$

The shear velocity is, noting that the density of seawater may be taken as 1025 kg/m^3,

$$u^* = \sqrt{\tau_0/\rho} = \sqrt{0.0868/1025} = 0.00920\,\text{m/s}$$

We may then use Clauser's correlation to solve for penetration depth:

$$D = \pi\sqrt{\frac{\nu}{\omega\sin\phi}} = \pi\sqrt{\frac{0.04 D u^*}{\omega\sin\phi}}$$

$$\therefore D^2 = \pi^2 \frac{0.04 D u^*}{\omega\sin\phi}$$

$$\therefore D = \pi^2 \frac{0.04 u^*}{\omega\sin\phi}$$

$$\therefore D = \pi^2 \times \frac{0.04 \times 0.00920}{\left(7.27 \times 10^{-5}\right) \times \sin 41^{\circ}} = \boxed{76.2\,\text{m}}$$

Using this penetration depth, we can compute Clauser's modified eddy viscosity:

$$\nu_{\text{turb}} \approx 0.04 D \left(\frac{\tau_0}{\rho}\right)^{1/2} = 0.04 D u^*$$

$$\therefore \nu_{\text{turb}} = 0.04 \times 76.2 \times 0.00920 = 0.0280\,\text{m}^2/\text{s}$$

It remains to compute the surface velocity:

$$V_o = \frac{\tau_0/\rho}{\sqrt{2\omega\nu \sin\phi}} = \frac{0.00920^2}{\sqrt{2 \times \left(7.27 \times 10^{-5}\right) \times 0.0280 \times \sin 41^{\circ}}} = \boxed{0.0518\,\text{m/s}}$$

These are realistic results, with a surface velocity of the order of a few centimeters per second and a penetration depth of the order of 100 meters.

P.3.28 ➡ *Solution*

We are to solve the same differential equation with one changed boundary condition:

$$\nu w'' - 2i\omega \sin(\phi) w = 0 \;\; ; \;\; w = u + iv \quad \left(i = \sqrt{-1}\right)$$

subject to $w(-h) = 0$, $w'(0) = iK$, with $K = \tau_0/\mu$. This linear ODE has a general solution based on hyperbolic functions:

$$w(z) = A\cosh(bz) + B\sinh(bz) \;\; ; \;\; b = \left(2i\omega\sin\phi/\nu\right)^{1/2}$$

Note that parameter b can be restated as:

$$b = (1+i)\left(\omega\sin\phi/\nu\right)^{1/2}$$

From the boundary conditions, we find that $A = B\tanh(bh)$, $B = iK/b$. The desired solution for (general) velocity and surface velocity is thus

$$w = \frac{iK}{b}\left[\tanh(bh)\cosh(bz) + \sinh(bz)\right] \text{ and } w(0) = \frac{iK}{b}\tanh(bh) = u_0 + iv_0$$

We use the identity $\tanh(x + iy) = [\sinh(2x) + i\sin(2y)]/[\cosh(2x) + \cos(2y)]$ to untangle the real and imaginary parts of the surface velocity:

$$u_0 = \frac{1}{\Theta}\left[\sinh(2\beta h) - \sin(2\beta h)\right] \;\; ; \;\; v_0 = \frac{1}{\Theta}\left[\sinh(2\beta h) + \sin(2\beta h)\right]$$

where $\beta = \pi/D$ and

$$\Theta = \frac{K}{2\beta\left[\cosh(2\beta h) + \cos(2\beta h)\right]}$$

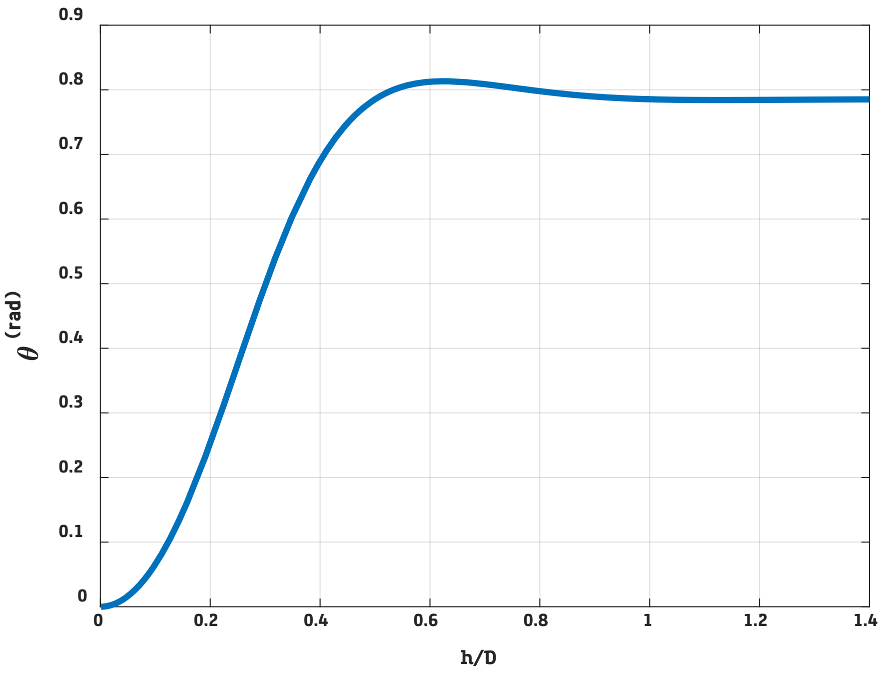
The ratio u_0/v_0 is the tangent of the desired surface flow angle:

$$\tan\Theta = \frac{u_0}{v_0} = \frac{\sinh(2\pi h/D) - \sin(2\pi h/D)}{\sinh(2\pi h/D) + \sin(2\pi h/D)}$$

We can then plot Θ versus h/D using the MATLAB code

```
fplot(@(b) (atan((sinh(2*pi*b)-
sin(2*pi*b))/(sinh(2*pi*b)+sin(2*pi*b)))), [0 1.4])
ylim([0 0.9])
grid on
```

The graph is shown below. The angle rises slowly from zero through an increasing overshoot at 47° at $h/D \approx 0.6$ to level off at 45° in "deep" water.



We can readily evaluate the function at θ = 20° (i.e., 20 × π/180 rad) with MATLAB's *feval* command:

```
feval(@(b) (atan((sinh(2*pi*b)-
sin(2*pi*b))/(sinh(2*pi*b)+sin(2*pi*b)))), 20*pi/180)

ans =

    0.6039
```

Clearly, $h/D \approx 0.604$ when θ = 20°.

P.3.31 ➡ *Solution*

The angular velocity of 1200 rpm can be converted to 1200 × 2π/60 =126 rad/s. The kinematic viscosity of air at 20°C and 1 atm can be taken as 1.50×10^{-5} m^2/s. One way (the hard way) to evaluate flow rate is to integrate the radial velocity at the edge of the disk:

$$Q = \int_0^\infty v_z\left(r = R\right) 2\pi R dz = 2\pi R^2 \sqrt{\nu\omega} \int_0^\infty F dz^*$$

However, the rightmost integral is not tabulated in the text. A much easier approach is to note that the flow rate should also equal the fluid 'pumped' axially toward the disk:

$$Q = \pi R^2 \left| v_z\left(\infty\right) \right|$$

where $v_z(\infty)$ can be obtained from equation (3-188), namely

$$v_z\left(\infty\right) = -0.8838\sqrt{\omega\nu}$$

so that

$$Q = \pi \times 0.25^2 \times \left| -0.8838 \times \sqrt{126 \times \left(1.50 \times 10^{-5}\right)} \right| = \boxed{0.00754\,\text{m}^3/\text{s}}$$

Next, the torque on one side of the disk is given by Eq. (3-190):

$$M = \frac{\pi}{2} \rho r_0^4 G_0' \sqrt{\nu\omega^3}$$

Function G_0' can be read from Table 3.5, noting that

$$z^* = 0$$

so that $|G_0'|$ = 0.61592. The torque we aim for is then

$$M = \frac{\pi}{2} \times 1.205 \times 0.25^4 \times 0.61592 \times \sqrt{\left(1.50 \times 10^{-5}\right) \times 126^3} = \boxed{0.0250\,\text{N}\cdot\text{m}}$$

The power required to drive one side of the disk is

$$\Pi = M\omega = 0.0250 \times 126 = \boxed{3.15\,\text{W}}$$

White notes that the rotational Reynolds number for the problem at hand,

$$\mathrm{Re} = \frac{\omega R^2}{\nu} = \frac{126 \times 0.25^2}{1.50 \times 10^{-5}} = 525,000$$

exceeds the threshold for turbulent flow, so the results we've obtained may not be immediately valid.

P.3.32 ➡ *Solution*

With *Re* = 0, Eq. (3-195) reduces to

$$f''' + \cancel{\mathrm{Re}\,\alpha f^2} + 4\alpha^2 f = \text{const.}$$

$$\therefore f''' + 4\alpha^2 f = \text{const.}$$

This is a third-order linear ODE for which the general solution can be shown to be

$$f = A_1 \sin(2\alpha\eta) + A_2 \cos(2\alpha\eta) + A_3$$

The boundary conditions are $f(0) = 1$, $f(+1) = 0$, and $f(-1) = 0$. Substituting the first BC yields

$$f(\eta = 0) = \underbrace{A_1 \sin(2\alpha \times 0)}_{=0} + A_2 \cos(2\alpha \times 0) + A_3 = 1$$

$$\therefore A_2 + A_3 = 1 \;\; \text{(I)}$$

Substituting the second BC,

$$f(\eta = 1) = A_1 \sin(2\alpha \times 1) + A_2 \cos(2\alpha \times 1) + A_3 = 0$$

$$\therefore A_1 \sin(2\alpha) + A_2 \cos(2\alpha) + A_3 = 0 \;\; \text{(II)}$$

Substituting the third BC,

$$A_1 \sin(-2\alpha) + A_2 \cos(-2\alpha) + A_3 = 0$$

$$\therefore -A_1 \sin(2\alpha) + A_2 \cos(2\alpha) + A_3 = 0 \;\; \text{(III)}$$

Equations (I) to (III) constitute a system of linear equations with three unknowns. Adding (II) to (III), we have

$$\cancel{A_1 \sin(2\alpha)} + A_2 \cos(2\alpha) + A_3 - \cancel{A_1 \sin(2\alpha)} + A_2 \cos(2\alpha) + A_3 = 0$$

$$\therefore 2A_2 \cos(2\alpha) + 2A_3 = 0$$

$$\therefore A_2 \cos(2\alpha) + A_3 = 0$$

$$\therefore A_3 = -A_2 \cos(2\alpha) \;\; \text{(IV)}$$

Substituting A_3 in (I),

$$A_2 + A_3 = 1 \rightarrow A_2 - A_2 \cos(2\alpha) = 1$$

$$\therefore A_2 \left[1 - \cos(2\alpha)\right] = 1$$

$$\therefore A_2 = \frac{1}{1 - \cos(2\alpha)}$$

From (IV),

$$A_3 = -A_2 \cos(2\alpha) = -\frac{\cos(2\alpha)}{1 - \cos(2\alpha)}$$

Substituting A_2 and A_3 in (I), we see that

$$A_1 \sin(2\alpha) + \frac{\cos(2\alpha)}{1 - \cos(2\alpha)} - \frac{\cos(2\alpha)}{1 - \cos(2\alpha)} = 0$$

$$\therefore A_1 \sin(2\alpha) = 0$$

$$\therefore A_1 = 0$$

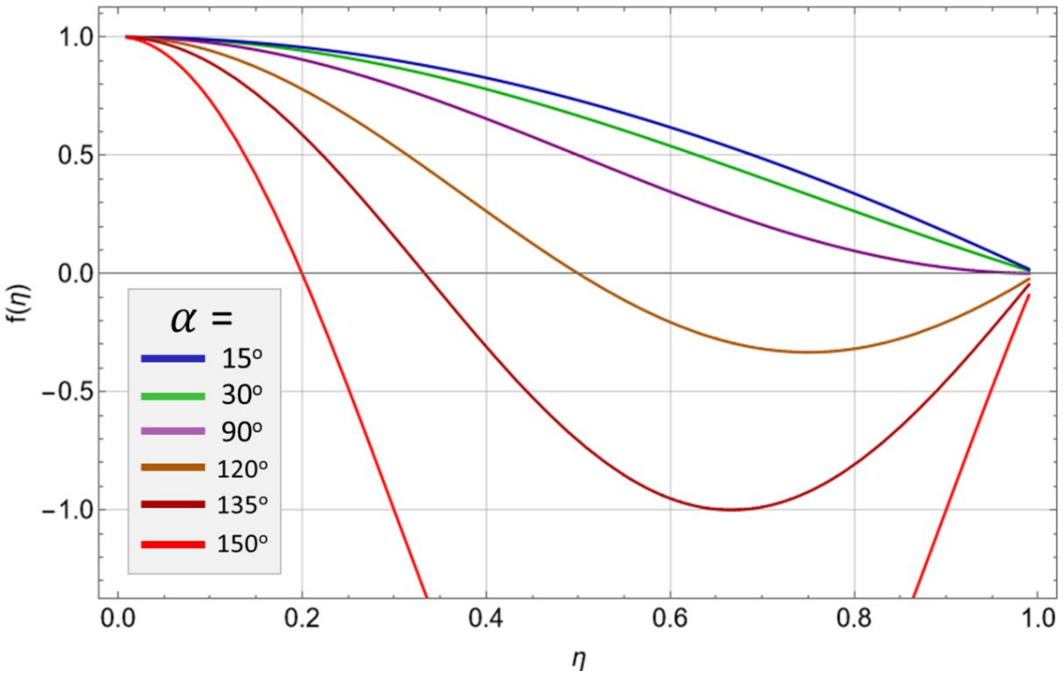
It follows that the desired solution for creeping Jeffery-Hamel wedge flow is

$$f(\eta)=A_1\sin(2\alpha\eta)+A_2\cos(2\alpha\eta)+A_3$$
$$\therefore f(\eta)=\frac{\cos(2\alpha\eta)}{1-\cos(2\alpha)}-\frac{\cos(2\alpha)}{1-\cos(2\alpha)}$$
$$\therefore f(\eta)=\frac{\cos(2\alpha\eta)-\cos(2\alpha)}{1-\cos(2\alpha)}$$

After a sequence of lengthy trigonometry passages (some of the steps are outlined here), function $f(\eta)$ is found to be

$$f(\eta)=1+\frac{1}{2}\csc^2\alpha\left[\sin\left(\frac{\pi}{2}-2\alpha\eta\right)-1\right]$$

Some representative velocity profiles are plotted below. The case {α = 0°} refers to Poiseuille channel flow and is not shown. The case {α = 90°} is the separation point. For α > 90°, separation or backflow must occur in a diverging flow even at zero Reynolds number.



P.3.33 ➡ *Solution*

As mentioned in the problem statement, the radial velocity is given by the general formula

$$u_r \equiv \frac{\partial\psi/\partial\theta}{r^2\sin\theta}$$

We can establish the partial in the numerator using Mathematica, and then apply the *Simplify* command:

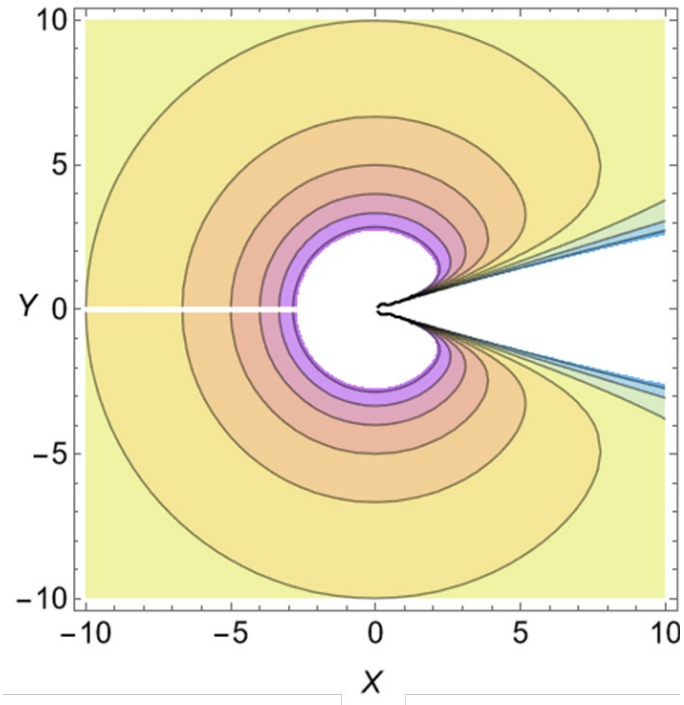
```
In[865]:= Simplify[D[(2*v*r*Sin[θ]^2)/(1 + a - Cos[θ]), θ] / (r^2 * Sin[θ])]

Out[865]= -(v (3 - 4 (1 + a) Cos[θ] + Cos[2 θ]))/(r (1 + a - Cos[θ])^2)
```

That is,

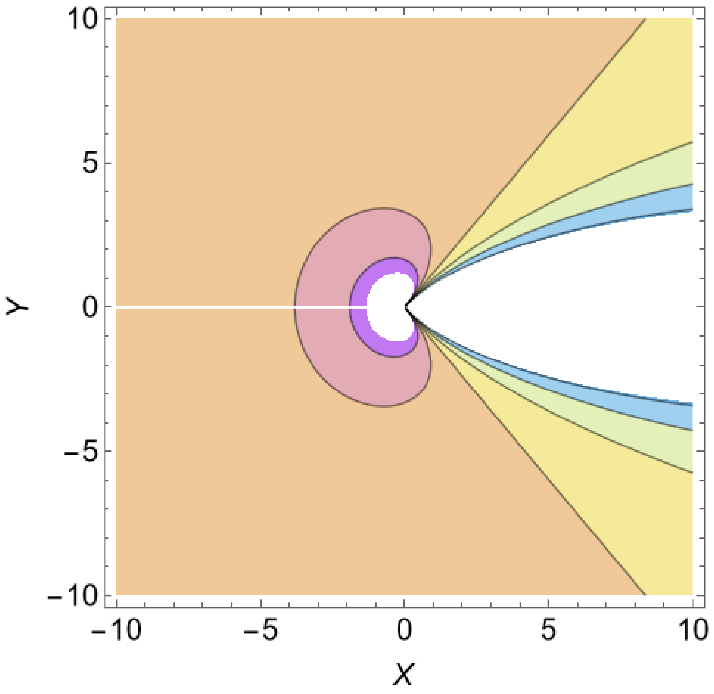
$$u_r=-\frac{v\left[3-4(1+a)\cos(\theta)+\cos(2\theta)\right]}{r\left[1+a-\cos(\theta)\right]^2}$$

Assuming v = 1 and a = 0.001, we may plot contours for u_r using Mathematica's *ContourPlot* function:

```
In[936]:= f = -(3 - 4 * (1 + 0.001) * Cos[θ] + Cos[2 θ])/(r * (1 + 0.001 - Cos[θ])^2);

In[925]:= Clear[x]

In[926]:= Clear[y]

In[937]:= tf = TransformedField["Polar" → "Cartesian", f, {r, θ} → {x, y}];

In[938]:= ContourPlot[tf, {x, -10, 10}, {y, -10, 10}, ColorFunction → "Pastel"]
```



The contours for a = 0.1 are shown next.



The jet width δ is defined as the point where u_r = $0.01u_{max}$. Setting θ in the equation above to zero, we see that the maximum velocity is u_{max} = $4\nu/(ra)$. Then, the jet velocity ratio is

```
In[900]:= Simplify[ ( -(ν (3-4 (1+a) Cos[θ]+Cos[2 θ]))/(r (1+a-Cos[θ])^2) ) / ((4 ν)/(a r)) ]

Out[900]= -(a (3 - 4 (1 + a) Cos[θ] + Cos[2 θ]))/(4 (1 + a - Cos[θ])^2)
```

Output 900 above gives the jet velocity ratio. We move on to equate this relationship to 0.01, so that

$$\frac{u_r}{u_{\max}}=-\frac{a\left[3-4(1+a)\cos(\theta)+\cos(2\theta)\right]}{4\left[1+a-\cos(\theta)\right]^2}=0.01$$

with a = 0.001, we have

```
In[899]:= Clear[a]

In[908]:= a = 0.001;

In[909]:= Solve[-(a (3 - 4 (1 + a) Cos[θ] + Cos[2 θ]))/(2 (1 + a - Cos[θ])^2) == 0.01, θ]

··· Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for con

Out[909]= {{θ → -0.158199}, {θ → 0. - 0.170015 i}, {θ → 0. + 0.170015 i}, {θ → 0.158199}}

In[911]:= 0.158 * 180 / Pi

Out[911]= 9.05273
```

The viable solution, $\theta \approx 0.158$ rad, is highlighted in red. Converting it to degrees, we find $\theta \approx 9.05°$. Proceeding similarly with other values of a, we prepare the following table:

a	0.001	0.003	0.006	0.009	0.012
θ	9.05°	15.0°	20.2°	23.6°	26.2°

The jet mass flow is obtained by integrating over the jet profile at a given value of *r*:

$$\dot{m} = \int_{\text{jet}} \rho u_r dA = \int_0^{\theta(\delta)} \rho \frac{f(\theta,a)}{r} 2\pi r (r d\theta)$$

$$\therefore \dot{m} = \text{const} \times r \times \int_0^{\theta(\delta)} f(\theta,a) d\theta$$

$$\therefore \dot{m} = \underbrace{\left[\text{const} \times \int_0^{\theta(\delta)} f(\theta,a) d\theta \right]}_{=\text{Also constant}} \times r$$

$$\therefore \dot{m} = \text{const.} \times r$$

Therefore, the (laminar) mass flow increases linearly with *r* along the jet axis due to "entrainment" from the ambient fluid outside the jet.

P.3.34 ➡ *Solution*

At terminal velocity, the sphere's net weight in oil equals its drag:

$$W_{\text{net}} = (\rho_{\text{sph}} - \rho) g \frac{\pi}{6} D^3 = \text{Drag} = C_D \frac{\rho V^2}{2} \frac{\pi}{4} D^2$$

Solving for *V* gives a modified form of Stokes' famous solution:

$$V = \left[\frac{4Dg(\rho_{\text{sph}}/\rho - 1)}{3C_D} \right]^{1/2} \quad \text{(I)}$$

The drag coefficient is a function of Reynolds number, i.e., $C_D = f(Re)$. For creeping (Stokes) motion, we have $C_D = 24/Re$ and a velocity given by $V = W/(3\pi D\mu)$, which could serve as a first estimate. At higher Reynolds numbers, the drag coefficient may be estimated with the correlation

$$C_D(\text{sphere}) = \frac{24}{\text{Re}} + \frac{6}{1+\sqrt{\text{Re}}} + 0.4 \quad \text{(II)}$$

Consider first the particle with diameter equal to 0.1 mm. This is the smallest sphere of the three we were told to assess, and hence should be the one most likely to settle via Stokes flow. Accordingly, we estimate the velocity as

$$V = \frac{W}{3\pi D\mu} = \frac{[(7.8 \times 0.88 \times 998) \times 9.81] \times \left[\frac{\pi}{6} \times (0.1 \times 10^{-3})^3\right]}{3\pi \times (0.1 \times 10^{-3}) \times 0.15} = \boxed{2.49 \times 10^{-4} \text{ m/s}}$$

To check if the creeping flow assumption is indeed valid, we may substitute this velocity and other parameters into the definition of Reynolds number:

$$\text{Re} = \frac{\rho V D}{\mu} = \frac{(0.88 \times 998) \times (2.49 \times 10^{-4}) \times (0.1 \times 10^{-3})}{0.15} = 1.46 \times 10^{-4}$$

Since $Re \ll 1$, our assumption is reasonable.

Consider now the 1.0-mm-diameter sphere. Assuming that Stokes flow is also valid for this sphere, we estimate the settling velocity with the usual relation

$$V = \frac{W}{3\pi D\mu} = \frac{[(7.8 \times 0.88 \times 998) \times 9.81] \times \left[\frac{\pi}{6} \times (1.0 \times 10^{-3})^3\right]}{3\pi \times (1.0 \times 10^{-3}) \times 0.15} = \boxed{0.0249 \text{ m/s}}$$

Checking for the assumption of creeping flow:

$$\mathrm{Re} = \frac{\rho VD}{\mu} = \frac{(0.88\times 998)\times 0.0249\times\left(1.0\times 10^{-3}\right)}{0.15} = 0.146$$

Since *Re* ≲ 1, Stokes flow applies to this sphere as well.

Lastly, we have the 10-mm-diameter sphere. Proceeding similarly to the two previous spheres, the settling velocity and the Reynolds number are found to be 2.5 m/s and 147, respectively. This latter result indicates that creeping flow is *not* valid in this particular case. Accordingly, we must estimate velocity and drag coefficient with equations (I) and (II), respectively. We can readily obtain these solutions by defining a MATLAB function and then applying the command *fsolve*:

```
function fcn = stok(x)
%x(1) is velocity
%x(2) is drag coefficient
reyn = 0.88*998*0.01*x(1)/0.15;
fcn(1) = x(2) - 24/reyn - 6/(1+sqrt(reyn)) - 0.4;
fcn(2) = x(1) - sqrt(4*0.01*9.81*(7.8/0.88-1)/(3*x(2)));

>> x0 = [1,1];
>> x = fsolve(fun,x0)

x =

    0.7774    1.7018
```

As can be seen, the code returns a velocity $V \approx$ 0.777 m/s and a drag coefficient $C_D \approx$ 1.70.

P.4.1 ➡ *Solution*

This profile is more realistic than the quadratic approximation, Eq. (4-11), because it satisfies not only $u(0) = 0$, $u(\delta) = U$, and $\partial u(\delta)/\partial y = 0$, but also $\partial^2 u(0)/\partial y^2 = 0$. The maximum thickness θ is approximated by

$$\theta = \int_0^{\delta}\frac{u}{U}\left(1-\frac{u}{U}\right)dy = \delta\int_0^1\left(\frac{3}{2}\eta-\frac{1}{2}\eta^3\right)\left(1-\frac{3}{2}\eta+\frac{1}{2}\eta^3\right)d\eta$$

where $\eta = y/\delta$. The integral above can be readily set up in Mathematica:

In[66]:= Integrate[(3/2 η - 1/2 η^3) * (1 - 3/2 η + 1/2 η^3), {η, 0, 1}] * δ

Out[66]= $\frac{39\,\delta}{280}$

In a similar manner, we can find δ^*, namely $\delta^* = \delta \int (1 - u/U)dy = 3\delta/8$.

Now, the wall shear is

$$\tau_w = -\mu\left.\frac{du}{dy}\right|_{y=0} = -\mu U\left.\left(\frac{3}{2\delta_u}-\frac{3}{2\delta_u}y^2\right)\right|_{y=0}$$

$$\therefore |\tau_w| = \frac{3\mu U}{2\delta}$$

and the friction factor follows as

$$C_f = \frac{2\tau_w}{\rho U^2} = \frac{2}{\rho U^2}\times\frac{3\mu U}{2\delta} = \frac{3\mu}{\rho U\delta} = 2\frac{d\theta}{dx}$$

$$\therefore \frac{3\mu}{\rho U\delta} = 2\frac{d}{dx}\left(\frac{39}{280}\delta\right)$$

Separating variables,

$$\delta d\delta = \frac{140\mu dx}{13\rho U}$$

Integrating and assuming $\delta = 0$ at $x = 0$ brings to

$$\frac{\delta^2}{2} = \frac{140\mu}{13\rho U}x$$

$$\therefore \frac{\delta^2}{2} = \frac{140\mu}{13\rho U}\frac{x^2}{x}$$

$$\therefore \frac{\delta^2}{x^2} = \frac{280\mu}{13\rho U x}$$

$$\therefore \frac{\delta^2}{x^2} = \frac{280}{13\mathrm{Re}_x}$$

$$\therefore \boxed{\frac{\delta}{x} = \frac{4.64}{\sqrt{\mathrm{Re}_x}}}$$

The other two thicknesses follow from θ = $39\delta/280$ = 0.139δ and δ^* = $3\delta/8$ = 0.375δ, so that

$$\therefore \frac{\delta}{x} = \frac{4.64}{\sqrt{\mathrm{Re}_x}} \to \frac{\sqrt{\mathrm{Re}_x}}{x}\delta = 4.64$$

$$\therefore \frac{\sqrt{\mathrm{Re}_x}}{x} \times \frac{\theta}{0.139} = 4.64$$

$$\therefore \boxed{\frac{\theta}{x}\sqrt{\mathrm{Re}_x} = 0.645}$$

For the other thickness, we write

$$\frac{\sqrt{\mathrm{Re}_x}}{x}\delta = 4.64 \to \frac{\sqrt{\mathrm{Re}_x}}{x} \times \frac{\delta^*}{0.375} = 4.64$$

$$\therefore \boxed{\frac{\delta^*}{x}\sqrt{\mathrm{Re}_x} = 1.74}$$

To compute the terms mentioned in (*d*) and (*e*), we use the approximation $\tau_w \approx 1.5\mu U/\delta$ and write

$$C_f = \frac{2\tau_w}{\rho U^2} = \frac{2}{\rho U^2} \times \frac{1.5\mu U}{\delta}$$

$$\therefore C_f = \frac{3\mu}{\rho U \delta}$$

$$\therefore C_f = \frac{3\mu}{\rho U \delta}$$

$$\therefore C_f\sqrt{\mathrm{Re}_x} = \frac{3}{4.64}$$

$$\therefore \boxed{C_f\sqrt{\mathrm{Re}_x} = 0.647}$$

Finally,

$$C_D = \frac{2\theta(L)}{L} = 2 \times \frac{0.645\not{L}}{\sqrt{\mathrm{Re}_L}\not{L}}$$

$$\therefore \boxed{C_D\sqrt{\mathrm{Re}_L} = 1.29}$$

P.4.3 ➡ *Solution*

In addition to satisfying the conditions $u(0)$ = 0, $u(\delta)$ = U, and $\partial u/\partial y(\delta)$ = 0, the sine-wave profile also satisfies the zero-pressure-gradient condition $\partial^2 u(0)/\partial y^2$ = 0 pertaining to flat-plate flow, which is not satisfied by the parabolic profile. The wall shear estimate is

$$\left|\tau_w\right| = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu \times \left[\frac{\pi U}{2\delta}\cos\left(\frac{\pi y}{2\delta}\right)\right]\Bigg|_{y=0}$$

$$\therefore \left|\tau_w\right| = \frac{\pi\mu U}{2\delta} \quad \text{(I)}$$

Then, the momentum thickness estimate is determined as

$$\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \delta\int_0^1 \sin\left(\frac{\pi}{2}\eta\right)\left[1-\sin\left(\frac{\pi}{2}\eta\right)\right]d\eta$$

$$\therefore \theta = \left(-\frac{1}{2}+\frac{2}{\pi}\right)\delta = \boxed{0.137\delta}$$

Next, the friction coefficient can be obtained as

$$C_f = \frac{2\tau_w}{\rho U^2} = \frac{2\left(\pi\mu U/2\delta\right)}{\rho U^2} = 2\frac{d\theta}{dx}$$

$$\therefore \frac{2\left(\pi\mu U/2\delta\right)}{\rho U^2} = 2\frac{d}{dx}\left(0.137\delta\right)$$

$$\therefore \frac{2\left(\pi\mu U/2\delta\right)}{\rho U^2} = 0.274\frac{d\delta}{dx}$$

$$\therefore \frac{\pi\mu}{\rho U}x = 0.274\frac{\delta^2}{2}$$

$$\therefore \frac{\pi\mu}{\rho Ux}x^2 = 0.274\frac{\delta^2}{2}$$

$$\therefore \frac{\pi}{\text{Re}_x} = 0.137\frac{\delta^2}{x^2}$$

$$\therefore \frac{\delta}{x}\sqrt{\text{Re}_x} = \left(\frac{\pi}{0.137}\right)^{1/2}$$

$$\therefore \frac{\delta}{x}\sqrt{\text{Re}_x} = 4.79$$

The accuracy is decent. Putting $\delta(x)$ back into the C_f estimate brings to

$$C_f\sqrt{\text{Re}_x} = \frac{\pi/2}{4.79} = \boxed{0.328}$$

P.4.4 ➡ *Solution*

For air at 20°C and 1 atm, we may take ρ_a = 1.205 kg/m³ and μ = 1.80×10⁻⁵ Pa·s. The density of water at the same temperature may be taken as ρ_w = 998 kg/m³. We begin by estimating the pressure drop Δp,

$$\Delta p = \left(\rho_w - \rho_a\right)gh = \left(998-1.205\right)\times 9.81\times 0.021 = 205\,\text{Pa}$$

so that, from the definition of dynamic pressure,

$$\Delta p = \frac{\rho_{\text{air}}u^2}{2} \rightarrow u = \sqrt{\frac{2\Delta p}{\rho_{\text{air}}}}$$

$$\therefore u = \sqrt{\frac{2\times 205}{1.205}} = 18.5\,\text{m/s}$$

Since u < 20 m/s, the inlet of the Pitot tube is somewhere *within* the boundary layer. The velocity ratio is $f' = u/U$ = 18.5/20 = 0.925. Table 4-1 gives $f'(\eta = 2.6)$ = 0.93060, which is reasonably close to 0.925. Hence, $\eta \approx$ 2.6 and, referring to the definition of this parameter and solving for x,

$$\eta = y\sqrt{\frac{U}{2\nu x}} = y\sqrt{\frac{\rho_a U}{2\mu x}}$$

$$\therefore \eta^2 = y^2 \frac{\rho_a U}{2\mu x}$$

$$\therefore x = \frac{\rho_a U}{2\mu}\left(\frac{y}{\eta}\right)^2$$

$$\therefore x = \frac{1.205\times 20}{2\times\left(1.80\times 10^{-5}\right)}\left(\frac{0.002}{2.6}\right)^2 = \boxed{0.396\,\text{m}}$$

That is, the Pitot tube is approximately 40 cm away from the leading edge of the plate. Lastly, we check the Reynolds number:

$$\text{Re}_x = \frac{\rho_a ux}{\mu} = \frac{1.205\times 20\times 0.396}{\left(1.80\times 10^{-5}\right)} = 530{,}000$$

For a smooth plate and a quiet freestream, transition to turbulence occurs beyond $Re_x \approx 1{,}000{,}000$. Thus, the flow in the vicinities of the tube is laminar, and our results are reasonable.

P.4.10 ➡ *Solution*

Knowing that, for Falkner-Skan flow, $\eta = y[(m+1)U/2\nu x]^{1/2}$, and $f'(\eta) = u/U$, we may compute the boundary-layer displacement thickness as

$$\delta^* = \int_0^\delta \left(1-\frac{u}{U}\right)dy = \left[\frac{2\nu x}{(m+1)U}\right]^{1/2} \underbrace{\int_0^\delta (1-f')\,d\eta}_{=\eta^*}$$

$$\therefore \delta^* = \left[\frac{2\nu x}{(m+1)U}\right]^{1/2}\eta^* \quad \text{(I)}$$

The wall shear, in turn, is

$$\tau_w = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu U f'(0)\left[\frac{(m+1)U}{2\nu x}\right]^{1/2} \quad \text{(II)}$$

Then, since $U = Kx^m$, the pressure gradient is

$$\frac{dp}{dx} = -\rho U\frac{dU}{dx} = -\rho\times Kx^m \times mKx^{m-1} = -\rho m K^2 x^{2m-1}$$

$$\therefore \frac{dp}{dx} = -\frac{\rho m U^2}{x} \quad \text{(III)}$$

Combining equations (I), (II), and (III), the Clauser parameter is found to be

$$\text{Clauser parameter} = \frac{\delta^*}{\tau_w}\frac{dp}{dx} = \frac{\left[\frac{2\nu x}{(m+1)U}\right]^{\frac{1}{2}}\eta^*}{\mu U f'(0)\left[\frac{(m+1)U}{2\nu x}\right]^{\frac{1}{2}}}\times\left(-\frac{\rho m U^2}{x}\right)$$

$$\therefore \text{Clauser p.} = -\frac{2\nu x\eta^*}{\mu f'(0)(m+1)U^2}\frac{\rho m U^2}{x}$$

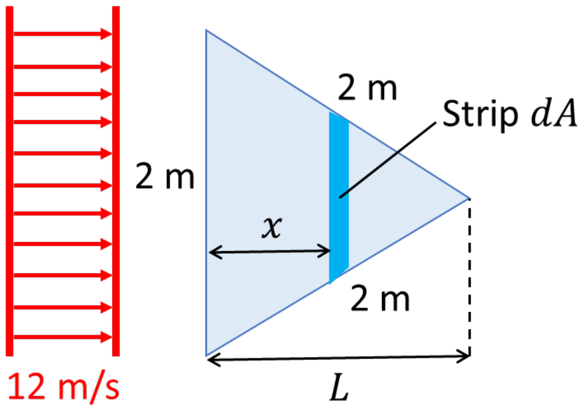
$$\therefore \boxed{\text{Clauser p.} = -\frac{2m\eta^*}{(m+1)f'(0)}}$$

All quantities in the right-hand side are independent of *x*, hence the Clauser parameter is a constant for a given wedge flow. For separating flow, $f'(0) = 0$ and the Clauser parameter → +∞.

P.4.12 ➡ *Solution*

Assume that the local wall shear stress varies only with distance *x* from the leading edge, in accordance with laminar flat-plate theory. Then the strip dA in the figure below would have a uniform shear stress given by Eq. (4-52), with $f'(0)$ = 0.4696:

$$\frac{2\tau_w}{\rho U^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \rightarrow \tau_w = 0.332\left(\frac{\rho\mu}{x}\right)^{\frac{1}{2}} U^{\frac{3}{2}}$$



For a triangle with side length a, the strip has an area $dA = Ldx(1 - x/L)$, where $L = a \times \sin 60°$ and a = 2 m. Then the total shear force on the plate is the integral of the local strip force:

$$F = \int \tau_w dA(2\,\text{sides}) = 2\times 0.332(\rho\mu)^{1/2} U^{3/2} \int_0^L \frac{L-x}{\sqrt{x}}dx$$

$$\therefore F = 0.664\left(\rho\mu U^3\right)^{1/2}\left(2L\sqrt{x} - \frac{2}{3}x^{3/2}\right)\Bigg|_{x=0}^{x=L}$$

$$\therefore F = 0.664\left(\rho\mu U^3\right)^{1/2} \times \frac{4L^{3/2}}{3}$$

$$\therefore F = 0.885\left(\rho\mu U^3\right)^{1/2} L^{3/2}$$

For the case at hand, U = 12 m/s, $L = 2\sin(60°)$ = 1.73 m and, for air under the given conditions, $\rho \approx$ 1.2 kg/m^3 and $\mu \approx 1.8\times10^{-5}$ Pa·s, so that

$$F = 0.885\times\left[1.2\times\left(1.8\times10^{-5}\right)\times12^3\right]^{1/2} \times 1.73^{3/2} = \boxed{0.389\,\mathrm{N}}$$

Lastly, we check the Reynolds number:

$$\mathrm{Re}_L = \frac{\rho UL}{\mu} = \frac{1.2\times12\times1.73}{1.8\times10^{-5}} = 1.38\times10^6$$

This is greater than the turbulent threshold $Re_L \approx 10^6$, hence our results may not be immediately valid.

P.4.13 ➡ *Solution*

Each square cell has four surfaces, and each $(L \times a)$ surface experiences a shear force given by laminar boundary-layer theory, Eq. (4-53):

$$C_D = \frac{1.328}{\sqrt{\mathrm{Re}_L}} = \frac{F}{\frac{1}{2}\rho U^2 aL}$$

$$\therefore \frac{1.328}{\frac{\rho^{1/2}U^{1/2}L^{1/2}}{\mu^{1/2}}} = \frac{F}{\frac{1}{2}\rho U^2 aL}$$

$$\therefore F = 0.664(\rho\mu L)^{1/2} U^{3/2} a$$

An $N \times N$ array of boxes has N^2 cells of four surfaces each, yielding a total shear force

$$\mathbf{F} = 0.664(\rho\mu L)^{1/2} U^{3/2} a \times 4N^2$$

$$\therefore \mathbf{F} = 2.66N^2(\rho\mu L)^{1/2} U^{3/2} a$$

A freebody around the entire array shows that this shear force has to be balanced by a pressure drop across the array, multiplied by the array frontal area:

$$\Delta p_{\text{array}} = \frac{\mathbf{F}}{(N\times a)^2} = \boxed{\frac{2.66}{a}(\rho\mu L)^{1/2}U^{3/2}}$$

Surprisingly, the pressure drop is found to be independent of the number *N* of ducts. The pressure drop achieved is modest: for air at 20°C, flowing in cells with a = 1 cm and L = 10 cm at a speed of 20 m/s, Δp_{array} becomes

$$\Delta p_{\text{array}} = \frac{2.66}{0.01}\times\left[1.20\times\left(1.80\times10^{-5}\right)\times0.1\right]^{1/2}\times20^{3/2} = \underline{35.0\,\text{Pa}}$$

P.4.17 ➡ *Solution*

The jet width can be estimated with equation (4-106),

$$b \approx 21.8\left(\frac{x^2\mu^2}{J\rho}\right)^{1/3} \quad \text{(I)}$$

but the momentum flux J is missing. To evaluate it, we appeal to equation (4-104):

$$u_{\max} \approx 0.4543\left(\frac{J^2}{\rho\mu x}\right)^{1/3} \to 0.20 = 0.4543\times\left[\frac{J^2}{1.20\times\left(1.80\times10^{-5}\right)\times0.50}\right]^{1/3}$$

```
In[105]:= Solve[0.20 == 0.4543 * (j^2 / (1.20 * 1.8 * 10^-5 * 0.5))^(1/3), j]
Out[105]= {{j -> -0.000959937}, {j -> 0.000959937}}
```

That is, $J = 9.60\times10^{-4}$ kg/s^2. Substituting this and other quantities into (I) yields

$$b \approx 21.8\left(\frac{x^2\mu^2}{J\rho}\right)^{1/3} = 21.8\times\left[\frac{0.50^2\times\left(1.8\times10^{-5}\right)^2}{\left(9.60\times10^{-4}\right)\times1.20}\right]^{1/3} = \boxed{0.0900\,\text{m}}$$

The jet has width equal to approximately 9 centimeters. Next, the mass flow can be determined with equation (4-107):

$$\dot{m} = 3.302(J\rho\mu x)^{1/3} = 3.302\times\left[\left(9.60\times10^{-4}\right)\times1.20\times\left(1.80\times10^{-5}\right)\times0.50\right]^{1/3}$$

$$\therefore \boxed{\dot{m} = 0.00720\,\text{kg/s/m}}$$

Lastly, the jet Reynolds number can be expressed in three essentially equivalent ways:

$$\text{Re}_{\text{Jet}} = \frac{\dot{m}}{\mu} = \frac{0.00720}{1.8\times10^{-5}} = \underline{400}$$

$$\text{Re}_{\text{Jet}} = \left(\frac{J\rho x}{\mu^2}\right)^{1/3} = \left[\frac{\left(9.60\times10^{-4}\right)\times1.20\times0.50}{\left(1.80\times10^{-5}\right)^2}\right]^{1/3} = \underline{121}$$

$$\text{Re}_{\text{Jet}} = \frac{\rho u_{\max} b}{\mu} = \frac{1.20\times0.20\times0.09}{1.80\times10^{-5}} = \underline{1200}$$

P.4.18 ➡ *Solution*

The Reynolds number for the flow at hand is

$$\text{Re}_L = \frac{\rho UL}{\mu} = \frac{1.2\times1.0\times0.3}{1.8\times10^{-5}} = 20{,}000$$

The wake velocity is given by equation (4-112),

$$\frac{u_1}{U_0}=C_D\left(\frac{\mathrm{Re}_L}{16\pi}\right)^{1/2}\left(\frac{L}{x}\right)^{1/2}\exp\left(-\frac{U_0 y^2}{4x\nu}\right)$$

At the centerline, y = 0, so that

$$\frac{u_1}{U_0}=C_D\left(\frac{\mathrm{Re}_L}{16\pi}\right)^{1/2}\left(\frac{L}{x}\right)^{1/2}\underbrace{\exp\left(-\frac{U_0\times 0^2}{4x\nu}\right)}_{=1.0}$$

$$\therefore \frac{u_1}{U_0}=C_D\left(\frac{\mathrm{Re}_L}{16\pi}\right)^{1/2}\left(\frac{L}{x}\right)^{1/2}$$

$$\therefore \frac{u_1}{1.0}=0.05\times\left(\frac{20,000}{16\pi}\right)^{1/2}\times\left(\frac{0.3}{3.0}\right)^{1/2}=0.315$$

$$\therefore \boxed{u_1(y=0)=0.315\,\mathrm{m/s}}$$

By analogy with jet theory, the wake half-thickness can be defined as the point where the defect velocity drops to 1% of its maximum velocity, so that, from the Gaussian profile,

$$\exp\left(-\frac{U_0 y^2}{4x\nu}\right)=0.01$$

$$\therefore \ln\left[\exp\left(-\frac{U_0 y^2}{4x\nu}\right)\right]=\ln(0.01)$$

$$\therefore -\frac{U_0 y^2}{4x\nu}=\ln(0.01)$$

$$\therefore y=\sqrt{-\frac{4x\nu\ln(0.01)}{U_0}}$$

$$\therefore y=\sqrt{-\frac{4\times 3.0\times\left(1.80\times 10^{-5}/1.2\right)\times\ln(0.01)}{1.0}}=0.0288\,\mathrm{m}$$

The wake width is twice this value, or $b = 2 \times 0.0288 = 0.0576$ m. Lastly, the wake Reynolds number can be expressed in terms of wake width and the maximum defect velocity:

$$\mathrm{Re}_{\mathrm{Wake}}=\frac{\rho\Delta u_{\max} b}{\mu}=\frac{1.2\times 0.315\times 0.0576}{1.8\times 10^{-5}}=\boxed{1210}$$

Note that the Reynolds number can be defined in other ways, as in the case of jet flow.

P.4.23 ➡ *Solution*

We select the inviscid-flow theory for freestream velocity, $U(x) = 2U_0 \sin(x/a)$, where a cylinder radius and x starts at the front stagnation point (see Fig. 4-24(a) for the geometry). The momentum thickness from Thwaites' method is given by Eq. (4-138), namely

$$\theta^2=\frac{0.45\nu}{U^6}\int_0^x U^5 dx \quad \text{(I)}$$

Thwaites' factor λ is given by Eq. (4-132), that is,

$$\lambda=\frac{\theta^2}{\nu}\left(\frac{dU}{dx}\right)$$

where $dU/dx = (2U_0/a)\cos(x/a)$, and θ^2 can be taken from the quadrature (I), giving

$$\lambda=\frac{\theta^2}{\nu}\left(\frac{dU}{dx}\right)=\frac{\frac{0.45\nu}{U^6}\int_0^x U^5 dx}{\nu}\times\frac{2U_0}{a}\cos\left(\frac{x}{a}\right)$$

$$\therefore \lambda = \frac{0.45}{\left[2U_0 \sin(x/a)\right]^6} \int_0^x \left[2U_0 \sin(x/a)\right]^5 dx \times \frac{2U_0}{a} \cos\left(\frac{x}{a}\right)$$

$$\therefore \lambda = \frac{0.45 \cos(\zeta)}{\sin^6(\zeta)} \int_0^\zeta \sin^5(\zeta)\, d\zeta \quad \text{(II)}$$

where $\zeta = x/a$ and the integral on the right-hand side can be evaluated to yield

$$\int_0^\zeta \sin^5(\zeta)\, d\zeta = \frac{4}{15} \sin^6\left(\frac{\zeta}{2}\right)\left[19 + 18\cos(\zeta) + 3\cos(2\zeta)\right]$$

Then, with λ known, the wall shear stress is given by Eq. (4-139):

$$\tau_w = \frac{\mu U}{\theta} S(\lambda)$$

where $S(\lambda)$ is determined with Eq. (3-140),

$$S(\lambda) \approx (\lambda + 0.09)^{0.62} \quad \text{(III)}$$

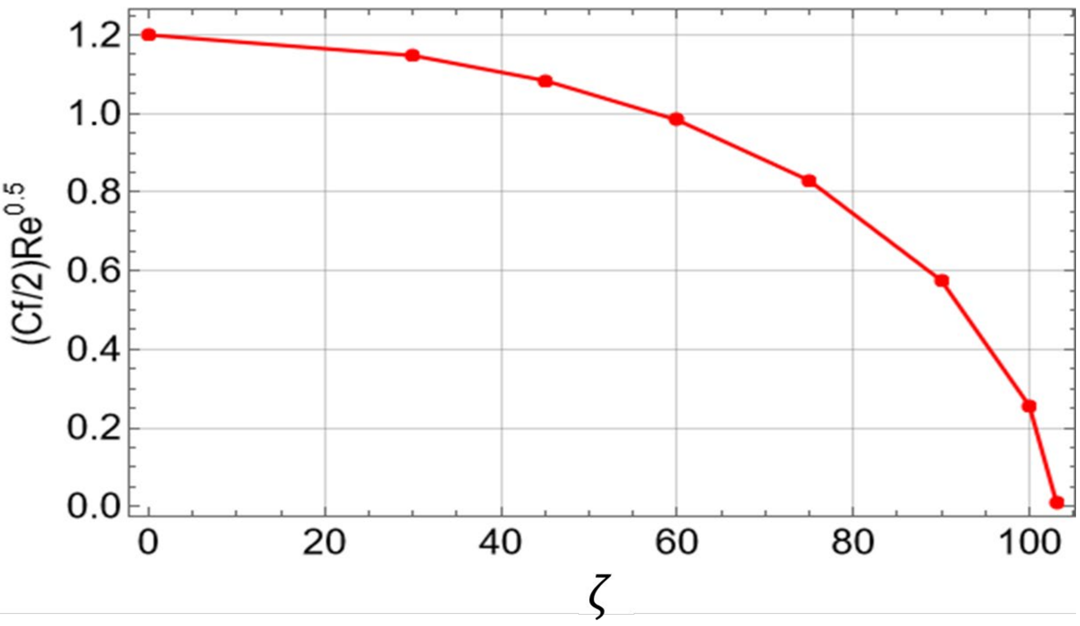
Lastly, we are asked to plot $(C_f/2)\sqrt{Ux/\nu}$, where $C_f/2 = \tau_w/(\rho U^2)$. Putting all this together from the above formulas, we obtain the expression

$$\frac{C_f}{2}\sqrt{\text{Re}_x} = \left[\frac{\zeta \sin^5(\zeta)}{0.45 I}\right]^{1/2} S(\lambda) \quad \text{(IV)}$$

where $I = \int_0^\zeta \sin^5 \zeta \, d\zeta$. Some values are tabulated below.

ζ (deg)	0°	30°	45°	60°	75°	90°	100°	103.1°
ζ (rad)	0	0.524	0.785	1.05	1.31	1.57	1.75	1.80
λ (Eq. II)	0.075	0.0722	0.0677	0.0589	0.0410	0	−0.0603	−0.0900
$S(\lambda)$ (Eq. III)	0.327	0.324	0.318	0.307	0.284	0.225	0.113	0.00357
$\frac{C_f}{2}\sqrt{Re}$ (Eq. IV)	1.20	1.15	1.08	0.984	0.830	0.575	0.256	0.00771

Then, we plot the red row versus the blue row, as shown. The skin friction values show good agreement with Terrill's digital computer results plotted in Figure 4-24(b).



P.4.24 ➡ *Solution*

We illustrate here the particular case of Tani (1949), namely

$$U = 1 - x^2$$

which, when taken to the fifth power, becomes

```
In[153]:= Expand[(1 - x^2)^5]
Out[153]= 1 - 5 x^2 + 10 x^4 - 10 x^6 + 5 x^8 - x^10
```

To compute θ, we take the square root of Eq. (4-138),

$$\theta \approx \sqrt{\frac{0.45\nu}{U^6}\underbrace{\int_0^x U^5 dx}_{=I}}$$

$$\therefore \theta \approx \frac{(0.45\nu I)^{1/2}}{U^3}$$

where

$$I = -\frac{x^{11}}{11} + \frac{5x^9}{9} - \frac{10x^7}{7} + 2x^5 - \frac{5x^3}{3} + x$$

so that

$$\lambda = -\frac{0.9xI}{U^6}$$

To find the value of λ that corresponds to the stagnation point, which is read from Table 4-5 to be x = 0.268, we substitute above and solve for λ:

```
In[159]:= U = 1 - X^2;

In[161]:= x = 0.268;

In[162]:= λ = - 0.9 * x * Integrate[U^5, {X, 0, x}] / (1 - 0.268^2)^6

Out[162]= -0.0899843
```

Note that $\lambda \approx -0.09$, as it should be (Sect. 4-6.9).

Now, the local skin friction coefficient is given by $C_f = 2\tau_w/\rho U^2$, where wall shear τ_w can be expressed as

$$\tau_w = \frac{\mu U}{\theta} S(\lambda)$$

with $S(\lambda)$ given by Eq. (4-140). We ultimately obtain

$$C_f\sqrt{\mathrm{Re}_x} = \frac{2[S(\lambda)](1-x^2)^{3/2}}{(0.45H)^{1/2}} \quad \text{(I)}$$

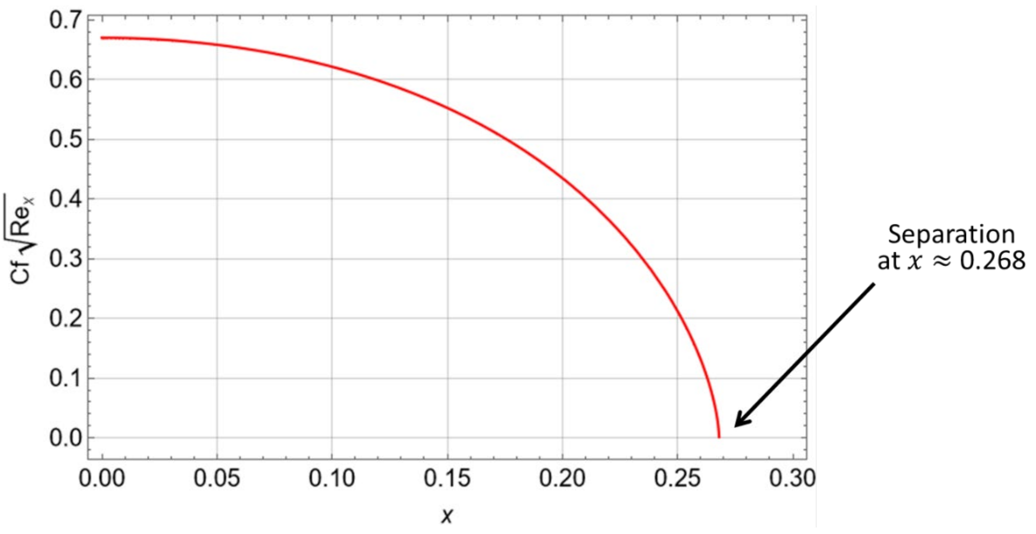
where $H = I/x$, or

$$H = \frac{I}{x} = -\frac{x^{10}}{11} + \frac{5x^8}{9} - \frac{10x^6}{7} + 2x^4 - \frac{5x^2}{3} + 1$$

Let us plot (I) with Mathematica:

```
II=Integrate[(1-x^2)^5,x]
x-(5 x^3)/3+2 x^5-(10 x^7)/7+(5 x^9)/9-x^11/11
λp = -((0.9*x*II)/(1-x^2)^6)
-((0.9 x (x-(5 x^3)/3+2 x^5-(10 x^7)/7+(5 x^9)/9-x^11/11))/(1-x^2)^6)
Sλ=(λp+0.09)^0.62;
H=II/x;
Plot[(2*Sλ*(1-x^2)^(3/2))/(0.45*H)^(1/2),{x,0,0.3}]
```

The ensuing plot is shown below and reproduces the computer-based solution quite well.



P.4.25 ➡ *Solution*

From one-dimensional continuity, the average velocity *U(x)* may be expressed as

$$U_0 bW = Ub\left(W + 2x\tan\theta\right) \to U_0 W = U\left(W + 2x\tan\theta\right)$$

$$\therefore U = \frac{U_0 W}{W + 2x\tan\theta}$$

$$\therefore U = \frac{U_0}{1+\dfrac{2x\tan\theta}{W}} \quad \text{(I)}$$

Following Thwaites' approach, we must find the position *x* at which $\lambda \approx -0.09$, noting that, from Eq. (4-132),

$$\lambda = \frac{\theta^2 U'}{\nu} = \frac{\theta^2}{\nu}\frac{dU}{dx}$$

Replacing θ with (4-138),

$$\lambda = \frac{\theta^2 U'}{\nu} = \frac{0.45}{U^6}\frac{dU}{dx}\int_0^x U^5 dx$$

with *U* given by Eq. (I). We could program the derivative and integral in the expression above with Mathematica, but, with reference to (I), note that the velocity profile has the form $(1 + x)^{-1}$; Table 4-5 indicates that the exact laminar-separation-point position for such a distribution occurs at $x \approx 0.158$. Accordingly, we set

$$\frac{2x\tan\theta}{W} \approx 0.158 \to x_{\text{sep}} = L_{\text{sep}} \approx \frac{0.079W}{\tan\theta_{\text{sep}}}$$

so that, if $L = 1.5W$,

$$L = \frac{0.079\times\left(L/1.5\right)}{\tan\theta_{\text{sep}}}$$

$$\therefore 1 = \frac{0.079\times\left(1/1.5\right)}{\tan\theta_{\text{sep}}}$$

$$\therefore \tan\theta_{\text{sep}} = 0.0527$$

$$\therefore \boxed{\theta_{\text{sep}} = 3.02^{\circ}}$$

Laminar flow has much less resistance to separation than turbulent flow.

P.5.1 ➡ *Solution*

Equation (5-9) in the textbook is stated as

$$\left(U_1 - U_2\right)^2 > \frac{\left[g\left(\rho_1 - \rho_2\right) + \alpha^2\mathfrak{T}\right]\left(\rho_1 + \rho_2\right)}{\alpha\rho_1\rho_2} \quad \text{(I)}$$

Differentiating with respect to α, we obtain

```
In[176]:= Simplify[D[((g (ρ1 - ρ2) + α^2 * T) * (ρ1 + ρ2))/(α * ρ1 * ρ2), α]]

Out[176]= ((ρ1 + ρ2) (T α^2 - g ρ1 + g ρ2))/(α^2 ρ1 ρ2)
```

Setting to zero the first term in parentheses in the numerator would lead to a negative density, which is absurd. Thus, we zero out the second term in parentheses instead, so that

$$\mathfrak{T}\alpha^2 - g\rho_1 + g\rho_2 = 0$$

$$\therefore \mathfrak{T}\alpha^2 - g(\rho_1 - \rho_2) = 0$$

$$\therefore \boxed{\alpha = \sqrt{\frac{g(\rho_1 - \rho_2)}{\mathfrak{T}}}}$$

as we intended to show. To see that this is indeed a minimum, we may obtain the second derivative of (I) and substitute the solution above,

```
In[177]:= Simplify[D[((g (ρ1 - ρ2) + α^2 * T) * (ρ1 + ρ2))/(α * ρ1 * ρ2), {α, 2}]] /. (α -> (g * (ρ1 - ρ2))/T)
```

$$\text{Out[177]=} \frac{2\,T^3\,(\rho_1^2 - \rho_2^2)}{g^2\,\rho_1\,(\rho_1 - \rho_2)^3\,\rho_2}$$

Note that all parameters in the numerator and denominator of the expression above are positive, which confirms that the optimum we've found is a minimum.

For air blowing over gasoline, $\rho_1 \approx 680$ kg/m^3, $\rho_2 \approx 1.2$ kg/m^3, and $\mathfrak{T}$ = 0.022 N/m. The minimum corresponds to an α such that

$$\therefore \alpha = \sqrt{\frac{9.81\times(680-1.2)}{0.022}} = 550\,\text{m}^{-1}$$

The corresponding wavelength is

$$\alpha = \frac{2\pi}{\lambda} \to \lambda = \frac{2\pi}{\alpha}$$

$$\therefore \lambda = \frac{2\pi}{550} = \boxed{0.0114\,\text{m}}$$

Lastly, the corresponding difference $U_1 - U_2$ is determined as

$$(U_1 - U_2) = \sqrt{\frac{\left[g(\rho_1 - \rho_2) + \alpha^2\mathfrak{T}\right](\rho_1 + \rho_2)}{\alpha\rho_1\rho_2}}$$

$$\therefore (U_1 - U_2) = \sqrt{\frac{\left[9.81\times(680-1.2) + 550^2\times 0.022\right]\times(680+1.2)}{550\times 680\times 1.2}} = 4.495\,\text{m/s}$$

$$\therefore \boxed{(U_1 - U_2) \approx 4.50\,\text{m/s}}$$

P.5.2 ➡ *Solution*

If surface tension and the velocities are negligible, then we are studying the effect of gravity on an interface between two still fluids of different density. The expression for wave frequency, from Eq. (5-8), reduces to

$$\sigma = \sqrt{-\frac{\alpha g(\rho_1 - \rho_2)}{(\rho_1 + \rho_2)}}$$

If an interfacial wave is produced, its propagation speed is

$$C = \frac{\lambda}{T} = \frac{2\pi/T}{2\pi/\lambda} = \frac{\sigma}{\alpha}$$

Combining the two foregoing expressions yields

$$\frac{\sigma}{\alpha} = \frac{\sqrt{-\frac{\alpha g(\rho_1 - \rho_2)}{(\rho_1 + \rho_2)}}}{\alpha} = \sqrt{-\frac{\alpha g(\rho_1 - \rho_2)}{\alpha^2(\rho_1 + \rho_2)}}$$

$$\therefore C = \sqrt{-\frac{g(\rho_1 - \rho_2)}{\alpha(\rho_1 + \rho_2)}}$$

$$\therefore C = i\sqrt{\frac{g(\rho_1-\rho_2)}{\alpha(\rho_1+\rho_2)}}$$

$$\therefore \boxed{C = i\sqrt{\frac{g\lambda(\rho_1-\rho_2)}{2\pi(\rho_1+\rho_2)}}}$$

This is the speed of a *deep-water* wave, far from any upper or lower boundaries. For air layered over fresh water, $\rho_1 \approx 1000$ kg/m^3, $\rho_2 \approx 1.2$ kg/m^3, and λ may be taken as, say, 3 m; accordingly,

$$\therefore C = i\sqrt{\frac{9.81\times 3.0\times(1000-1.2)}{2\pi\times(1000+1.2)}} = \underline{2.16\,\text{m/s}}$$

If $\rho_1 < \rho_2$, then the argument of the square root is *positive* and the wave is unstable. Presumably the two layers will overturn so that the heavy fluid goes to the bottom.

P.5.7 ➡ *Solution*

Referring to Table 5-1 of the textbook, we see that stagnation flow (β = 1.0) becomes unstable at Re_θ = 5636. Meanwhile, from the Falkner-Skan solutions in Table 4-2, the dimensionless momentum thickness in stagnation flow is θ^* = 0.29235, as shown below.

β	$Re_{\delta^*,\text{crit}}$	$Re_{\theta,\text{crit}}$	$c_{i,\max}$	$\left(-\frac{\alpha\delta^*}{Re_{\delta^*}}\right)_{\max}\times 10^7$
+1.0	12,490	5,636	0.0065	1.14
0.8	10,920	4,874	0.0070	1.35
0.6	8,890	3,909	0.0075	1.67
0.5	7,680	3,344	0.0080	1.92
0.4	6,230	2,679	0.0085	2.42
0.3	4,550	1,927	0.0095	3.45
0.2	2,830	1,174	0.0104	6.0
0.1	1,380	556	0.0129	15.7
0.05	865	342	0.0154	32

η								
	β	−0.19884	−0.18	0.0	0.3	1.0	2.0	10.0
	f_0''	0.0	0.12864	0.46960	0.77476	1.23259	1.68722	3.67523
	η^*	2.35885	1.87157	1.21678	0.91099	0.64790	0.49743	0.24077
	θ^*	0.58544	0.56771	0.46960	0.38574	0.29235	0.23079	0.11523
0.0		0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1		0.00099	0.01376	0.04696	0.07597	0.11826	0.15876	0.31843
0.2		0.00398	0.02933	0.09391	0.14894	0.22661	0.29794	0.54730
0.3		0.00895	0.04668	0.14081	0.21886	0.32524	0.41854	0.70496
0.4		0.01591	0.06582	0.18761	0.28569	0.41446	0.52190	0.81043
0.5		0.02485	0.08673	0.23423	0.34938	0.49465	0.60964	0.87954

As a result, we may write, with m = 1,

$$\theta\left(\frac{m+1}{2}\frac{U}{\nu x}\right)^{1/2} = 0.294 \rightarrow \theta\left(\frac{U}{\nu x}\right)^{1/2} = 0.294$$

$$\therefore \theta = 0.294\left(\frac{\nu x}{U}\right)^{1/2}$$

Dividing both sides by x:

$$\frac{\theta}{x} = \frac{0.294}{(Ux/\nu)^{1/2}}$$

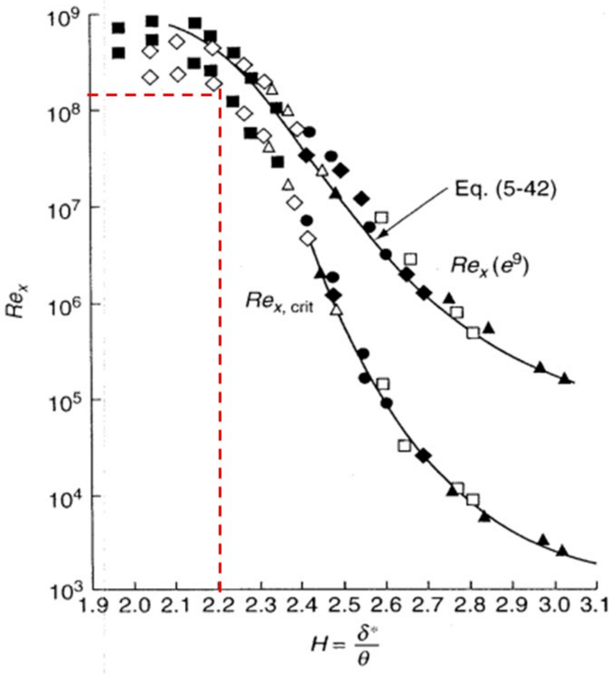
Solving for Re_θ and setting it to the critical value mentioned above,

$$\frac{\theta}{x} = \frac{0.294}{(Ux/\nu)^{1/2}} \rightarrow \text{Re}_\theta = 0.294\,\text{Re}_x^{1/2}$$

$$\therefore 0.294\,\text{Re}_{x,\text{crit}}^{1/2} = 5936$$

$$\therefore \text{Re}_{x,\text{crit}} = \left(\frac{5936}{0.294}\right)^2 = \boxed{4.08\times 10^8}$$

This very large Reynolds number, of the order of 100 million, underscores the stability of the favorable-gradient stagnation-flow velocity profile in focus. We can check this result by comparing it to the Wazzan *et al.* (1981) shape factor correlation in Fig. 5-31. From Table 5-1, $Re_{\delta^*_{\text{crit}}}$ = 12,490, so that $H = \delta^*/\theta$ = 12,490/5636 = 2.22. Entering this value into Fig. 5-31, we read $Re_{x,\text{crit}} \approx 2\times10^8$, which is in good agreement with the estimate above.



P.5.9 ➡ *Solution*

This is not a similarity solution, so we need estimates of $\theta(x)$ and $H(x)$ along the wall, assuming that instability occurs before boundary-layer separation. We could use Thwaites' method from Sect. 4-6.7, noting that, for Howarth decelerating flow,

$$\theta^2 = 0.075\frac{\nu L}{U_0}\left[\left(1-\frac{x}{L}\right)^{-6} - 1\right]$$

but

$$\lambda = -0.075\left[\left(1-\frac{x}{L}\right)^{-6} - 1\right] \quad \text{(I)}$$

so that

$$\text{Re}_\theta = \left(-\lambda\,\text{Re}_L\right)^{1/2} = \left(-\lambda\times10^6\right)^{1/2} = 1000\left(-\lambda\right)^{1/2} \quad \text{(II)}$$

Suppose that x/L = 0.01. The corresponding λ can be obtained from eq. (I), giving

$$\lambda = -0.075\left[\left(1-0.01\right)^{-6} - 1\right] = -0.00466$$

so that, substituting in (II),

$$\text{Re}_\theta = 1000\left[-\left(-0.00466\right)\right]^{1/2} = 68.3$$

Noting that $z = 0.25 - \lambda = 0.25 + 0.00466 = 0.2547$, shape factor $H(\lambda)$ can be determined with Eq. (4-141),

$$H(\lambda) \approx 2.0 + 4.14z - 83.5z^2 + 854z^3 - 33337z^4 + 4576z^5$$

$$\therefore H(\lambda = 0.2547) \approx 2.0 + 4.14\times0.2547 - 83.5\times0.2547^2 + 854\times0.2547^3$$

$$-3337\times0.2547^4 + 4576\times0.2547^5 = 2.610$$

The corresponding $Re_{\delta\star} = H \times Re_\theta$ = 2.610 × 68.3 = 178. Then, we may enter H = 2.610 into Fig. 5-12 to read a critical Reynolds number $Re_{\delta\star,\text{crit}} \approx 300$, which is substantially greater than 178; accordingly, instability does not occur at x/L = 0.01. We proceed to another value of x/L – say, x/L = 0.03. The corresponding λ is calculated as (equation (I))

$$\lambda = -0.075\left[\left(1-0.03\right)^{-6} - 1\right] = -0.0150$$

Next, Reynolds number Re_θ is computed as (equation (II))

$$\mathrm{Re}_\theta = 1000\left[-(-0.0150)^{1/2}\right] = 122$$

Next, z = 0.25 − λ = 0.25 + 0.0150 = 0.265, and shape factor H follows as

$$\therefore H(\lambda = 0.2650) \approx 2.0 + 4.14\times 0.2650 - 83.5\times 0.2650^2 + 854\times 0.2650^3$$
$$-3337\times 0.2650^4 + 4576\times 0.2650^5 = 2.650$$

so that $Re_{\delta\star}$ = H × Re_θ = 2.650 × 122 = 323. Entering shape factor H = 2.650 onto Fig. 5-12, we read a critical Reynolds number $Re_{\delta\star,\mathrm{crit}}$ ≈ 350; this is within 8% of our calculated $Re_{\delta\star}$ of 323, so we may surmise that instability takes hold at a position close to x/L = 0.03; of course, we could continue to iterate to improve the solution further, but the resolution of Fig. 5-12 is too poor to allow for substantial improvement.

P.5.11 ➡ *Solution*

This is a variation of Prob. 5-9, in that the goal is the same – namely, to find the position at which instability ensues – but the velocity distribution is different. We've already discussed a similar velocity profile in Problem 4.23. Recall that Thwaites' factor is expressed as

$$\lambda = \frac{\theta^2}{\nu}\left(\frac{dU}{dx}\right)$$

which, noting that $dU/dx = 2U_0\cos(x/a)/a$ and solving for momentum thickness,

$$\lambda = \frac{\theta^2}{\nu}\times\frac{2U_0}{a}\cos\left(\frac{x}{a}\right)$$
$$\therefore \theta^2 = \frac{\nu\lambda a}{2U_0\cos(\zeta)}$$

where we have used the substitution $\zeta = x/a$. Also note that λ was found to be

$$\lambda = \frac{0.45\cos(\zeta)}{\sin^6(\zeta)}\int_0^\zeta \sin^5(\zeta)\,d\zeta \quad \text{(II)}$$

where

$$\int_0^\zeta \sin^5(\zeta)\,d\zeta = \frac{4}{15}\sin^6\left(\frac{\zeta}{2}\right)\left[19 + 18\cos(\zeta) + 3\cos(2\zeta)\right]$$

Assume first that ζ = 30° = 0.524 rad. Substituting in (II), λ is calculated to be

$$\lambda = \frac{0.45\cos(0.524)}{\sin^6(0.524)}\int_0^{0.524}\sin^5(\zeta)\,d\zeta = 0.0721$$

Noting that $Re_D = 10^6$ as given, we may determine Re_θ as

$$\mathrm{Re}_\theta = \sin(\zeta)\left[\lambda\,\mathrm{Re}_D/\cos(\zeta)\right]^{1/2} = \sin 30^\circ\times\left(0.0721\times 10^6/\cos 30^\circ\right)^{1/2} = 144$$

Noting that z = 0.25 − λ = 0.25 − 0.0721 = 0.178, we appeal to Eq. (4-141) and compute the shape factor H,

$$H(z) \approx 2.0 + 4.14z - 83.5z^2 + 854z^3$$
$$-3337z^4 + 4576z^5$$

$$\therefore H(z = 0.0721) \approx 2.0 + 4.14\times 0.0721 - 83.5\times 0.0721^2 + 854\times 0.0721^3$$
$$-3337\times 0.0721^4 + 4576\times 0.0721^5 = 2.103$$

so that

$$\mathrm{Re}_{\delta^*} = H\times\mathrm{Re}_\theta = 2.103\times 144 = 303$$

and

$$\mathrm{Re}_x = \mathrm{Re}_D\,\zeta\sin(\zeta) = 10^6 \times 1.05 \times \sin(1.05) = 9.11 \times 10^5$$

Now, we turn to Fig. 5-12; entering a shape factor $H \approx 2.10$, we read $Re_{\delta\star,\mathrm{crit}} \approx 30{,}000$, which is much greater than the $Re_{\delta\star}$ calculated above, so we may surmise that instability has not ensued. Also, we can enter H into Fig. 5-31 to read $Re_{x,\mathrm{crit}} \approx 8\times10^8$, which is also much greater than the Re_x computed above, so instability likewise has not been attained relatively to Wazzan's correlation.

Now, let ζ equal, say, 60° = 1.05 rad. Substituting in (II) gives parameter λ:

$$\lambda = \frac{0.45\cos(1.05)}{\sin^6(1.05)}\int_0^{1.05}\sin^5(\zeta)\,d\zeta = 0.0588$$

Reynolds number Re_θ is, in turn,

$$\mathrm{Re}_\theta = \sin(\zeta)\left[\lambda\,\mathrm{Re}_D/\cos(\zeta)\right]^{1/2} = \sin 60^\circ \times \left(0.0588\times10^6/\cos 60^\circ\right)^{1/2} = 298$$

Noting that $z = 0.25 - \lambda = 0.25 - 0.0588 = 0.191$, we proceed to determine shape factor $H(z)$,

$$H(z=0.191) \approx 2.0 + 4.14\times0.191 - 83.5\times0.191^2 + 854\times0.191^3$$
$$-3337\times0.191^4 + 4576\times0.191^5 = 2.417$$

so that

$$\mathrm{Re}_{\delta^*} = H\times\mathrm{Re}_\theta = 2.417\times298 = 720$$

and

$$\mathrm{Re}_x = \mathrm{Re}_D\,\zeta\sin(\zeta) = 10^6 \times 1.05 \times \sin(1.05) = 9.11 \times 10^5$$

With reference to Fig. 5-12, we enter shape factor H and read $Re_{\delta\star,\mathrm{crit}} \approx 4000$, which is substantially greater than our $Re_{\delta\star}$ and hence indicates that instability has not ensued. Also, referring to Fig. 5-31, we read $Re_{x,\mathrm{crit}} \approx 5\times10^6$, which is substantially greater than our Re_x and likewise indicates that instability has not been attained.

Now, let ζ equal, say, 80° = 1.40 rad. The corresponding λ is

$$\lambda = \frac{0.45\cos(1.40)}{\sin^6(1.40)}\int_0^{1.40}\sin^5(\zeta)\,d\zeta = 0.0306$$

so that

$$\mathrm{Re}_\theta = \sin(\zeta)\left[\lambda\,\mathrm{Re}_D/\cos(\zeta)\right]^{1/2} = \sin 80^\circ \times \left(0.0306\times10^6/\cos 80^\circ\right)^{1/2} = 418$$

Noting that $z = 0.25 - \lambda = 0.25 - 0.0306 = 0.219$, we proceed to update the shape factor $H(z)$,

$$H(z=0.219) \approx 2.0 + 4.14\times0.219 - 83.5\times0.219^2 + 854\times0.219^3$$
$$-3337\times0.219^4 + 4576\times0.219^5 = 2.501$$

so that

$$\mathrm{Re}_{\delta^*} = H\times\mathrm{Re}_\theta = 2.501\times418 = 1050$$

and

$$\mathrm{Re}_x = \mathrm{Re}_D\,\zeta\sin(\zeta) = 10^6 \times 1.40 \times \sin(1.40) = 1.38 \times 10^6$$

Referring yet again to Fig. 5-12, we enter shape factor H and read $Re_{\delta\star,\mathrm{crit}} \approx 1100$, which is reasonably close to our estimate of $Re_{\delta\star}$. As in the case of the previous problem, further improvement is made difficult by the poor resolution of Fig. 5-12. In turn, entering H into Fig. 5-31, we read $Re_{x,\mathrm{crit}} \approx 8\times10^5$, which is also reasonably close to the Re_x computed above. We make no further improvement to the solution and take $\zeta \approx 80°$ or $(x/a)_{crit}$ = 1.40 as our final result.

P.5.15 ➡ *Solution*

Referring to Table 5-1, we see that for separating flow β = −0.1988 the system becomes unstable when Re_θ = 17. In turn, we can refer to Table 4-2 and read that the dimensionless momentum thickness for separating flow is θ^* = 0.58544.

TABLE 4-2
Numerical values of the streamwise velocity $f'(\eta)$ for Falkner–Skan similarity flows

η								
	β	−0.19884	−0.18	0.0	0.3	1.0	2.0	10.0
	f''_0	0.0	0.12864	0.46960	0.77476	1.23259	1.68722	3.67523
	η^*	2.35885	1.87157	1.21678	0.91099	0.64790	0.49743	0.24077
	θ^*	0.58544	0.56771	0.46960	0.38574	0.29235	0.23079	0.11523
0.0		0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1		0.00099	0.01376	0.04696	0.07597	0.11826	0.15876	0.31843
0.2		0.00398	0.02933	0.09391	0.14894	0.22661	0.29794	0.54730

Noting that m = −0.09043 for the stagnation point, we can refer to Eq. (4-70) and write

$$\theta\left(\frac{m+1}{2}\frac{U}{\nu x}\right)^{1/2}=0.585\rightarrow\theta\left(\frac{-0.09043+1}{2}\frac{U}{\nu x}\right)^{1/2}=0.585$$

$$\therefore\theta\times 0.674\times\left(\frac{U}{\nu x}\right)^{1/2}=0.585$$

$$\therefore\frac{\theta}{x}\times\mathrm{Re}_x^{1/2}=0.868$$

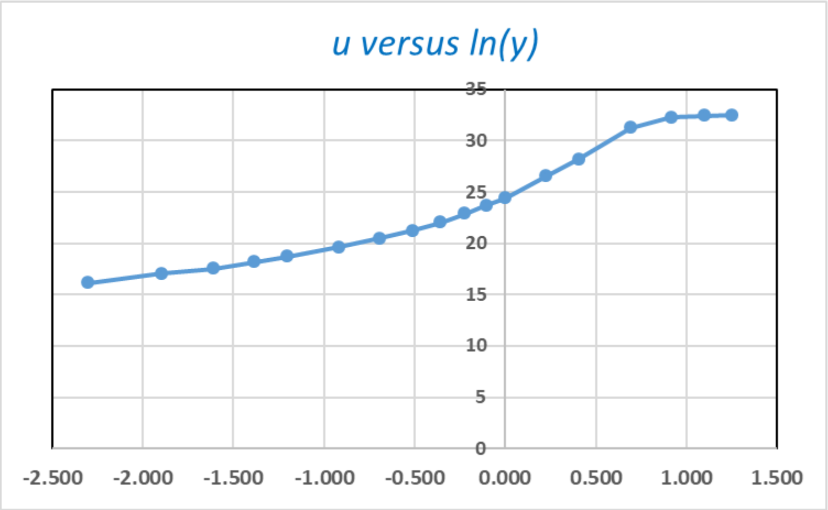
$$\therefore\mathrm{Re}_\theta=0.868\,\mathrm{Re}_x^{1/2}=17$$

$$\therefore\mathrm{Re}_x=\left(\frac{17}{0.868}\right)^2=\boxed{384}$$

This awfully low Reynolds number underscores the fact that the *S*-shaped separating-flow velocity profile is remarkably unstable. Note that we cannot check this result against the Wazzan *et al.* (1981) correlation because *H* in Fig. 5-31 only goes up to 3.1, and the shape factor for Falkner-Skan separating flow is $H = \delta^*/\theta$ = 2.35885/0.58544 = 4.03.

P.6.4 ➡ *Solution*

For air at 24°C and 1 atm, take ρ = 0.00230 slug/ft^3 and ν = 1.65×10^{-4} ft^2/s. We begin by plotting *u* versus *log(y)* to get a preliminary understanding of the data at hand:



The plot shows a linear trend for the first five or six points – the logarithmic overlap. Thus, we may appeal to the log-law correlation,

$$\frac{u}{u^*}=\frac{1}{\kappa}\ln\left(\frac{u^*y}{\nu}\right)+B$$

where von Kármán's constant $\kappa \approx$ 0.41 and $B \approx$ 5.0. To obtain an estimate of friction velocity, we may substitute one of the first few points of *u(y)* versus *y* data in the log-law and solve for u^*. Take, for instance, y = 0.15 in. and *u(y)* = 17.02 ft/s; noting that ν = 1.66×10^{-4} ft^2/s and using Mathematica's *FindRoot* command:

```
FindRoot[17.02/uStar-1/0.41*Log[(uStar*0.15/12)/(1.66*10-4)]-5.0,{uStar,0.5}]
{uStar->1.08186}
```

That is, $u^* \approx 1.08$ ft/s. It is appropriate to check the inner variable y^+ that corresponds to this friction velocity:

$$y^+ = \frac{yv^*}{\nu} = \frac{(0.15/12)\times 1.08}{1.66\times 10^{-4}} = 81.3$$

Since $30 < y^+ < 300$, we are in the logarithmic overlap range. We proceed to compute the wall shear stress:

$$\tau_w = \rho\left(v^*\right)^2 = 0.00230\times 1.08^2 = 0.00268\text{ psf}$$

Before computing Clauser's parameter β, we need an estimate of the thickness δ^*. To obtain such an estimate, we may integrate the velocity data numerically to obtain δ^*; in doing so, we find $\delta^* \approx 0.60$ in. Accordingly, we substitute the pertaining variables into the definition of Clauser's parameter to obtain

$$\beta = \frac{\delta^*}{\tau_w}\frac{dp}{dx} = \frac{\delta^*}{\rho\left(v^*\right)^2}\left(-\rho U\frac{dU}{dx}\right)$$

$$\therefore \beta = \frac{(0.60/12)}{1.08^2}\times\left[-32.50\times(-1.06)\right] = \boxed{1.48}$$

Coles and Hirst (1968) found $\beta = 1.358$ for the same dataset. Now, to find Coles' wake parameter Π, we evoke Eq. (6-47):

$$u^+ \approx \frac{1}{\kappa}\ln\left(y^+\right) + B + \frac{2\Pi}{\kappa}f\left(\frac{y}{\delta}\right)$$

where we take $f(y/\delta) \approx \sin^2(\pi\eta/2)$ as recommended in Eq. (6-46). Let us fit parameter Π with Mathematica's *FindFit* function. We first normalize the data, as shown below; then, we evoke *FindFit* to compute the parameter Π:

```
uPlus = u/1.08;
yPlus = y*1.08/(1.66*10-4)/122;
dataList=Transpose@{yPlus,uPlus};
FindFit[dataList,1/0.41*Log[Y]+5.0+(2Π)/0.41*Sin[Pi/2 Y/(3.5/12)]2,{Π},Y]
{Π->2.4234}
```

That is, the value of the Coles parameter for the data at hand is close to 2.42.

P.6.6 ➡ *Solution*

The desired formula is hidden within Eq. (6-53), which computes average pipe-flow velocity by integrating across the entire pipe:

$$u_{\text{avg}} = v^*\left[\underline{\frac{1}{\kappa}\ln\left(\frac{av^*}{\nu}\right) + B} - \frac{3}{2\kappa}\right]$$

The underscored terms are actually equal to u_{max}/v^* (neglecting the slight 'wake' at the centerline). Accordingly, we may rearrange the equation as follows:

$$u_{\text{avg}} = v^*\left[\frac{u_{\max}}{v^*} - \frac{3}{2\kappa}\right] = u_{\max} - \frac{3}{2\kappa}v^*$$

$$\therefore u_{\text{avg}} + \frac{3}{2\kappa}v^* = u_{\max}$$

$$\therefore \frac{u_{\max}}{u_{\text{avg}}} = 1 + \frac{3}{2\kappa}\frac{v^*}{u_{\text{avg}}}$$

However, $v^*/u_{avg} = \sqrt{\Lambda/8}$, so that

$$\frac{u_{\max}}{u_{\text{avg}}} = 1 + \frac{3}{2\kappa}\sqrt{\frac{\Lambda}{8}}$$

$$\therefore \frac{u_{\max}}{u_{\text{avg}}} = 1 + \frac{3}{2\times 0.41}\sqrt{\frac{\Lambda}{8}}$$

$$\therefore \boxed{\frac{u_{\max}}{u_{\text{avg}}} = 1 + 1.29\sqrt{\Lambda}}$$

as we intended to show.

P.6.7 ➡ *Solution*

Dividing the flow rate *Q* by the cross-sectional area *A* gives the average velocity of the flow:

$$u_{\text{avg}} = \frac{30/3600}{\left(\frac{\pi\times 0.03^2}{4}\right)} = 11.8\,\text{m/s}$$

Checking the Reynolds number:

$$\text{Re}_D = \frac{\rho u_{\text{avg}} D}{\mu} = \frac{1000\times 11.8\times 0.03}{0.001} = 354{,}000$$

That is, $Re_D \gg 2000$ and the flow is well into the turbulent regime. We proceed to apply the friction factor correlation

$$\frac{1}{\sqrt{\Lambda}} = 2.0\log_{10}\left(\text{Re}_D\sqrt{\Lambda}\right) - 0.8$$

We could use a design chart or solve the equation above numerically. Taking the latter approach, we employ the MATLAB code

```
function fct = darcy(L)
%L is the friction factor and ReD is the Reynolds number
ReD = 354000;
fct = L^(-0.5) - 2*log10(ReD*L^0.5) + 0.8;
>> fun = @darcy;
>> x0 = [0.1];
>> x = fsolve(fun,x0)
x =

    0.0140
```

That is, Λ = 0.0140. The wall shear is then

$$\tau_w = \Lambda\frac{1}{8}\rho u_{\text{avg}}^2 = 0.0140\times\frac{1}{8}\times 998\times 11.8^2 = \boxed{243\,\text{Pa}}$$

The pressure drop per unit length Δ*x* = 1 m is, in turn,

$$\Delta p = \tau_w \times \frac{4\Delta x}{D} = 243\times\frac{4\times 1.0}{0.03} = 32{,}400\,\text{Pa/m} = \boxed{32.4\,\text{kPa/m}}$$

Lastly, the centerline velocity can be obtained by dint of the equation we were told to demonstrate in the previous problem:

$$u_{\max} = u_{\text{avg}}\left(1+1.29\sqrt{\Lambda}\right) = 11.8\times\left(1+1.29\times\sqrt{0.0140}\right) = \boxed{13.6\,\text{m/s}}$$

To find the maximum flow rate for which laminar flow holds, we first set the Reynolds number to 2000 and solve for average velocity:

$$\text{Re}_D = \frac{\rho u_{\text{avg}} D}{\mu} \le 2000 \rightarrow \frac{998\times u_{\text{avg}}\times 0.03}{0.001} = 2000$$

$$\therefore u_{\text{avg,max}} = 0.0668\,\text{m/s}$$

This corresponds to a flow rate such that

$$Q_{\max} = u_{\text{avg,max}} A = 0.0668\times\left(\frac{\pi}{4}\times 0.03^2\right) = 4.72\times 10^{-5}\,\text{m}^3/\text{s}$$

$$\therefore \boxed{Q_{\max} = 0.170\,\text{m}^3/\text{h}}$$

Finding the flow rate at which the wall shear equals 100 Pa is more intricate, because at first we have neither the velocity nor the friction factor Λ. We shall use two equations; the first one is the wall shear, which can be expressed as

$$\tau_w = \Lambda \frac{1}{8} \rho u_{\text{avg}}^2 = 100 \rightarrow \frac{1}{8} \times 998 \times \Lambda u_{\text{avg}}^2 = 100$$

$$\therefore \Lambda u_{\text{avg}}^2 = 0.802 \;\; \text{(I)}$$

The second one is the Prandtl formula

$$\frac{1}{\sqrt{\Lambda}} = 2.0 \log_{10}\left(\text{Re}_D \Lambda^{1/2}\right) - 0.8$$

where Re_D = 998 × 0.03 × u_{avg}/0.001 = 29,940u_{avg}. Accordingly, we set up the following MATLAB code:

```
function ff = prandtl(x)
%x(1) is the average velocity, and x(2) is the friction factor
ReD = 29940*x(1);
ff(1) = x(2)*x(1)^2 - 0.802;
ff(2) = x(2)^(-0.5) - 2.0*log10(ReD*x(2)^0.5) + 0.8;
>> fun = @prandtl;
>> x0 = [1, 0.1];
>> x = fsolve(fun, x0)
x =

    7.2151    0.0154
```

That is, $u_{avg} \approx$ 7.22 m/s and Λ = 0.0154. The corresponding flow rate is

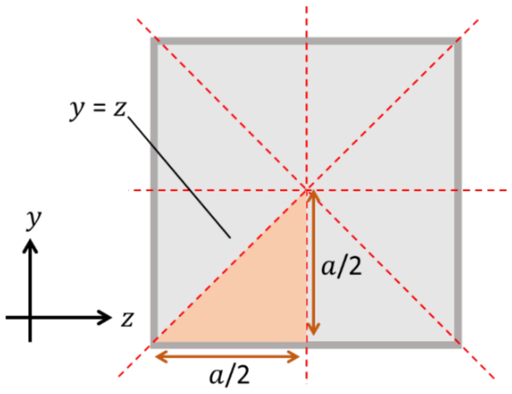
$$Q = u_{\text{avg}} A = 7.22 \times \left(\frac{\pi}{4} \times 0.03^2\right) = 0.00510 \, \text{m}^3/\text{s}$$

$$\boxed{Q_{\tau_w \to 100} = 18.36 \, \text{m}^3/\text{h}}$$

A flow rate of approximately 18.4 cubic meters per hour would yield a wall shear of 100 Pa.

P.6.9 ➡ *Solution*

By double symmetry, we need to consider only one-eighth of the duct cross-section, as shown to the side. Although the flow in this region is three-dimensional (owing to secondary circulation toward the corners), the mean velocity u in the duct-axis direction is well-represented by the log-law, Eq. (6-38a), based on the distance y from the wall, as sketched in the figure. In general, however, the wall shear stress varies with z along the wall, being zero in the corners and rather flat near z = a/2.



The total volume flow rate in the duct is 8 times the flow rate in the shaded area:

$$Q = 8\int_0^{a/2} dz \int_0^z \left\{ v^*(z) \left[\frac{1}{\kappa} \ln\left(\frac{y v^*}{\nu}\right) + B \right] dy \right\}$$

If, as a first approximation, we assume a constant shear velocity $v^*(z)$ around the cross-section, so that the integration yields

$$u_{\text{avg}} = \frac{Q}{a^2} = v_{\text{avg}}^* \left[\frac{1}{\kappa} \ln\left(\frac{a^+}{2}\right) + B - \frac{3}{2\kappa} \right] \;\; ; \text{ where } a^+ = \frac{a v^*}{\nu}$$

Now, by definition, the friction factor Λ for any duct is such that $v^*_{\text{avg}}/u_{\text{avg}} = \sqrt{\Lambda/8}$. Therefore, the above relation may be restated as a friction factor formula:

$$\sqrt{\frac{8}{\Lambda}} = \frac{1}{\kappa}\ln\left(\mathrm{Re}_a\sqrt{\frac{\Lambda}{32}}\right) + B - \frac{3}{2\kappa}$$

where Re_a is a Reynolds number based on the side a of the square cross-section. With κ = 0.41 and B = 5.0, the equation may be simplified as

$$\sqrt{\frac{8}{\Lambda}} = \frac{1}{\kappa}\ln\left(\mathrm{Re}_a\sqrt{\frac{\Lambda}{32}}\right) + B - \frac{3}{2\kappa} \rightarrow \frac{2.83}{\sqrt{\Lambda}} = \frac{1}{0.41}\ln\left(\mathrm{Re}_a \times 0.177\sqrt{\Lambda}\right) + 5.0 - \frac{3}{2\times 0.41}$$

$$\therefore \frac{2.83}{\sqrt{\Lambda}} = 2.44\left[\ln\left(\mathrm{Re}_a\sqrt{\Lambda}\right) + \ln\left(0.177\right)\right] + 1.34$$

$$\therefore \frac{2.83}{\sqrt{\Lambda}} = 2.44\ln\left(\mathrm{Re}_a\sqrt{\Lambda}\right) + 2.44\ln\left(0.177\right) + 1.34$$

$$\therefore \frac{2.83}{\sqrt{\Lambda}} = 2.44\ln\left(\mathrm{Re}_a\sqrt{\Lambda}\right) - 2.89$$

$$\therefore \frac{1}{\sqrt{\Lambda}} = 0.862\ln\left(\mathrm{Re}_a\sqrt{\Lambda}\right) - 1.02$$

$$\therefore \frac{1}{\sqrt{\Lambda}} = 1.99\log_{10}\left(\mathrm{Re}_a\sqrt{\Lambda}\right) - 1.02$$

Interestingly, these are the same numbers as those for turbulent pipe flow, as shown in the simplified form of Eq. (6-53) in the textbook. The difference, of course, is that the Reynolds number Re_a is based on the side of the square cross-section, a. This is reasonable in view of the fact that the hydraulic diameter of such a cross-section is

$$D_h = \frac{4\times\text{Area}}{\text{Perimeter}} = \frac{4\times a^2}{(4a)} = a$$

as expected. The friction formula may be made more accurate if we modified our approach to account for a *variable* $v^{\star}(z)$ that satisfies the condition that τ_w = 0 (or shear velocity $v^{\star}$ = 0) in the duct corners. However, this requires modelling how $v^{\star}(z)$ varies; one possibility would be to use the power law $v^{\star}/v^{\star}_{\max} = (2z/a)^{1/7}$.

P.6.13 ➡ *Solution*

We could equate total thickness δ or momentum thickness θ at the transition point, thereby defining an *effective* origin x_0 for the ensuing turbulent boundary layer:

$$\underline{\text{Equate }\delta}:\ 5.0\mathrm{Re}_{\mathrm{tr}}^{1/2} = 0.16\,\mathrm{Re}_{x_0}^{6/7}\quad \text{(I)}$$

$$\underline{\text{Equate }\theta}:\ 0.664\mathrm{Re}_{\mathrm{tr}}^{1/2} = 0.01555\,\mathrm{Re}_{x_0}^{6/7}\quad \text{(II)}$$

Both give about the same estimate of the "virtual" turbulent origin x_0, but equating θ is more realistic. Downstream of Re_{tr}, local boundary layer parameters are computed on the basis of the effective local Reynolds number

$$\mathrm{Re}_{x,\mathrm{eff}} = \mathrm{Re}_x - \mathrm{Re}_{\mathrm{tr}} + \mathrm{Re}_{x_0}$$

This Reynolds number can then be substituted into thickness and coefficient correlations, as in

$$\frac{\delta}{x_{\mathrm{eff}}} = \frac{0.16}{\mathrm{Re}_{x,\mathrm{eff}}^{1/7}}\ ;\ C_f = \frac{0.027}{\mathrm{Re}_{x,\mathrm{eff}}^{1/7}}$$

Schlichting's classic textbook *Boundary Layer Theory* suggests modifying turbulent drag formulas as follows:

$$C_D\left(L > x_{\mathrm{tr}}\right) = \frac{0.031}{\mathrm{Re}_L^{1/7}} - \frac{A}{\mathrm{Re}_L}$$

One computes "*A*" by equating the above formula to the laminar drag at transition, $C_D = 1.328/Re_{tr}^{1/2}$.

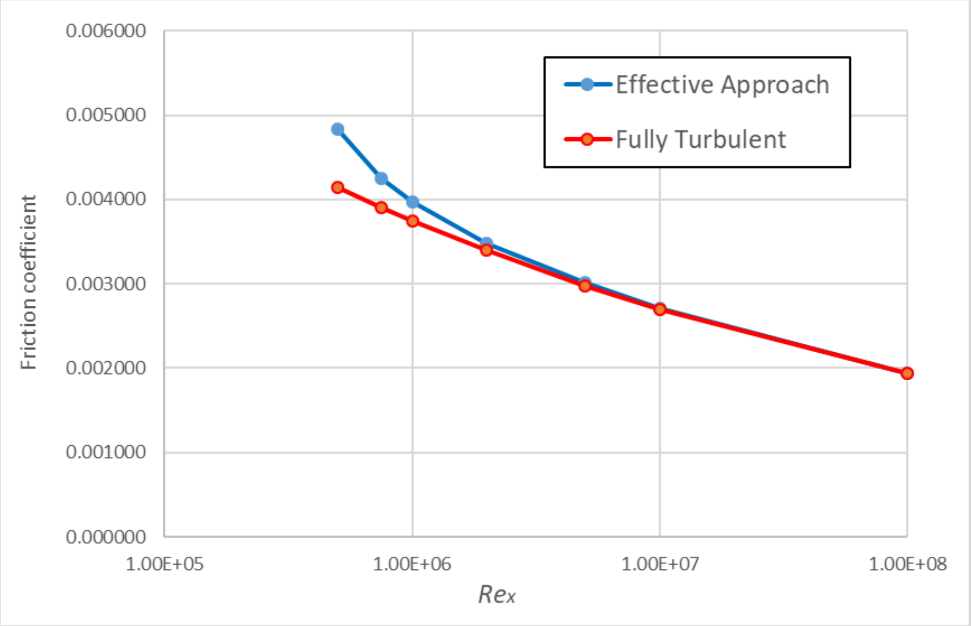
Let us compare the two approaches for an assumed value of $Re_{\text{tr}}^{1/2}$ = 500,000. Equating θ as written in (II), we obtain

$$0.664\times 500{,}000^{1/2} = 0.01555\,\text{Re}_{x_0}^{6/7}$$

$$\therefore \text{Re}_{x_0} = \left(\frac{0.664\times 500{,}000^{1/2}}{0.01555}\right)^{\frac{7}{6}} = 168{,}000$$

We proceed to list some tabulated values of friction coefficient as computed from the 'effective' Reynolds number approach (blue column) or from the fully-turbulent flow assumption (red column). A graph comparing the two approaches is also provided.

Re_x	Re_{x0}	$Re_{x,eff}$	C_f (Eff.)	C_f (F.T.)
5.00E+05	1.68E+05	1.68E+05	0.004838	0.004142
7.50E+05	1.68E+05	4.18E+05	0.004249	0.003909
1.00E+06	1.68E+05	6.68E+05	0.003974	0.003752
2.00E+06	1.68E+05	1.67E+06	0.003487	0.003398
5.00E+06	1.68E+05	4.67E+06	0.003010	0.002981
1.00E+07	1.68E+05	9.67E+06	0.002713	0.002700
1.00E+08	1.68E+05	9.97E+07	0.001944	0.001943



Clearly, there is substantial disagreement between the two approaches near the onset of turbulence; as the Reynolds number becomes larger, however, the results become increasingly similar. We leave the analysis of $\delta(x)$ to the reader.

P.6.14 ➡ *Solution*

For water at 20°C and 1 atm, we may take ρ = 998 kg/m³ and μ = 0.001 Pa·s. The corresponding Reynolds number is

$$\text{Re}_L = \frac{\rho u L}{\mu} = \frac{998\times 6.0\times 1.0}{0.001} = 5.99\times 10^6$$

Since $Re_L \approx$ 6,000,000, the first one-sixth of the plate flow is laminar, whereas the rest is turbulent. Out of convenience, we ignore laminar flow and assume that flow is fully turbulent. Then, we appeal to Eqs. (6-70) and write

$$\frac{\delta}{x} = \frac{0.16}{\text{Re}_L^{1/7}} \to \delta(x=L) = \frac{0.16L}{\text{Re}_L^{1/7}}$$

$$\therefore \delta(x=1.0) = \frac{0.16\times 1.0}{\left(5.99\times 10^6\right)^{1/7}} = 0.0172\,\text{m}$$

$$\delta^* \equiv \frac{\delta}{8} = \frac{0.0172}{8} = 0.00215\,\text{m}$$

$$C_f(x=L)=\frac{0.027}{\mathrm{Re}_L^{1/7}}=\frac{0.027}{\left(5.99\times10^6\right)^{1/7}}=0.00291$$

The wall shear at $x = L$ is then

$$\tau_w(x=L)=\left[C_f(x=L)\right]\frac{\rho u^2}{2}=0.00291\times\underbrace{\frac{998\times6.0^2}{2}}_{18{,}000}=52.3\,\mathrm{Pa}$$

Referring to Eq. (6-80), the drag coefficient is one-sixth larger than the trailing-edge value of C_f:

$$C_D=\frac{7}{6}C_f(x=L)=\frac{7}{6}\times0.00291=0.00340$$

so that

$$D=0.00340\times18{,}000\times(1.0\times0.6)=36.7\,\mathrm{N}$$

Alternatively, we may compute C_f with Eq. (6-78):

$$C_f\approx\frac{0.455}{\ln^2(0.06\,\mathrm{Re}_L)}=\frac{0.455}{\ln^2\left[0.06\times\left(5.99\times10^6\right)\right]}=0.00278$$

so that

$$\tau_w(x=L)=0.00278\times18{,}000=\boxed{49.9\,\mathrm{Pa}}$$

and $C_D = 7C_f/6 = 7 \times 0.00278/6 = 0.00324$, with the result that

$$D=0.00324\times18{,}000\times(1.0\times0.6)=\boxed{35.0\,\mathrm{N}}$$

P.6.15 ➡ *Solution*

From Prob. 6-14, we found the Reynolds number to be $Re_L = 5.99\times10^6$. With roughness height k = 0.1 mm = 10^{-4} m, we have the ratio $L/k = 1.0/10^{-4}$ = 10,000, and it is difficult to verify whether the flow is 'fully rough'. From Eq. (6-61), the criterion for fully-rough flow would be

$$k^+=\frac{ku^*}{\nu}=\underbrace{\frac{kU}{\nu}}_{=\mathrm{Re}_k}\underbrace{\frac{u^*}{U}}_{=1/\lambda}=\frac{\mathrm{Re}_k}{\lambda}>60$$

That is,

$$\mathrm{Re}_k>60\lambda$$

With λ in the range of 20 to 40, the inequality above becomes $Re_k \gtrsim$ 1200 or so. For the case at hand, the roughness Reynolds number $Re_k = \rho uk/\mu = 998 \times 6.0 \times 10^{-4}/0.001 = 598.8$, which places us in the "intermediate roughness" regime, for which Eq. (6-82) is valid. The equation in question is

$$\mathrm{Re}_x\approx1.73\left(1+0.3k^+\right)e^Z\left[Z^2-4Z+6-\frac{0.3k^+}{1+0.3k^+}(Z-1)\right]$$

where $Z = \kappa\lambda = 0.41\lambda$ and $k^+ = Re_x(k/x)/\lambda = (5.99\times10^6) \times (10^{-4}/1.0)/\lambda = 599/\lambda$. Substituting above brings to

$$5.99\times10^6\approx1.73(1+0.3\times599/\lambda)\times e^{0.41\lambda}\times\left[(0.41\lambda)^2-4\times(0.41\lambda)+6-\frac{0.3\times599/\lambda}{1+0.3\times599/\lambda}(0.41\lambda-1)\right]$$

We can solve this nonlinear equation with Mathematica's *FindRoot* command:

```
In[308]:= FindRoot[
   5.99 * 10^6 - 1.73 * (1 + 0.3 * 599 / λ) * Exp[0.41 * λ] *
    ((0.41 * λ)^2 - 4 * 0.41 * λ + 6 - (0.3 * 599 / λ)/(1 + 0.3 * 599 / λ) * (0.41 * λ - 1)), {λ, 10}]

Out[308]= {λ → 22.0701}
```

Thus, $\lambda(L)$ = 22.07. The corresponding shear velocity u^* is determined as

$$\lambda(L)=\frac{U}{u^*(L)}\rightarrow u^*(L)=\frac{U}{\lambda(L)}$$

$$\therefore u^*(L)=\frac{6.0}{22.07}=0.272\text{ m/s}$$

The wall shear is then

$$u^*=\sqrt{\frac{\tau_w}{\rho}}\rightarrow\tau_w=\rho\left(u^*\right)^2$$

$$\therefore\tau_w=998\times0.272^2=\boxed{73.8\text{ Pa}}$$

Further, $k^+(L) = ku^*/\nu = 10^{-4}\times 0.272/10^{-6}$ = 27.2, which is between 4 and 60 and hence places us in the transitional-roughness regime. We may calculate ΔB from Eq. (6-62), namely

$$\Delta B(L)=\frac{1}{\kappa}\ln\left(1+0.3k^+\right)=\frac{1}{0.41}\times\ln\left(1+0.3\times27.2\right)=\boxed{5.40}$$

Then, appealing to the law-of-the-wall, Eq. (6-60), and neglecting any 'wake', we may solve for the thickness δ:

$$\frac{U}{u^*}=\frac{1}{\kappa}\ln\left(\frac{\delta u^*}{\nu}\right)+B-\Delta B=22.07$$

$$\therefore\frac{1}{0.41}\ln\left(\frac{\delta\times0.272}{10^{-6}}\right)+5.0-5.40=22.07$$

$$\therefore 2.44\ln\left(272{,}000\delta\right)=22.07+0.40$$

$$\therefore\ln\left(272{,}000\delta\right)=9.209$$

$$\therefore\delta=\frac{\exp(9.209)}{272{,}000}=\boxed{0.0367\text{ m}}$$

This is about 2.1 times greater than the thickness predicted in Prob. 6-14 ($\delta\approx$ 0.0172 m).

Now, there is no simple method to calculate the drag in the intermediate-roughness regime. One way to go is to compute τ_w for several positions x along the plate and check for some mathematical trend. The data we need are tabulated below.

x	Re_x	λ	u^*	τ_w
0.01	59880	13.37	0.449	200.988
0.05	299400	16.23	0.370	136.394
0.1	598800	17.53	0.342	116.915
0.2	1197600	18.86	0.318	101.007
0.4	2395200	20.23	0.297	87.789
0.6	3592800	21.04	0.285	81.160
0.8	4790400	21.62	0.278	76.864
1	5988000	22.07	0.272	73.761

We can attempt to fit this τ_w versus x data to a power law with Mathematica's *FindFit* command:

```
In[330]:= τw = {200.988, 136.394, 116.915, 101.007, 87.789, 81.160, 76.864, 73.761}
Out[330]= {200.988, 136.394, 116.915, 101.007, 87.789, 81.16, 76.864, 73.761}

In[331]:= x = {0.01, 0.05, 0.1, 0.2, 0.4, 0.6, 0.8, 1.0}
Out[331]= {0.01, 0.05, 0.1, 0.2, 0.4, 0.6, 0.8, 1.}

In[332]:= Clear[data]

In[333]:= data = Transpose@{x, τw}
Out[333]= {{0.01, 200.988}, {0.05, 136.394}, {0.1, 116.915},
  {0.2, 101.007}, {0.4, 87.789}, {0.6, 81.16}, {0.8, 76.864}, {1., 73.761}}

In[439]:= dragFit = FindFit[data, a * X^b, {a, b}, X]
Out[439]= {a → 71.546, b → -0.222012}
```

Thus, the data are found to follow the power law

$$\tau_w(x) = 71.55x^{-0.222}$$

with $x \in (0.01;\ 1.0)$ m. Integration of the above formula from $x = 0$ to $x = L = 1.0$ m should yield the drag force on one side of the plate:

$$\text{Drag} = \int_0^{L=1\,\text{m}} \left(71.55x^{-0.222} \times \text{Width}\right) dx = \int_0^{L=1\,\text{m}} \left(71.55x^{-0.222} \times 0.6\right) dx$$

$$\therefore \boxed{\text{Drag} = 55.2\ \text{N}}$$

Recall that the drag on the plate without roughness was 35.0 N; the result above is about 58% higher.

P.6.17 ➡ *Solution*

First, we note that the quantity $v_w^+ u^+$ in Eq. (6-86), when evaluated at $y = \delta$, $u = U$, actually yields the blowing parameter β:

$$v_w^+ U^+ = \frac{v_w}{v^*}\frac{U}{v^*} = \frac{v_w}{U}\frac{U^2}{\left(v^*\right)^2} = \frac{v_w}{U}\left(\frac{2}{C_f}\right) = \beta$$

Further, the quantity v_w^+ itself is related to local skin friction:

$$v_w^+ = \frac{v_w}{v^*} = \frac{v_w}{U}\frac{U}{v^*} = \frac{v_w}{U}\left(2/C_f\right)^{1/2}$$

Accordingly, Eq. (6-86) can be stated as

$$\frac{2}{v_w^+}\left[\left(1 + v_w^+ u^+\right)^{1/2} - 1\right] = \frac{2\left(2/C_f\right)^{1/2}}{\beta}\left[\left(1+\beta\right)^{1/2} - 1\right] = \frac{1}{\kappa}\ln\left(\delta^+\right) + B$$

or

$$\frac{2\left(2/C_f\right)^{1/2}}{\beta}\left[\left(1+\beta\right)^{1/2} - 1\right] = \left(\frac{2}{C_{f,0}}\right)^{1/2} \text{ if } \delta^+ \approx \delta_0^+$$

where this latter substitution follows since $[\ln(\delta^+)/\kappa + B]$ exactly equals the no-blowing ratio U/v^* – we assume, without proof, that this dimensionless thickness is not affected by the blowing. Bear in mind that the Greek letter β refers to the 'blowing parameter', whereas Latin B refers to the log-law constant $B \approx 5.0$. We can solve the equation above for the skin friction ratio C_f/C_{f0}, giving:

$$\left\{\frac{2\left(2/C_f\right)^{1/2}}{\beta}\left[\left(1+\beta\right)^{1/2} - 1\right]\right\}^2 = \left[\left(\frac{2}{C_{f,0}}\right)^{1/2}\right]^2$$

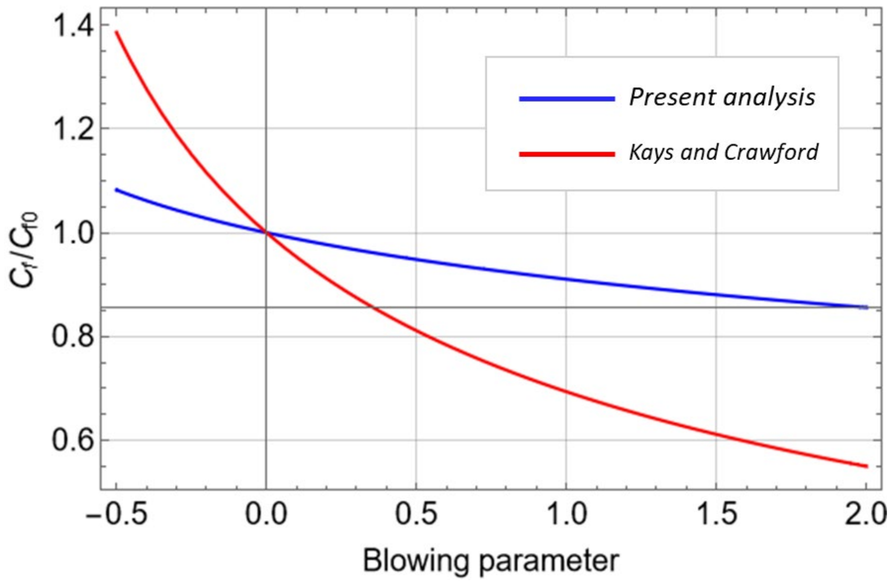
$$\therefore \frac{C_f}{C_{f,0}} = \left\{\frac{2}{\beta}\left[\left(1+\beta\right)^{1/2} - 1\right]\right\}^2$$

This is to be compared to the result by Kays and Crawford, namely

$$\frac{C_f}{C_{f,0}} = \frac{\ln\left(1+\beta\right)}{\beta}$$

We can then plot the two relations in the range $-0.5 < \beta < 2.0$ with the following Mathematica code:

```
p1 = Plot[(2/\[Beta]*((1 + \[Beta])^(1/2) - 1))^( 1/2), {\[Beta], -0.5,
2.0}, PlotStyle -> Blue]

p2 = Plot[Log[1 + \[Beta]]/\[Beta], {\[Beta], -0.5, 2.0}, PlotStyle ->
Red]

Show[p1, p2, PlotRange -> All, Frame -> True, GridLines -> Automatic]
```



We see that suction ($\beta < 0$) increases friction (and heat transfer), while blowing ($\beta > 0$) decreases friction (and heat transfer). This was also the case for laminar flat-plate flow, as shown in Fig. 4-15*b*, but turbulent flow does not "blow off" at finite β.

P.6.18 ➡ *Solution*

As usual, for water at 20°C, ρ = 998 kg/m³ and μ = 0.001 Pa·s. We first compute the average flow velocity,

$$V = \frac{Q}{A} = \frac{0.06}{\pi \times \dfrac{0.08^2}{4}} = 11.9\,\text{m/s}$$

We were told that the ratio of average to centerline velocity is 0.85, so the maximum flow velocity is V_{max} = 11.9/0.85 = 14.0 m/s. Now, we evoke Stevenson's logarithmic-law of the wall with suction or blowing:

$$\frac{2}{v_w^+}\left[\left(1+v_w^+u^+\right)^{1/2}-1\right] \approx \frac{1}{\kappa}\ln\left(y^+\right)+B$$

$$\therefore \frac{2}{v_w^+}\left[\left(1+v_w^+\times\frac{V_{\text{max}}}{v^*}\right)^{1/2}-1\right] = \frac{1}{0.41}\ln\left(\frac{\rho R v^*}{\mu}\right)+5.0$$

$$\therefore \frac{2}{v_w^+}\left[\left(1+v_w^+\times\frac{14.0}{v^*}\right)^{1/2}-1\right] = \frac{1}{0.41}\ln\left(\frac{998\times 0.04\times v^*}{0.001}\right)+5.0$$

$$\therefore \frac{2}{v_w^+}\left[\left(1+14.0\frac{v_w^+}{v^*}\right)^{1/2}-1\right] = 2.44\ln\left(39{,}900v^*\right)+5.0 \quad \text{(I)}$$

with $v_w^+ = v_w/v^\star = +0.01/v^\star$, We can easily solve the equation above with Mathematica's *FindRoot* command:

```
vw=0.01/vs;
FindRoot[2/vw*((1+14.0*vw/vs)^1/2-1)-2.44*Log[39900*vs]-
5.0,{vs,0.1}]
{vs->0.415881}
```

That is, $v^\star$ = 0.416 m/s. The corresponding wall shear stress is:

$$\tau_w = \rho\left(v^*\right)^2 = 998\times 0.416^2 = \boxed{173\,\text{Pa}}$$

Now, we set v_w^+ to 0.0 and apply *FindRoot* a second time; note that, in order to avoid a division by zero in equation (I), we write $v_w^+ = 0.000001/v^\star$ instead of $v_w^+ = 0/v^\star$:

```
vw=0.000001/vs;
FindRoot[2/vw*((1+14.0*vw/vs)^1/2-1)-2.44*Log[39900*vs]-
5.0,{vs,0.1}]
{vs->0.481637}
```

That is, $v^\star$ = 0.482 m/s. The corresponding wall shear stress is:

$$\tau_w = \rho\left(v^*\right)^2 = 998\times 0.482^2 = \boxed{232\,\text{Pa}}$$

Finally, we set v_w^+ to −0.01 m/s and appeal to *FindRoot* a third time:

```
FindRoot[2/vw*((1+14.0*vw/vs)^1/2-1)-2.44*Log[39900*vs]-
5.0,{vs,0.1}]
{vs->0.549825 +1.8084*10^-24 I}
```

The imaginary part is tiny and can be attributed to the solution algorithm; thus, v^* = 0.550 m/s, so that

$$\tau_w = \rho\left(v^*\right)^2 = 998\times 0.550^2 = \boxed{302\,\text{Pa}}$$

P.6.20 ➡ *Solution*

The two formulas we need to compare are Eq. (6-62),

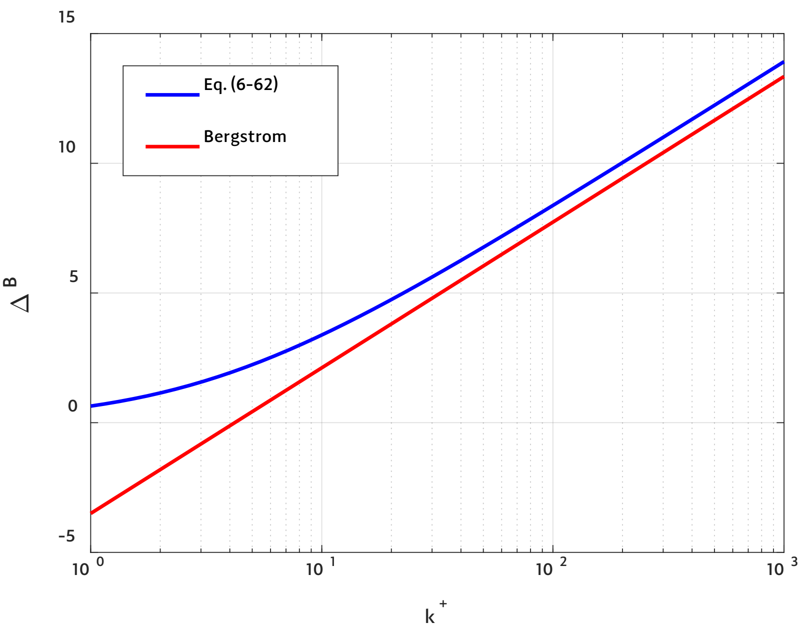
$$\Delta B \approx \frac{1}{\kappa}\ln\left(1+0.3k^+\right)$$

and Bergstrom's expression

$$\Delta B \approx \frac{1}{\kappa}\ln\left(k^+\right)-3.5\ ;\ k^+ \geq 4.2$$

The pertaining MATLAB code is shown below. We use *logspace* to create a set of data from 10^0 to 10^3 and *semilogx* to create a logarithmic-scale *x*-axis.

```
x = logspace(0,3);
eq1 = 1/0.41*log(1+0.3.*x);
semilogx(x,eq1,'b','LineWidth',2);
hold on
grid on
eq2 = 1/0.41*log(x) - 3.5;
semilogx(x,eq2,'r','LineWidth',2);
```



Note that the line for Bergstrom's equation includes some $k^+ < 4.2$, but the correlation is not valid below this range because it may yield negative ΔB's. Now, recall that the average velocity in the pipe can be described by the simple log-law (6-53):

$$u_{\text{avg}} = v^*\left[\frac{1}{\kappa}\ln\left(\frac{Rv^*}{\nu}\right)+B-\Delta B-\frac{3}{2\kappa}\right]$$

where we have included the ΔB correction and used the pipe radius R as the reference dimension a. Replacing ΔB with Bergstrom's correlation, we write

$$u_{\text{avg}} = v^*\left[\frac{1}{\kappa}\ln\left(\frac{Rv^*}{\nu}\right)+B-\frac{1}{\kappa}\ln\left(k^+\right)+3.5-\frac{3}{2\kappa}\right]$$

At this point, we employ the relations

$$\frac{u_{\text{avg}}}{v^*} = \sqrt{\frac{8}{\Lambda}} \;;\; \frac{Rv^*}{\nu} = \text{Re}_D\sqrt{\frac{\Lambda}{32}} \;;\; k^+ = \frac{k}{D}\text{Re}_D\sqrt{\frac{\Lambda}{32}}$$

and note that κ = 0.41 and B = 5.0, so that

$$\frac{u_{\text{avg}}}{v^*} = \frac{1}{\kappa}\ln\left(\frac{Rv^*}{\nu}\right) + B - \frac{1}{\kappa}\ln\left(k^+\right) + 3.5 - \frac{3}{2\kappa}$$

$$\therefore \sqrt{\frac{8}{\Lambda}} = \frac{1}{0.41}\ln\left(\text{Re}_D\sqrt{\frac{\Lambda}{32}}\right) + 5.0 - \frac{1}{0.41}\ln\left(\frac{k}{D}\text{Re}_D\sqrt{\frac{\Lambda}{32}}\right) + 3.5 - \frac{3}{2\times 0.41}$$

$$\therefore \frac{2.83}{\sqrt{\Lambda}} = 2.44\ln\left(\text{Re}_D\sqrt{\frac{\Lambda}{32}}\right) + 5.0 - 2.44\left[\ln\left(\frac{k}{D}\right) + \ln\left(\text{Re}_D\sqrt{\frac{\Lambda}{32}}\right)\right] - 0.16$$

$$\therefore \frac{2.83}{\sqrt{\Lambda}} = \cancel{2.44\ln\left(\text{Re}_D\sqrt{\frac{\Lambda}{32}}\right)} + 5.0 - 2.44\ln\left(\frac{k}{D}\right) - \cancel{2.44\ln\left(\text{Re}_D\sqrt{\frac{\Lambda}{32}}\right)} - 0.16$$

$$\therefore \frac{2.83}{\sqrt{\Lambda}} = 2.44\ln\left(\frac{D}{k}\right) + 4.84$$

$$\therefore \frac{1}{\sqrt{\Lambda}} = 0.862\ln\left(\frac{D}{k}\right) + 1.71$$

$$\therefore \boxed{\frac{1}{\sqrt{\Lambda}} = 1.99\log_{10}\left(\frac{D}{k}\right) + 1.71}$$

Importantly, the Reynolds number vanished along the derivation. Accordingly, Bergstrom's model predicts a *fully rough wall friction* for all Reynolds numbers. Its predictions, however, are considerably less, by 10 to 25%, than the fully rough predictions of the Colebrook formula as $Re_D \to \infty$.

P.6.21 ➡ *Solution*

Eq. (6-41) is Spalding's generalized correlation for the turbulent boundary layer:

$$y^+ = u^+ + e^{-\kappa B}\left[e^{\kappa u^+} - 1 - \kappa u^+ - \frac{\left(\kappa u^+\right)^2}{2} - \frac{\left(\kappa u^+\right)^3}{6}\right]$$

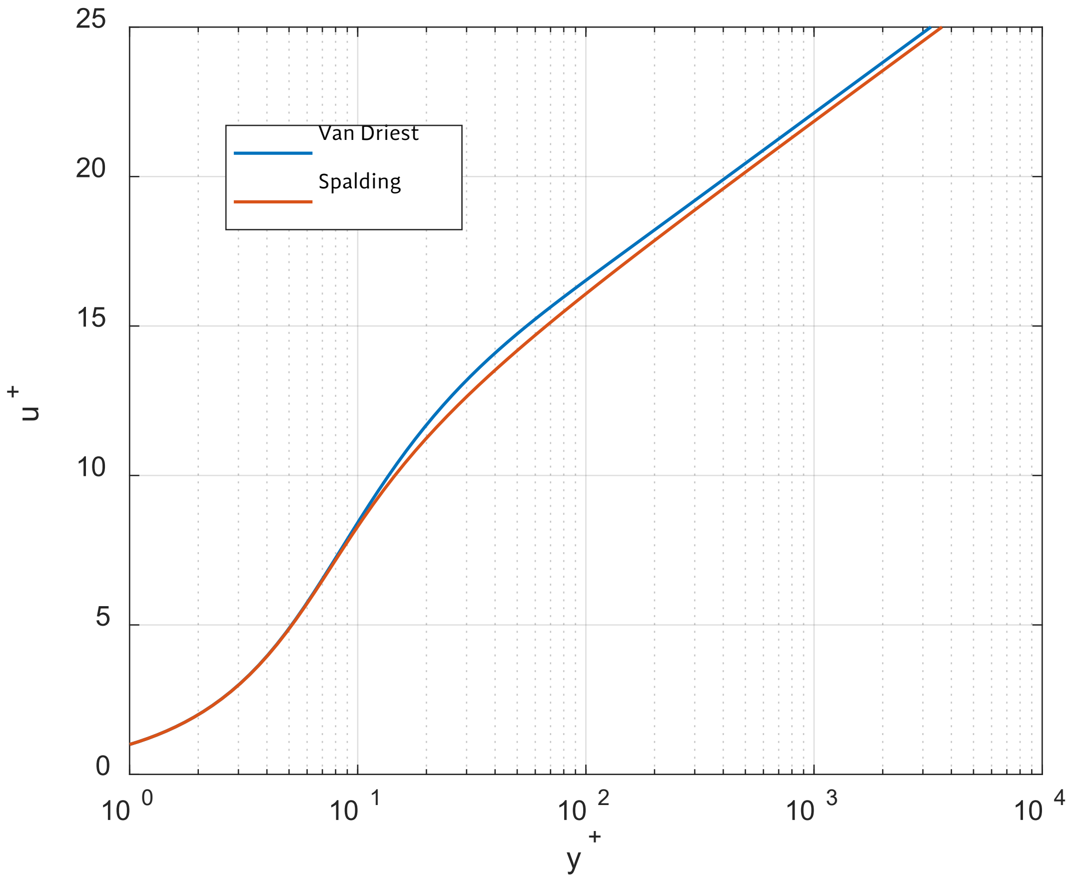
Eq. (6-96) is van Driest's integral for turbulent boundary layer in flat-plate flow,

$$u^+ = \int_0^{y^+} \frac{2dy^+}{1 + \left\{1 + 4\kappa^2\left(y^+\right)^2\left[1 - \exp\left(-y^+/A\right)\right]^2\right\}^{1/2}}$$

with A generally taken as 26. We can plot both expressions with the following MATLAB code:

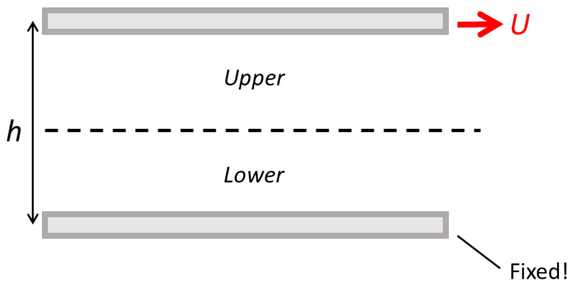
```
fun = @(y) (2./(1 + (1 + 4.*0.41.^2.*y.^2.*(1 - exp(-y./26)).^2).^0.5));
x = logspace(0,4,100);
spaldingY = [0];
i = 1;
Y = [1];
for i = 1:100
    Y(i) = integral(fun,0,x(i));
    Z = 0.41.*Y(i);
    spaldingY(i) = Y(i) + exp(-0.41*5.0).*(exp(Z)-1-Z-(Z.^2)/2-(Z.^3)/6);
    if i == 100
        Y(100) = integral(fun,0,x(100));
        Z = 0.41*Y(100);
        spaldingY(100) = Y(100) + exp(-0.41*5.0).*(exp(Z)-1-Z-(Z^2)/2-
(Z^3)/6);
        semilogx(x,Y)
        grid on
        hold on
        semilogx(spaldingY,Y);
        ylim([0 25]);
        hold off
    end
end
```

The resulting plot is shown below. Notice that the two law-of-the-wall approaches yield very similar results.



Although the two models yield very similar results, the van Driest model is more flexible, in that it can be used to fit other conditions – pressure gradient, blowing/suction, roughness, etc. – by changing the value of the damping constant *A*.

P.6.22 ➡ ***Solution***



The profile is broken into an upper and a lower part, as shown to the side. The lower log-law begins at u = 0 and rises to u = U/2 at the center, y = h/2. The upper part begins at u = U and drops down to u = U/2 at the center. In both the upper and lower region, the log-law must satisfy the centerline condition

$$\frac{U/2}{v^*} = \frac{1}{\kappa}\ln\left(\frac{hv^*}{2\nu}\right) + B$$

This is a relation between shear stress, velocity, and plate separation distance. In dimensionless friction-factor/Reynolds-number form, we may rewrite the above as

$$\left(\frac{\mathrm{Re}_h}{4\phi}\right)^{1/2} = \frac{1}{\kappa}\ln\left(\frac{1}{2}\sqrt{\mathrm{Re}_h\,\phi}\right) + B$$

where Re_h = Uh/ν and ϕ = $\tau_w h/\mu U$. Setting the Reynolds number Re_h to 10^5 and solving for ϕ, we employ the Mathematica code

```
FindRoot[(Subscript[Re, h]/(4*φ))^(1/2)-1/0.41*Log[1/2 √(Re_h * φ) ]-5.0,{φ,10}]
{φ->50.983}
```

That is, $\phi \approx 50.98$. Solving for other Reynolds numbers, we prepare the following table.

Re_h	10^5	10^6	10^7	10^8	10^9	10^{10}
ϕ	51.0	337	2370	17,600	135,000	1.07×10^6

Now, the velocity profiles may be obtained by computing $U^+ = \sqrt{Re_h/\phi}$ and $h^+ = \sqrt{Re_h \times \phi}$ and then referring to the following two log-laws:

Lower layer $0 < y < h/2$	$u^+ = \frac{1}{\kappa}\ln\left(h^+ \times \frac{y}{h}\right) + B$
Upper layer $h/2 < y < h$	$U^+ - u^+ = \frac{1}{\kappa}\ln\left(h^+ \times \frac{h-y}{h}\right) + B$

Consider, as we were asked, $Re_h = 10^5$. We first compute

$$U^+ = \sqrt{\frac{\mathrm{Re}_h}{\phi}} = \sqrt{\frac{100,000}{51.0}} = 44.3$$

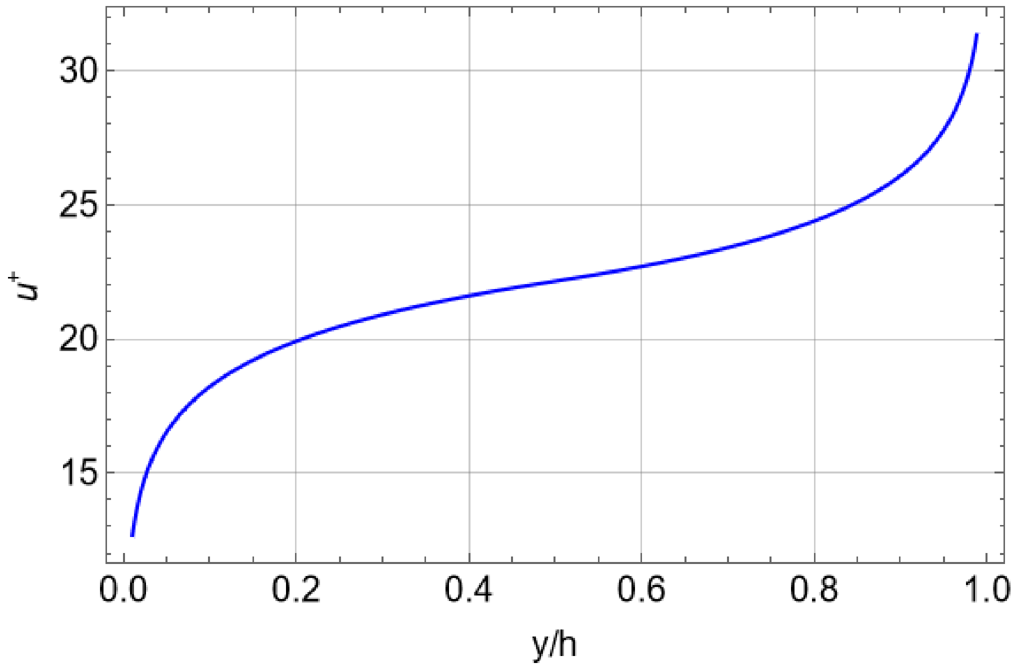
and

$$h^+ = \sqrt{\mathrm{Re}_h \times \phi} = \sqrt{100,000 \times 51.0} = 2260$$

Then, we define a piecewise function with Mathematica's *Piecewise* command and proceed to plot it:

```
In[409]:= f1 = Piecewise[{{1/0.41 * Log[2260 * yh] + 5.0, 0 < yh < 0.5}, {44.3 - (1/0.41 * Log[2260 * (1 - yh)] + 5.0), 0.5 < yh < 1.0}}]

Out[409]= { 5. + 2.43902 Log[2260 yh]          0 < yh < 0.5
            39.3 - 2.43902 Log[2260 (1 - yh)]  0.5 < yh < 1.
            0                                  True

In[412]:= p1 = Plot[f1, {yh, 0, 1}, PlotStyle → Blue, Frame → True, GridLines → Automatic]
```

The velocity profile is indeed *S*-shaped:



P.6.30 ➡ *Solution*

Noting that the mean velocity may be expressed as $\bar{u} = U_{max} \times sech^2(\eta)$ (Eq. (6-151)), we integrate to obtain the mass flow $\dot{m}$:

$$\dot{m} = \int_{-\infty}^{+\infty} \rho u dy = \rho U_{\max} \int_{-\infty}^{+\infty} \mathrm{sech}^2(\eta) d(x\eta/\sigma)$$

Here, we may use the following result,

```
In[414]:= Integrate[Sech[η]^2, {η, -∞, +∞}]

Out[414]= 2
```

so that

$$\dot{m} = \frac{2\rho U_{\max} x}{\sigma} \rightarrow x = \frac{\dot{m}\sigma}{2\rho U_{\max}}$$

$$\therefore x = \frac{800 \times 7.67}{2 \times 998 \times 3.0} = 1.02\,\mathrm{m}$$

Then, at 2 m further downstream, we have x = 1.02 + 2 = 3.02 m; the formulas from Sect. 6-9.1.1 may be used to evaluate the desired flow conditions. First, the width b is

$$\frac{b}{x} = \tan 13^{\circ} \rightarrow b = x \tan 13^{\circ}$$

$$\therefore b = 3.02 \times \tan 13^{\circ} = 0.697\,\text{m} = \boxed{69.7\,\text{cm}}$$

The maximum velocity is, in turn,

$$U_{\max} = U_{x=1.02}\left(\frac{1.02}{3.02}\right)^{1/2} = 3.0 \times \left(\frac{1.02}{3.02}\right)^{1/2} = \boxed{1.74\,\text{m/s}}$$

Finally, the mass flow is determined as

$$\text{Mass flow} = \frac{2\rho U_{\max} x}{\sigma} = \frac{2 \times 998 \times 1.74 \times 3.02}{7.67} = \boxed{1370\,\text{kg/s}\cdot\text{m}}$$

P.6.31 ➡ *Solution*

For air, take $\rho \approx 1.2$ kg/m^3. The air velocity may be obtained from the expression for mass flow rate:

$$\dot{m} = \rho A U_0 \rightarrow U_0 = \frac{\dot{m}}{\rho A}$$

$$\therefore U_0 = \frac{0.001}{1.20 \times \left(\frac{\pi}{4} \times 0.004^2\right)} = 66.3\,\text{m/s}$$

To estimate the jet momentum issuing from the orifice, in turn, we make use of the integral

$$J = \int_{\text{orifice}} \rho u^2 dA = \rho U_0^2 A_0 = 1.20 \times 66.3^2 \times \left(\frac{\pi}{4} \times 0.004^2\right) = 0.0663\,\text{kg}\cdot\text{m/s}$$

Then, the maximum velocity 1 m downstream from the jet may be estimated with Eq. (6-152):

$$U_{\max} \approx 7.4 \frac{(J/\rho)^{1/2}}{x} = 7.4 \times \frac{(0.0663/1.20)^{1/2}}{1.0} = \boxed{1.74\,\text{m/s}}$$

Also from (6-152), we may determine the width of the jet 1.0 m away from the entrance:

$$\eta \approx 15.2\frac{y}{x} \rightarrow b_{1\%} = \frac{x\eta_{1\%}}{15.2}$$

$$\therefore b_{1\%} = \frac{1.0 \times 2.993}{15.2} = 0.197\,\text{m} = \boxed{19.7\,\text{cm}}$$

Lastly, the viscosity ratio, noting that K = 0.018 and μ = 1.8×10^{-5} Pa·s for air at 20°C, is calculated to be:

$$\frac{\mu_t}{\mu} = \frac{K\rho U_{\max} b}{\mu} = \frac{0.018 \times 1.20 \times 1.74 \times 0.197}{1.8 \times 10^{-5}} = \boxed{411}$$

P.6.32 ➡ *Solution*

With reference to Sect. 6-9.3 of the text, we see that

$$y_{1/2} \approx 0.30(x\theta)^{1/2} \;\; ; \;\; \Delta u_{\max} = 1.63U\left(\frac{\theta}{x}\right)^{1/2} \quad \text{(I, II)}$$

We have x = 1000 m, so it remains to calculate the momentum thickness θ of the wake. Recall that the momentum thickness is related to the drag on the cylinder by the simple expression

$$F = \rho U^2 \theta$$

However, the drag is also given by the product of drag coefficient and dynamic pressure, that is,

$$F = C_D \frac{1}{2} \rho U^2 D$$

so that, equating the two expressions and solving for θ, we obtain

$$F = \cancel{\rho U^2} \theta = C_D \frac{1}{2} \cancel{\rho U^2} D$$

$$\therefore \theta = \frac{C_D D}{2} \quad \text{(III)}$$

Noting that $\rho \approx$ 1025 kg/m^3 and $\mu \approx$ 0.0011 Pa·s for seawater at 20°C, we compute the Reynolds number

$$\mathrm{Re}_D = \frac{\rho U D}{\mu} = \frac{1025 \times 0.6 \times 5.0}{0.0011} = 2.80 \times 10^6$$

Entering this Reynolds number into the chart in Figure 3-38(a), we read a drag coefficient $C_D \approx$ 0.5, so that, substituting in (III), we obtain

$$\theta = \frac{C_D D}{2} = \frac{0.5 \times 5.0}{2} = 1.25\,\mathrm{m}$$

Then, we substitute the pertaining variables into (I) to compute the wake width $y_{1/2}$:

$$y_{1/2} = 0.30 (x\theta)^{1/2} = 0.30 \times (1000 \times 1.25)^{1/2} = \boxed{10.6\,\mathrm{m}}$$

Likewise for the wake velocity defect,

$$\Delta u_{\max} = 1.63 U \left(\frac{\theta}{x}\right)^{1/2} = 1.63 \times 0.60 \times \left(\frac{1.25}{1000}\right)^{1/2} = 0.0346\ \mathrm{m/s}$$

$$\therefore \boxed{\Delta u_{\max} \approx 3.5\ \mathrm{cm/s}}$$

These results were obtained via uncertain correlations, to which we fed a simple estimate of the momentum thickness θ. Thus, at best, our results are crude approximations of wake width and velocity defect.

REFERENCE

- WHITE, F.M. (2006). *Viscous Fluid Flow – International Edition*. 3rd edition. New York: McGraw-Hill.